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The Discovering of the New Option Price of the Stock Price Related to the Nobel Prize Work of F. Black and M. Scholes

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Abstract : In this paper, we studied the new option price of the stock price which is the another solution of the Black-Scholes Equation. Such option price can be related to the Black-Scholes Formula which is the Nobel prize work of F. Black and M. Scholes. Moreover we also obtain the kernel of such option price which is the interesting new results. However, we hope that such new results of this paper may be useful in the research area of Financial Mathematics.

Keywords : Black-Scholes Equation; option price.
2010 Mathematics Subject Classification : 35K05; 91G20.

1 Introduction

Back to the year 1973 that F. Black and M. Scholes, [see [1], pp.637-659] introduced the Black-Scholes Formula which is the solution of the Black-Scholes Equation. Such Black-Scholes Formula involes the knowledge of the option price and is accepted as the fair price for trading in the stock market of the European option, Because the outstanding of their work then after that they recieved the Nobel prize as the reward.

Starting with the Black-Scholes Equation

$$\frac{\partial u}{\partial t}u(s,t) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2}u(s,t) + rs\frac{\partial}{\partial s}u(s,t) - ru(s,t) = 0 \tag{1.1}$$

with the call payoff

$$u(s_T, T) = \max(s_T - p, 0) = (s_T - p)^+ \tag{1.2}$$

where $u(s,t)$ is the option price of the stock for $0 \leq t \leq T$, s is the price of stock at time t, σ is the volatility of stock, r is the interest, s_T is the price of stock at the expiration time T and p is the strike price. The solution of (1.1) satisfies (1.2) is the Black-Scholes Formula which is the option price and given by

$$u(s,t) = sN(d_1) - pe^{-r(T-t)}N(d_2) \tag{1.3}$$

where

$$d_1 = \frac{\ln\left(\frac{s}{p}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{1.4}$$

$$d_2 = \frac{\ln\left(\frac{s}{p}\right) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{1.5}$$

and denote $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy$. In this work, we discovered the new option price in the convolution form

$$u(s,t) = V(R,\tau) = K(R,\tau) * f(R) \tag{1.6}$$

where $R = \ln s,\ \tau = T - t,\ f$ is the continuous function and

$$K(R,\tau) = \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}} \exp\left[-\frac{(R + (r - \frac{1}{2}\sigma^2)\tau)^2}{2\sigma^2\tau}\right]. \tag{1.7}$$

Moreover (1.6) can be computed as

$$u(s,t) = s - e^{r(T-t)}p. \tag{1.8}$$

Now (1.8) is the new option price which is simple formula. Compare with (1.3), we see that (1.3) is more complicated formula. We czn also show that (1.6) can be related to (1.3). That means the option price in (1.6) can be related to the Nobel prize work in (1.3). Moreover we can also study the properties of the kernel in (1.7).

2 Preliminaries

The following definitions and some lemmas are need.

Definition 2.1. Let f be an integrable function on the set real of number $\mathcal{R}$. Then the Fourier transform of f is defined by

$$\mathfrak{F}f(x) = \widehat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f(x)dx \tag{2.1}$$

and the inverse Fourier transform also defined by

$$f(x) = \mathfrak{F}^{-1}\left(\mathfrak{F}f(x)\right) = \mathfrak{F}^{-1}\widehat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\widehat{f}(\omega)d\omega. \tag{2.2}$$

Definition 2.2. (The Dirac-delta distribution)
The Dirac-delta distribution or the impulse function is denoted by δ and also defined by

$$< \delta(x), \varphi(x) >\equiv \int_{-\infty}^{\infty} \delta(x)\varphi(x)dx = \varphi(0) \tag{2.3}$$

where $\varphi(x)$ is the testing function of infinitely differentiable with compact support.

Lemma 2.3. *Recall the equation* (1.1) *and the call payoff* (1.2). *By changing the variable* $R = \ln s,\ \tau = T - t$ *and write* $u(s,t) = V(R,\tau)$ *then* (1.1) *and* (1.2) *is transformed to*

$$\frac{\partial}{\partial \tau}V(R,\tau) - \frac{1}{2}\sigma^2\frac{\partial^2}{\partial R^2}V(R,\tau) - (r - \frac{1}{2}\sigma^2)\frac{\partial}{\partial R}V(R,\tau) + rV(R,\tau) = 0 \tag{2.4}$$

with the call payoff

$$V(R,0) = (e^R - p)^+ \equiv f(R) \tag{2.5}$$

where f *is the continuous function of* R *and* $\tau = 0$ *corresponds to* $t = T$. *Moreover* (2.4) *has the solution in convolution form*

$$V(R,\tau) = K(R,\tau) * f(R) \tag{2.6}$$

where

$$K(R,\tau) = \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}}\exp\left[-\frac{(R + (r - \frac{1}{2})\tau)^2}{2\sigma^2\tau}\right] \tag{2.7}$$

as the kernel of (2.4).

Proof. Since we have $u(s,t) = V(R,\tau)$ where $R = \ln s$ and $\tau = T - t$. By using chain rules and then substitute into (1.1). Then we obtain (2.4) and the call payoff (1.2) is also transformed to (2.5). Take the Fourier transform with respect to R to (2.4) then we obtain

$$\frac{\partial}{\partial \tau}\widehat{V}(\omega,\tau) + \frac{1}{2}\sigma^2\omega^2\widehat{V}(\omega,\tau) - i\omega(r - \frac{1}{2}\sigma^2)\widehat{V}(\omega,\tau) + r\widehat{V}(\omega,\tau) = 0 \tag{2.8}$$

and from (2.5),

$$\widehat{V}(\omega,0)=\widehat{f}(\omega). \tag{2.9}$$

Now

$$\widehat{V}(\omega,\tau)=C(\omega)\exp\left[-\frac{1}{2}\sigma^2\omega^2+i\omega(r-\frac{1}{2}\sigma^2)-r\right]\tau$$

is the solution (2.8) and also from (2.9) we obtain $C(\omega)=\widehat{f}(\omega)$. Thus

$$\widehat{V}(\omega,\tau)=\widehat{f}(\omega)\exp\left[-\frac{1}{2}\sigma^2\omega^2+i\omega(r-\frac{1}{2}\sigma^2)-r\right]\tau.$$

Since

$$V(R,\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega R}\widehat{V}(\omega,\tau)d\omega,$$

thus

$$\begin{aligned}V(R,\tau)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega R}\widehat{f}(\omega)\exp\left[-\frac{1}{2}\sigma^2\omega^2+i\omega(r-\frac{1}{2}\sigma^2)-r\right]\tau d\omega\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i\omega R}e^{-i\omega y}\exp\left[-\frac{1}{2}\sigma^2\omega^2+i\omega(r-\frac{1}{2}\sigma^2)-r\right]\tau f(y)dyd\omega.\end{aligned}$$

where $\widehat{f}(\omega)=\int_{-\infty}^{\infty}e^{-i\omega y}f(y)dy$. Thus we have

$$V(R,\tau)=\frac{1}{2\pi}e^{-r\tau}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\sigma^2\omega^2\tau+i\left((r-\frac{1}{2}\sigma^2)\tau+R-y\right)\omega\right]f(y)dyd\omega.$$

By completing the square then we have

$$\begin{aligned}V(R,\tau)=&\frac{e^{-r\tau}}{2\pi}\int_{-\infty}^{\infty}\exp\left[-\frac{((r-\frac{1}{2}\sigma^2)\tau+R-y)^2}{2\sigma^2\tau}\right]f(y)dy\times\\&\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\sigma^2\tau(\omega-\frac{i((r-\frac{1}{2}\sigma^2)\tau+R-y)^2}{\sigma^2\tau})\right]d\omega.\end{aligned}$$

Put $z=\frac{\sigma\sqrt{\tau}}{\sqrt{2}}\left(\omega-i\frac{((r-\frac{1}{2}\sigma^2)\tau+R-y)}{\sigma^2\tau}\right)$ then $d\omega=\frac{\sqrt{2}}{\sigma\sqrt{\tau}}dz$. Thus

$$\begin{aligned}V(R,\tau)&=\frac{e^{-r\tau}}{2\pi}\frac{\sqrt{2}}{\sigma\sqrt{\tau}}\int_{-\infty}^{\infty}\exp\left[-\frac{((r-\frac{1}{2}\sigma^2)\tau+R-y)^2}{2\sigma^2\tau}\right]f(y)dy\int_{-\infty}^{\infty}e^{-z^2}dz\\&=\frac{e^{-r\tau}}{2\pi}\frac{\sqrt{2}}{\sigma\sqrt{\tau}}\sqrt{\pi}\int_{-\infty}^{\infty}\exp\left[-\frac{((r-\frac{1}{2}\sigma^2)\tau+R-y)^2}{2\sigma^2\tau}\right]f(y)dy.\end{aligned}$$

[Note that $\int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}$]. Thus we have

$$V(R,\tau) = \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}} \int_{-\infty}^{\infty} \exp\left[-\frac{((r-\frac{1}{2}\sigma^2)\tau + R - y)^2}{2\sigma^2\tau}\right] f(y)dy \tag{2.10}$$

as the solution of (2.4). Let

$$K(R,\tau) = \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}} \exp\left[-\frac{((r-\frac{1}{2}\sigma^2)\tau + R - y)^2}{2\sigma^2\tau}\right] f(y)dy$$

then $K(R,\tau)$ is the kernel or Green function of (2.4). Now (2.10) can be written in the convolution form

$$V(R,\tau) = K(R,\tau) * f(R).$$

Thus we obtain (2.6) and (2.7) as required. □

Lemma 2.4. *The equation* (2.10) *can be computed as the new option price*

$$V(R,\tau) = e^R - e^{-r\tau}p$$

which is the solution of (2.4) *and*

$$u(s,t) = V(\ln s, T-t) = s - e^{T-t}p$$

is the solution of (1.1). *Moreover, the equation* (2.10) *is related to the Black-Scholes Formula given by* (1.3)

Proof. From (2.10)

$$V(R,\tau) = \frac{e^{-t\tau}}{\sqrt{2\pi\sigma^2\tau}} \int_{-\infty}^{\infty} \exp\left[-\frac{(R + (r-\frac{1}{2}\sigma^2)\tau - y)^2}{2\sigma^2\tau}\right] (e^y - p)^+ dy$$

where $f(y) = (e^y - p)^+$ from (2.5). Thus

$$\begin{aligned} V(R,\tau) =& \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}} \left(\int_{-\infty}^{\infty} \exp\left[-\frac{(R + (r-\frac{1}{2}\sigma^2)\tau - y)^2}{2\sigma^2\tau}\right] e^y dy \right. \\ & \left. -p \int_{-\infty}^{\infty} \exp\left[-\frac{(R + (r-\frac{1}{2}\sigma^2)\tau - y)^2}{2\sigma^2\tau}\right] dy \right). \end{aligned} \tag{2.11}$$

Since

$$\int_{-\infty}^{\infty}\exp\left[-\frac{(R+(r-\frac{1}{2}\sigma^2)\tau-y)^2}{2\sigma^2\tau}\right]dy=\int_{-\infty}^{\infty}\exp\left[-\frac{(y-R-(r-\frac{1}{2}\sigma^2)\tau)^2}{2\sigma^2\tau}\right]dy.$$

Put $w=\frac{1}{\sigma\sqrt{2\tau}}(y-R-(r-\frac{1}{2}\sigma^2)\tau)\tau$ then $dy=\sigma\sqrt{2\tau}dw$ and $y=\sigma\sqrt{2\tau}w+R+(r-\frac{1}{2}\sigma^2)\tau$. Thus

$$\begin{aligned}
V(R,\tau) &= \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}}\left(\int_{-\infty}^{\infty}\sigma\sqrt{2\tau}e^{-w^2}\exp\left[\sigma\sqrt{2\tau}w+R+(r-\frac{1}{2}\sigma^2)\tau\right]dw\right. \\
&\qquad \left. -p\int_{-\infty}^{\infty}\sigma\sqrt{2\tau}e^{-w^2}dw\right) \\
&= \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}}\sigma\sqrt{2\tau}e^{R+(r-\frac{1}{2}\sigma^2)\tau}\int_{-\infty}^{\infty}e^{-w^2+\sigma\sqrt{2\tau}w}dw-\frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}}\sigma\sqrt{2\tau}\sqrt{\pi}p \\
&= \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau\tau}}\sigma\sqrt{2\tau}e^{R+(r-\frac{1}{2}\sigma^2)\tau}e^{\frac{1}{2}\sigma^2\tau}\int_{-\infty}^{\infty}e^{-(w-\frac{1}{2}\sigma\sqrt{2\tau})^2}d(w-\frac{1}{2}\sigma\sqrt{2}\tau)-e^{-r\tau}p \\
&= \frac{e^R\sigma\sqrt{2\tau}\sqrt{\pi}}{\sqrt{2\pi\sigma^2\tau}}-e^{-r\tau}p \\
&= e^R-e^{-r\tau}p.
\end{aligned}$$

Thus we obtain $V(R,\tau)=e^R-e^{-r\tau}p$ as the solution of (2.4) and also $u(s,t)=V(\ln s,T-t)=s-e^{-r(T-t)}p$ is the solution of (1.1) as required.
Next we can relate (2.11) to the Black-Scholes Formula in (1.3).
Let $A=\exp\left[-\frac{(R+(r-\frac{1}{2}\sigma^2)\tau-y)^2}{2\sigma^2\tau}\right]e^y$
and $B=\exp\left[-\frac{(R+(r-\frac{1}{2}\sigma^2)\tau-y)^2}{2\sigma^2\tau}\right]$ then (2.11) can be written as

$$V(R,\tau)=\frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}}\left(\int_{-\infty}^{\infty}Ady-p\int_{-\infty}^{\infty}Bdy\right).$$

Now

$$\int_{-\infty}^{\infty}Bdy=\int_{-\infty}^{\infty}\exp\left[-\frac{(R+(r-\frac{1}{2}\sigma^2)\tau-y)^2}{2\sigma^2\tau}\right]dy$$

choose $y \geq \ln p$ then $-y \leq -\ln p$ and put $\dfrac{z}{\sqrt{2}} = \dfrac{R + (r - \frac{1}{2}\sigma^2)\tau - y}{\sqrt{2\sigma^2\tau}}$. Thus

$$-\infty < z \leq \frac{R + (r - \frac{1}{2}\sigma^2)\tau - \ln p}{\sqrt{\sigma^2\tau}} \text{ and } d(-y) = dy = \sqrt{\sigma^2\tau}dz.$$

Let $\alpha = \dfrac{R + (r - \frac{1}{2}\sigma^2)\tau - \ln p}{\sqrt{\sigma^2\tau}}$ and by restriction the integration with the interval of z then

$$\int_{-\infty}^{\infty} Bdy = \int_{-\infty}^{\alpha} e^{-\frac{z^2}{2}} \sqrt{\sigma^2\tau}dz = \sqrt{\sigma^2\tau} \int_{-\infty}^{\alpha} e^{-\frac{z^2}{2}} dz.$$

Next consider the integral $\int_{-\infty}^{\infty} Ady$. Now

$$\int_{-\infty}^{\infty} Ady = \int_{-\infty}^{\infty} \exp\left[-\frac{(R + (r - \frac{1}{2}\sigma^2)\tau - y)^2}{2\sigma^2\tau}\right] e^y dy.$$

Let $u = \dfrac{1}{\sqrt{2\sigma^2\tau}}(R+(r-\frac{1}{2}\sigma^2)\tau-y)$ then $y = R+(r-\frac{1}{2}\sigma^2)\tau-\sqrt{2\sigma^2\tau}u$ and $d(-y) = dy = \sqrt{2\sigma^2\tau}du$. Choose $y \geq \ln p$ then $-\infty < u \leq \dfrac{R + (r - \frac{1}{2}\sigma^2)\tau - \ln p}{\sqrt{2\sigma^2\tau}}$. By restriction the integration with the interval of u and let $\beta = \dfrac{R + (r - \frac{1}{2}\sigma^2)\tau - \ln p}{\sqrt{2\sigma^2\tau}}$ then

$$\begin{aligned}
\int_{-\infty}^{\infty} Ady &= \int_{-\infty}^{\beta} \sqrt{2\sigma^2\tau} Adu \\
&= \sqrt{2\sigma^2\tau} \int_{-\infty}^{\beta} e^{-u^2} \exp\left[R + (r - \frac{1}{2}\sigma^2)\tau - \sqrt{2\sigma^2\tau}u\right] du \\
&= \sqrt{2\sigma^2\tau} e^{R+(r-\frac{1}{2}\sigma^2)\tau} e^{\frac{1}{2}\sigma^2\tau} \int_{-\infty}^{\beta} e^{-(u+\frac{1}{\sqrt{2}}\sqrt{\sigma^2\tau})^2} du
\end{aligned}$$

put $\frac{\theta}{\sqrt{2}} = u + \frac{1}{\sqrt{2}}\sqrt{\sigma^2\tau}$ then $du = \frac{1}{\sqrt{2}}d\theta$ and $\theta = \sqrt{2}u + \sqrt{\sigma^2\tau}$. Thus

$$-\infty < \theta \leq \frac{R + (r - \frac{1}{2}\sigma^2)\tau - \ln p}{\sqrt{\sigma^2\tau}} + \sqrt{\sigma^2\tau}$$
$$-\infty < \theta \leq \frac{R + (r + \frac{1}{2}\sigma^2)\tau - \ln p}{\sqrt{\sigma^2\tau}}.$$

Let $\gamma = \dfrac{R + (r + \frac{1}{2}\sigma^2)\tau - \ln p}{\sqrt{\sigma^2\tau}}$ then

$$\int_{-\infty}^{\infty} A dy = \sqrt{2\sigma^2\tau} e^{R+(r-\frac{1}{2}\sigma^2)\tau} e^{\frac{1}{2}\sigma^2\tau} \int_{-\infty}^{\gamma} e^{-\frac{\theta^2}{2}} \frac{1}{\sqrt{2}} d\theta.$$

Thus we have

$$V(R,\tau) = \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}} \left(\sqrt{2\sigma^2\tau} e^{R+(r-\frac{1}{2}\sigma^2)\tau} e^{\frac{1}{2}\sigma^2\tau} \int_{-\infty}^{\gamma} e^{-\frac{\theta^2}{2}} \frac{1}{\sqrt{2}} d\theta \right).$$

Thus we have

$$\begin{aligned} V(R,\tau) &= \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}} \left(\sqrt{\sigma^\tau} e^{R+(r-\frac{1}{2}\sigma^2)\tau} e^{\frac{1}{2}\sigma^2\tau} \int_{-\infty}^{\gamma} e^{-\frac{\theta^2}{2}} d\theta \right) \\ &\quad - \frac{e^{-r\tau}\sqrt{\sigma^2\tau}}{\sqrt{2\pi\sigma^2\tau}} p \int_{-\infty}^{\alpha} e^{-\frac{z^2}{2}} dz \\ &= e^R \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\gamma} e^{-\frac{\theta^2}{2}} d\theta \right) - e^{-r\tau} p \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\frac{z^2}{2}} dz \right) \\ &= e^R N(\gamma) - e^{r\tau} p N(\alpha). \end{aligned}$$

Since $u(s,t) = V(\ln s, T-t)$.
Hence $u(s,t) = sN(\gamma) - e^{r(T-t)}N(\alpha)$
Since $\gamma = \dfrac{R + (r + \frac{1}{2}\sigma^2)\tau - \ln p}{\sqrt{\sigma^2\tau}}$
and $\alpha = \dfrac{R + (r - \frac{1}{2}\sigma^2)\tau - \ln p}{\sqrt{\sigma^2\tau}}$.

Thus

$$\gamma = \frac{R + (r + \frac{1}{2}\sigma^2)\tau - \ln p}{\sqrt{\sigma^2\tau}} = \frac{\ln\left(\frac{s}{p}\right) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}},$$

$$\alpha = \frac{R + (r - \frac{1}{2}\sigma^2)\tau - \ln p}{\sqrt{\sigma^2\tau}} = \frac{\ln\left(\frac{s}{p}\right) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}.$$

Let $d_1 = \gamma$ and $d_2 = \alpha$ thus we have the Black-Scholes formula

$$u(s,t) = sN(d_1) - e^{-r(T-t)}pN(d_2).$$

So we obtain (1.3) as required. □

Lemma 2.5.

$$\lim_{\tau\to 0} K(R,\tau) = \delta(R)$$

where $K(R,\tau)$ *is the kernel in* (2.7) *and* $\delta(R)$ *is the Dirac-delta distribution from definition 2.2. Moreover from* (2.6) $V(R,0) = f(R)$*. That implies* (2.5) *holds.*

Proof. Since $K(R,\tau) = \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}} \exp\left[-\frac{(R + (r - \frac{1}{2}\sigma^2)\tau)^2}{2\sigma^2\tau}\right]$, hence $\lim_{\tau\to 0} K(R,\tau) = \delta(R)$, [see [2], pp.36-37]. Now from (2.6), $V(R,\tau) = K(R,\tau) * f(R)$ thus

$$V(R,0) = \lim_{\tau\to 0} V(R,\tau) = \lim_{\tau\to 0} K(R,\tau) * f(R) = \delta(R) * f(R) = f(R)$$

by the property of $\delta(R)$. That implies (2.5) holds. □

3 Main Results

The results of definitions and lemmas can be obtained the following theorems.

Theorem 3.1. *The Black-Scholes Equation in* (1.1) *with the call payoff* (1.2) *has a solution as the option price*

$$u(s,t) = K(\ln s, T-t) * f(\ln s) \tag{3.1}$$

where

$$K(\ln s, T-t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}} \exp\left[-\frac{(\ln s + (r - \frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)}\right] \tag{3.2}$$

is the kernel. Moreover from (3.1),

$$u(s,t) = s - e^{-r(T-t)}p \tag{3.3}$$

and also (3.1) *can be related to the Black-Scholes Formula in* (1.3).

Proof. By Lemma 2.3 with (2.6) and (2.7)

$$V(R,\tau) = K(R,\tau) * f(R)$$

where

$$K(\ln s,\tau) = \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}} \exp\left[-\frac{(R+(r-\frac{1}{2}\sigma^2)\tau)^2}{2\sigma^2\tau}\right].$$

Since $u(s,t) = V(R,\tau) = V(\ln s, T-t)$ where $R = \ln s$ and $\tau = T - t$. Thus we obtain (3.1) and (3.2) are required. Next, from Lemm 2.4 we also obtain (3.3) and the Black-Scholes formula in (1.3). Moreover, from (3.3) for $t = T$

$$u(s,T) = s - e^{-r(T-t)}p = (s-p)^+.$$

That implies the call payoff (1.2) holds. Compare with the Black-Scholes Formula in (1.3)

$$u(s,t) = sN(d_1) - pe^{-r(T-t)}N(d_2).$$

Now, form (1.4) and (1.3) $d_1 = d_2 = \infty$ for $t = T$. Thus $N(d_1 = N(d_2) = 1$ for $t = T$. It follows that the call payoff (3.1) is the same as the call payoff of the Black-Scholes Formula in (1.3). We can say that (3.3) is another option price which is new one and is more simple form compare with (1.3). □

Theorem 3.2. (The properties of the kernel $K(R,\tau)$)
The kernel $K(R,\tau)$ *in* (2.7) *have the following properties*

(1) $K(R,\tau)$ *satisfies the equation* (2.4).

(2) $K(R,\tau) > 0$ *for* $\tau \geq 0$.

(3) $\lim_{\tau\to 0} K(R,\tau) = \delta(R)$ *that* $\tau \to 0$ *corresponds to* $t \to T$.

(4) $\int_{-\infty}^{\infty} e^{r\tau}K(R,\tau)dR = 1.$

(5) $K(R,\tau)$ *is the Gaussian function or Normal distribution with mean* $e^{-r\tau}(\frac{1}{2}\sigma^2 - r)\tau$ *and variance* $e^{-2r\tau}\sigma^2\tau$.

(6) $K(R,\tau)$ *is the tempered distribution, that is* $K(R,\tau) \in S'(\mathcal{R})$ *is the space of tempered distribution on the set of real number.*

Proof. (1) Since $K(R,\tau)$ is an elementary solution of (2.4). Thus by varification $K(R,\tau)$ satisfies the equation (2.4).

(2) $K(R,\tau) > 0$ for $\tau \geq 0$ is obvious.

(3) By Lemma 2.5 $\lim_{\tau\to 0} K(R,\tau) = \delta(R)$.

(4) Since $\int_{-\infty}^{\infty} e^{r\tau}\frac{1}{\sqrt{2\tau}}K(R,\tau)dR = \frac{1}{\sqrt{2\pi\sigma^2\tau}}\int_{-\infty}^{\infty}\exp\left[-\frac{(R+(r-\frac{1}{2}\sigma^2)\tau)^2}{2\sigma^2\tau}\right]dR$

put $u = \frac{1}{\sqrt{2\tau\sigma^2}}(R+(r-\frac{1}{2}\sigma^2)\tau)$ then $dR = \sigma\sqrt{2\tau}du$. Thus

$$\int_{-\infty}^{\infty} e^{r\tau}K(R,\tau)dR = \frac{\sigma\sqrt{2\tau}}{\sqrt{2\pi\sigma^2\tau}}\int_{-\infty}^{\infty} e^{-u^2}du = \frac{\sigma^2\sqrt{2\tau}\sqrt{\pi}}{\sqrt{2\pi\sigma^2\tau}} = 1.$$

(Note that $\int_{-\infty}^{\infty} e^{-u^2}du = 1$).

(5)

$$\begin{aligned}\text{Mean } &= E(K(R,\tau) = e^{-r\tau}E\left(\frac{1}{\sqrt{2\pi\sigma^2\tau}}\exp\left[-\frac{(R+(r-\frac{1}{2}\sigma^2)\tau)^2}{2\sigma^2\tau}\right]\right)\\ &= e^{-r\tau}E\left(\exp\left[-\frac{(R-(\frac{1}{2}\sigma^2-r)\tau)^2}{2\sigma^2\tau}\right]\right)\\ &= e^{-r\tau}(\frac{1}{2}\sigma^2-r)\tau\end{aligned}$$

where E is the expected value.

$$\begin{aligned}\text{Variance } &= V(K(R,\tau))\\ &= e^{-2r\tau}V\left(\frac{1}{\sqrt{2\pi\sigma^2\tau}}\exp\left[-\frac{(R-(\frac{1}{2}\sigma^2-r)\tau)^2}{2\sigma^2\tau}\right]\right)\\ &= e^{-2r\tau}\sigma^2\tau.\end{aligned}$$

(6) $K(R,\tau) \in S'(\mathcal{R})$, see [[3], pp.135-136]. □

4 Conclusion

The main results of this work is the new option price given by (3.1) and (3.3). We see that such new option price can be related to the Black-Scholes Formula given (1.3). Such Black-Scholes Formula is the option price which is very popular for the investment in the stock market. Because this work M. Black and F. Sholes recieved the Nobel prize in the year 1973. We se that the Black-Scholes Formula in (1.3) is very complicated formula and is not comfortable for using. But the new option price given by (3.3) is more simple and very comfortable to be applied for the investment.

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