



Some Results on Coincidence Points for Contractions in Intuitionistic Fuzzy n -Normed Linear Space

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Abstract : In this paper, we prove some new coupled coincidence point theorems for two types of contraction mappings - one having commutative condition and other having non-commutative condition in an intuitionistic fuzzy n -Banach space. Our results extend and generalize some important existing classical coupled coincidence point theorems in literature. Some newly constructed examples support the non-triviality of our results.

Keywords : intuitionistic fuzzy n -Banach space; coupled fixed point; commutative mapping; non-commutative mapping.

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1 Introduction

In numerous problems in mathematics, the existence of a solution becomes analogous to the existence of a fixed point for a particular map. Thus the existence of a fixed point, coupled fixed point and coupled coincidence point of two mappings are of immense importance in various areas of mathematics such as

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Chaos Theory, Game Theory, Linear Programing, Differential Equations, Periodic Boundary Value Problems, Dynamical Systems etc. Fixed point theorems and coincidence point theorems provide conditions under which maps have solutions. Thus the study of fixed points remain a well motivated area of research in both classical and fuzzy settings.

In order to explain the situations where data are imprecise or vague Zadeh [1] introduced fuzzy set theory in 1965. On the other hand in 1986, considering both the degree of membership (belongingness) and non-membership (non-belongingness) of an element within a set, Atanassaov [2] introduced a generalized form of fuzzy set called intuitionistic fuzzy set. In 1984 Katsaras [3] established the idea of a fuzzy norm on a linear space for dealing with situations where the classical norm cannot measure the length of a vector accurately. In 2003, following Cheng and Mordeson [4], Bag and Samanta [5] introduced the concept of fuzzy normed linear space (FNLS). After the systematic development of fuzzy normed linear space, one of the important development over FNLS is the notion of intuitionistic fuzzy normed linear space (IFNLS) [6]. Vijayabalaji and Narayanan [7] extended n -normed linear space to fuzzy n -normed linear space while the concept of intuitionistic fuzzy n -normed linear space (IFnNLS) was introduced by Vijayabalaji et al [8].

The study of fixed points in fuzzy metric spaces was introduced by Heilpern [9]. He established fixed point theorems for fuzzy contraction mappings, which was a fuzzy extension of the Banach's contraction principle in metric linear space. This work was further extended by Burtnariu [10]. By using the concept of semi-compatibility and reciprocal continuity of mappings, Badshah and Joshi [11] established a common fixed point theorem for six mappings. In 1988, Grabiec [12] proved Banach contraction principle in complete fuzzy metric space and Edelstein contraction principle in compact fuzzy metric space. Extending Grabiec's work, Alaca et al. [13] introduced fixed point theorems of Banach and Edelstein in intuitionistic fuzzy metric space. In 2013, Ionescu et al. [14] found fixed points for some new contractions on intuitionistic fuzzy metric spaces. Manro and Tomar [15] established existence of fixed point of compatibility maps on fuzzy metric space. For binary mappings in partially ordered metric spaces the coupled fixed point theorems and their applications were introduced by Bhaskar and Lakshmikantham [16]. After that Lakshmikantham and Ćirić [17] introduced some more coupled fixed point theorems in partially ordered sets. In 2011, Gordji et al. [18] introduced coupled coincidence point theorems for contraction mappings in partially complete IFNLS. Some breakthrough works in fixed point theory was carried out by Petrusel and Rus [19] and Rus et al. [20]. For some more significant work in this direction we refer to [21–32].

In the current paper we are going to establish new coupled coincidence point theorems for some contraction mappings having commutative property and some having non-commutative property in an intuitionistic fuzzy n -Banach space (IFnBS).

2 Preliminaries

First we recall some basic definitions and examples which are useful for the current work.

Definition 2.1. [33] Let $n \in \mathbb{N}$ and X be a real linear space of dimension $d \geq n$ (d may be infinite). A real valued function $\|\cdot\|$ on $\underbrace{X \times X \times \cdots \times X}_n = X^n$ is called an n -norm on X if it satisfies the following properties:

- (i) $\|x_1, x_2, \dots, x_n\| = 0$ if and only if x_1, x_2, \dots, x_n are linearly dependent,
- (ii) $\|x_1, x_2, \dots, x_n\|$ is invariant under any permutation,
- (iii) $\|x_1, x_2, \dots, \alpha x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$ for any $\alpha \in \mathbb{R}$,
- (iv) $\|x_1, x_2, \dots, x_{n-1}, y + z\| \leq \|x_1, x_2, \dots, x_{n-1}, y\| + \|x_1, x_2, \dots, x_{n-1}, z\|$,
and the pair $(X, \|\cdot\|)$ is called an n -normed linear space.

Definition 2.2. [8] An IFnNLS is the five-tuple $(X, \mu, \nu, *, \circ)$, where X is a linear space over a field \mathbb{R} , $*$ is a continuous t-norm, \circ is a continuous t-conorm, μ, ν are fuzzy sets on $X^n \times (0, \infty)$, μ denotes the degree of membership and ν denotes the degree of non-membership of $(x_1, x_2, \dots, x_n, t) \in X^n \times (0, 1)$ satisfying the following conditions for every

$(x_1, x_2, \dots, x_n) \in X^n$ and $s, t > 0$:

- (i) $\mu(x_1, x_2, \dots, x_n, t) + \nu(x_1, x_2, \dots, x_n, t) \leq 1$,
- (ii) $\mu(x_1, x_2, \dots, x_n, t) > 0$,
- (iii) $\mu(x_1, x_2, \dots, x_n, t) = 1$ if and only if x_1, x_2, \dots, x_n are linearly dependent,
- (iv) $\mu(x_1, x_2, \dots, x_n, t)$ is invariant under any permutation of x_1, x_2, \dots, x_n ,
- (v) $\mu(x_1, x_2, \dots, cx_n, t) = \mu(x_1, x_2, \dots, x_n, \frac{t}{|c|})$ if $c \neq 0, c \in F$,
- (vi) $\mu(x_1, x_2, \dots, x_n + x'_n, s + t) \geq \min\{\mu(x_1, x_2, \dots, x_n, s), \mu(x_1, x_2, \dots, x'_n, t)\}$,
- (vii) $\mu(x_1, x_2, \dots, x_n, \cdot)$ is non-decreasing function of R^+ and
 $\lim_{t \rightarrow \infty} \mu(x_1, x_2, \dots, x_n, t) = 1$,
- (viii) $\nu(x_1, x_2, \dots, x_n, t) < 1$,
- (ix) $\nu(x_1, x_2, \dots, x_n, t) = 0$ if and only if x_1, x_2, \dots, x_n are linearly dependent,
- (x) $\nu(x_1, x_2, \dots, x_n, t)$ is invariant under any permutation of x_1, x_2, \dots, x_n ,
- (xi) $\nu(x_1, x_2, \dots, cx_n, t) = \nu(x_1, x_2, \dots, x_n, \frac{t}{|c|})$ if $c \neq 0, c \in F$,
- (xii) $\nu(x_1, x_2, \dots, x_n + x'_n, s + t) \leq \max\{\nu(x_1, x_2, \dots, x_n, s), \nu(x_1, x_2, \dots, x'_n, t)\}$,
- (xiii) $\nu(x_1, x_2, \dots, x_n, \cdot)$ is non-increasing function of R^+ and
 $\lim_{t \rightarrow \infty} \nu(x_1, x_2, \dots, x_n, t) = 0$.

Also assume that

- (xiv) $\mu(x_1, x_2, \dots, x_n, t) > 0$ and $\nu(x_1, x_2, \dots, x_n, t) < 1$, for all $t > 0$ implies $x = 0$.
- (xv) For $x \neq 0$, $\mu(x_1, x_2, \dots, x_n, \cdot)$ and $\nu(x_1, x_2, \dots, x_n, \cdot)$ are continuous functions of \mathbb{R} and μ and ν are respectively strictly increasing and strictly decreasing on the subset $\{t : 0 < \mu(x_1, x_2, \dots, x_n, t), \nu(x_1, x_2, \dots, x_n, t) < 1\}$ of \mathbb{R} .

Definition 2.3. [34] Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. We say that a sequence $x = \{x_k\}$ in X is convergent to $l \in X$ with respect to the intuitionistic fuzzy n -norm $(\mu, \nu)^n$ if, for every $\epsilon > 0$, $t > 0$ and $y_1, y_2, \dots, y_{n-1} \in X$, there exists $k_0 \in \mathbb{N}$ such that $\mu(y_1, y_2, \dots, y_{n-1}, x_k - L, t) > 1 - \epsilon$ and $\nu(y_1, y_2, \dots, y_{n-1}, x_k - l, t) < \epsilon$ for all $k \geq k_0$. It is denoted by $(\mu, \nu)^n - \lim x = l$ or $x_k \xrightarrow{(\mu, \nu)^n} l$ as $k \rightarrow \infty$.

Definition 2.4. [34] Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. Then the sequence $x = \{x_k\}$ in X is called a Cauchy sequence with respect to the intuitionistic fuzzy n -norm $(\mu, \nu)^n$ if, for every $\epsilon > 0$, $t > 0$ and $y_1, y_2, \dots, y_{n-1} \in X$, there exists $k_0 \in \mathbb{N}$ such that $\mu(y_1, y_2, \dots, y_{n-1}, x_k - x_m, t) > 1 - \epsilon$ and $\nu(y_1, y_2, \dots, y_{n-1}, x_k - x_m, t) < \epsilon$ for all $k, m \geq k_0$.

Definition 2.5. [5] Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. Then $(X, \mu, \nu, *, \circ)$ is said to be complete if any Cauchy sequence in X is convergent to a point in X . A complete IFnNLS is called a intuitionistic fuzzy n -Banach space (IFnBS).

Definition 2.6. [35, 36] Let $(X, \mu, \nu, *, \circ)$ and $(Y, \mu, \nu, *, \circ)$ be two IFnNLS. A function $f : X \rightarrow Y$ is said to be continuous at a point $x_0 \in X$ if, for any sequence $x = \{x_n\}$ in X converging to a point $x_0 \in X$, then the sequence $f(x_n)$ in Y convergence to a point $f(x_0) \in Y$. If $f : X \rightarrow Y$ is continuous at each $x \in X$, then $f : X \rightarrow Y$ is said to be continuous on X .

Definition 2.7. [18] Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. Then (μ, ν) is said to satisfy the n -property on $X \times (0, \infty)$ if

$$\begin{aligned} \lim_{n \rightarrow \infty} [\mu(x_1, x_2, \dots, x_{n-1}, x, k^n t)]^{n^p} &= 1, \\ \lim_{n \rightarrow \infty} [\nu(x_1, x_2, \dots, x_{n-1}, x, k^n t)]^{n^p} &= 0 \end{aligned}$$

whenever $x \in X$, $k > 1$ and $p > 0$.

Definition 2.8. [16] Let X be a non-empty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $f : X \times X \rightarrow X$ if

$$x = f(x, y), y = f(y, x)$$

Definition 2.9. [17] Let X be a non-empty set. An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $f : X \times X \rightarrow X$ and $g : X \rightarrow X$ if

$$g(x) = f(x, y), g(y) = f(y, x)$$

Definition 2.10. [17] Let (X, \leq) be a partially ordered set and $T_1 : X \times X \rightarrow X$, $T_2 : X \rightarrow X$ be two functions. Then T_1 is said to have the mixed T_2 -monotone property, if T_1 is monotone T_2 -non-decreasing in the first argument and is monotone T_2 -non-increasing in the second argument.

i.e. for any $x, y \in X$

$$x_1, x_2 \in X, T_2(x_1) \leq T_2(x_2) \Rightarrow T_1(x_1, y) \leq T_1(x_2, y) \quad (2.1)$$

and

$$y_1, y_2 \in X, T_2(y_1) \leq T_2(y_2) \Rightarrow T_1(x, y_1) \leq T_1(x, y_2). \quad (2.2)$$

If $T_2 = I$, then T_1 is said to have the mixed monotone property.

Definition 2.11. [17] Let X be a non-empty set and $f : X \times X \rightarrow X$, $g : X \rightarrow X$ be two mappings. The mappings f and g are said to be commutative if

$$g(f(x, y)) = f(g(x), g(y)), \text{ for all } x, y \in X.$$

Lemma 2.12. [17] Let X be a nonempty set and $T_2 : X \rightarrow X$ be a mapping. Then there exists a subset $E \subseteq X$ such that $T_2(E) = T_2(X)$ and $T_2 : E \rightarrow X$ is one-one.

3 Main Results

Now we are ready to discuss the main results. First we prove the existence of coupled coincidence points by considering commutative condition of the mappings.

Theorem 3.1. Suppose $(X, \mu, \nu, *, \circ)$ is an IFnBS with (μ, ν) has n -property where (X, \preceq) is partially ordered and $a * b \geq ab$, $a \circ b \leq ab$ for all $a, b \in [0, 1]$. Let $T_1 : X \times X \rightarrow X$ and $T_2 : X \rightarrow X$ be two mapping such that T_1 has the mixed T_2 -monotone property and for all $x, y, u, v \in X$, $t \in \mathbb{R}$ and $x_1, x_2, \dots, x_{n-1} \in X$

$$\begin{aligned} \mu(x_1, x_2, \dots, x_{n-1}, T_1(x, y) - T_1(u, v), kt) \geq & \mu(x_1, x_2, \dots, x_{n-1}, T_2(x) - T_2(u), t) \\ & * \mu(x_1, x_2, \dots, x_{n-1}, T_2(y) - T_2(v), t) \end{aligned} \quad (3.1)$$

and

$$\begin{aligned} \nu(x_1, x_2, \dots, x_{n-1}, T_1(x, y) - T_1(u, v), kt) \leq & \nu(x_1, x_2, \dots, x_{n-1}, T_2(x) - T_2(u), t) \\ & \circ \nu(x_1, x_2, \dots, x_{n-1}, T_2(y) - T_2(v), t), \end{aligned} \quad (3.2)$$

for which $T_2(x) \leq T_2(u)$ and $T_2(y) \geq T_2(v)$, where $0 < k < 1$, $T_1(X \times X) \subseteq T_2(X)$, T_2 is continuous and commuting with T_1 .

Suppose either

(a) T_1 is continuous or

(b) X has the following property:

(i) if x_n is a non-decreasing sequence and $\lim_{n \rightarrow \infty} x_n = x$, then $T_2(x_n) \leq T_2(x)$ for all $n \in \mathbb{N}$.

(ii) if y_n is a non-increasing sequence and $\lim_{n \rightarrow \infty} y_n = y$, then $T_2(y_n) \geq T_2(y)$ for all $n \in \mathbb{N}$.

If there exist $x_0, y_0 \in X$ such that

$$T_2(x_0) \leq T_1(x_0, y_0), T_2(y_0) \geq T_1(y_0, x_0)$$

Then there exist $x, y \in X$ such that

$$T_2(x) \leq T_1(x, y), T_2(y) \geq T_1(y, x)$$

i.e. T_1 and T_2 have a couple coincidence point in X .

Proof. Let $x_0, y_0 \in X$ be such that

$$T_2(x_0) \leq T_1(x_0, y_0) \text{ and } T_2(y_0) \geq T_1(y_0, x_0)$$

Since $T_1(X \times X) \subseteq T_2(X)$, we can construct two sequences x_n and y_n in X such that

$$T_2(x_{n+1}) = T_1(x_n, y_n), T_2(y_{n+1}) = T_1(y_n, x_n), \text{ for all } n \geq 0. \quad (3.3)$$

Next we show that

$$T_2(x_n) \leq T_2(x_{n+1}) \text{ and } T_2(y_n) \geq T_2(y_{n+1}), \text{ for all } n \geq 0. \quad (3.4)$$

We prove this by using Mathematical Induction.

For $n = 0$, since $T_2(x_0) \leq T_1(x_0, y_0)$, $T_2(y_0) \geq T_1(y_0, x_0)$ and $T_2(x_1) = T_1(x_0, y_0)$, $T_2(y_1) = T_1(y_0, x_0)$ we have,

$$T_2(x_0) \leq T_2(x_1) \text{ and } T_2(y_0) \geq T_2(y_1). \quad (3.5)$$

Thus 3.4 holds for $n = 0$.

Suppose 3.4 holds for some fixed $n \geq 0$.

$$T_2(x_n) \leq T_2(x_{n+1}) \text{ and } T_2(y_n) \geq T_2(y_{n+1}). \quad (3.6)$$

and T_1 has the mixed T_2 -monotone property, then we check for $n + 1$ as

$$T_2(x_{n+1}) = T_1(x_n, y_n) \leq T_1(x_{n+1}, y_n), T_2(y_{n+1}) = T_1(y_n, x_n) \geq T_1(y_{n+1}, x_n). \quad (3.7)$$

From 3.3 and 2.1 we obtain,

$$T_2(x_{n+2}) = T_1(x_{n+1}, y_{n+1}) \geq T_1(x_{n+1}, y_n),$$

$$T_2(y_{n+2}) = T_1(y_{n+1}, x_{n+1}) \leq T_1(y_{n+1}, x_n), \quad (3.8)$$

i.e.

$$T_2(x_{n+1}) \leq T_2(x_{n+2}), T_2(y_{n+1}) \geq T_2(y_{n+2}). \quad (3.9)$$

Therefore, 3.4 holds for $n + 1$. Hence, by Mathematical Induction 3.4 holds for all $n \geq 0$. Therefore, we have

$$T_2(x_0) \leq T_2(x_1) \leq T_2(x_2) \leq \dots \leq T_2(x_n) \leq T_2(x_{n+1}) \leq \dots \quad (3.10)$$

and

$$T_2(y_0) \geq T_2(y_1) \geq T_2(y_2) \geq \dots \geq T_2(y_n) \geq T_2(y_{n+1}) \geq \dots \quad (3.11)$$

Consider

$$\begin{aligned} \beta_n(t) &= \mu(x_1, x_2, \dots, x_{n-1}, T_2(x_n) - T_2(x_{n+1}), t) \\ &\quad * \mu(x_1, x_2, \dots, x_{n-1}, T_2(y_n) - T_2(y_{n+1}), t). \end{aligned} \quad (3.12)$$

Using 3.4 and 3.1 we have

$$\begin{aligned} &\mu(x_1, x_2, \dots, x_{n-1}, T_2(x_n) - T_2(x_{n+1}), kt) \\ &= \mu(x_1, x_2, \dots, x_{n-1}, T_1(x_{n-1}, y_{n-1}) - T_1(x_n, y_n), kt) \\ &\geq \mu(x_1, x_2, \dots, x_{n-1}, T_2(x_{n-1}) - T_2(x_n), t) \\ &\quad * \mu(x_1, x_2, \dots, x_{n-1}, T_2(y_{n-1}) - T_2(y_n), t) \\ &= \beta_{n-1}(t) \end{aligned} \quad (3.13)$$

and

$$\begin{aligned} &\mu(x_1, x_2, \dots, x_{n-1}, T_2(y_n) - T_2(y_{n+1}), kt) \\ &= \mu(x_1, x_2, \dots, x_{n-1}, T_1(y_{n-1}, x_{n-1}) - T_1(y_n, x_n), kt) \\ &\geq \mu(x_1, x_2, \dots, x_{n-1}, T_2(y_{n-1}) - T_2(y_n), t) \\ &\quad * \mu(x_1, x_2, \dots, x_{n-1}, T_2(x_{n-1}) - T_2(x_n), t) \\ &= \beta_{n-1}(t). \end{aligned} \quad (3.14)$$

From the t -norm property we have,

$$\beta_n(kt) \geq \beta_{n-1}(t) * \beta_{n-1}(t) \geq [\beta_{n-1}(t)]^2.$$

Repeating this process we have,

$$\beta_n(t) \geq [\beta_{n-1}(\frac{t}{k})]^2 \geq \dots \geq [\beta_0(\frac{t}{k^n})]^{2^n}.$$

Implies that

$$\begin{aligned} &\mu(x_1, x_2, \dots, x_{n-1}, T_2(x_n) - T_2(x_{n+1}), kt) \\ &\quad * \mu(x_1, x_2, \dots, x_{n-1}, T_2(y_n) - T_2(y_{n+1}), kt) \\ &\geq [\mu(x_1, x_2, \dots, x_{n-1}, T_2(x_0) - T_2(x_1), \frac{t}{k^n})]^{2^n} \\ &\quad * [\mu(x_1, x_2, \dots, x_{n-1}, T_2(y_0) - T_2(y_1), \frac{t}{k^n})]^{2^n}. \end{aligned} \quad (3.15)$$

Again we have

$$t(1-k)(1+k+\dots+k^{m-n-1}) < t, \text{ for all } m > n, 0 < k < 1.$$

By t -norm property, we have

$$\begin{aligned}
& \mu(x_1, x_2, \dots, x_{n-1}, T_2(x_n) - T_2(x_m), t) \\
& * \mu(x_1, x_2, \dots, x_{n-1}, T_2(y_n) - T_2(y_m), t) \\
& \geq \mu(x_1, x_2, \dots, x_{n-1}, T_2(x_n) - T_2(x_m), t(1-k)(1+k+\dots+k^{m-n-1})) \\
& \quad * \mu(x_1, x_2, \dots, x_{n-1}, T_2(y_n) - T_2(y_m), t(1-k)(1+k+\dots+k^{m-n-1})) \\
& \geq \mu(x_1, x_2, \dots, x_{n-1}, T_2(x_{n+1}) - T_2(x_n), t(1-k)) \\
& \quad * \mu(x_1, x_2, \dots, x_{n-1}, T_2(y_{n+1}) - T_2(y_n), t(1-k)) \\
& \quad * \mu(x_1, x_2, \dots, x_{n-1}, T_2(x_{n+1}) - T_2(x_{n+2}), t(1-k)k) \\
& \quad * \mu(x_1, x_2, \dots, x_{n-1}, T_2(y_{n+1}) - T_2(y_{n+2}), t(1-k)k) * \dots * \\
& \quad \mu(x_1, x_2, \dots, x_{n-1}, T_2(x_{m-1}) - T_2(x_m), t(1-k)k^{m-n-1}) \\
& \quad * \mu(x_1, x_2, \dots, x_{n-1}, T_2(y_{m-1}) - T_2(y_m), t(1-k)k^{m-n-1}) \\
& \geq \mu(x_1, x_2, \dots, x_{n-1}, T_2(x_0) - T_2(x_1), (1-k)\frac{t}{k^n}) \\
& \quad * \mu(x_1, x_2, \dots, x_{n-1}, T_2(y_0) - T_2(y_1), (1-k)\frac{t}{k^n}) * \dots * \\
& \quad \mu(x_1, x_2, \dots, x_{n-1}, T_2(x_0) - T_2(x_1), (1-k)\frac{t}{k^n}) \\
& \quad * \mu(x_1, x_2, \dots, x_{n-1}, T_2(y_0) - T_2(y_1), (1-k)\frac{t}{k^n}) \\
& \geq [\mu(x_1, x_2, \dots, x_{n-1}, T_2(x_0) - T_2(x_1), (1-k)\frac{t}{k^n})]^{m-n} \\
& \quad * [\mu(x_1, x_2, \dots, x_{n-1}, T_2(y_0) - T_2(y_1), (1-k)\frac{t}{k^n})]^{m-n} \\
& \geq [\mu(x_1, x_2, \dots, x_{n-1}, T_2(x_0) - T_2(x_1), (1-k)\frac{t}{k^n})]^m \\
& \quad * [\mu(x_1, x_2, \dots, x_{n-1}, T_2(y_0) - T_2(y_1), (1-k)\frac{t}{k^n})]^m \\
& \geq [\mu(x_1, x_2, \dots, x_{n-1}, T_2(x_0) - T_2(x_1), (1-k)\frac{t}{k^n})]^{n^p} \\
& \quad * [\mu(x_1, x_2, \dots, x_{n-1}, T_2(y_0) - T_2(y_1), (1-k)\frac{t}{k^n})]^{n^p},
\end{aligned} \tag{3.16}$$

where $p > 0$ such that $m < n^p$. Since (μ, ν) has the n -property, we have

$$\lim_{n \rightarrow \infty} [\mu(x_1, x_2, \dots, x_{n-1}, T_2(x_0) - T_2(x_1), (1-k)\frac{t}{k^n})]^{n^p} = 1$$

and so

$$\lim_{n \rightarrow \infty} [\mu(x_1, x_2, \dots, x_{n-1}, T_2(x_n) - T_2(x_m), t) * \mu(x_1, x_2, \dots, x_{n-1}, T_2(y_n) - T_2(y_m), t)] = 1.$$

Next we show that

$$\lim_{n \rightarrow \infty} [\nu(x_1, x_2, \dots, x_{n-1}, T_2(x_n) - T_2(x_m), t) \circ \nu(x_1, x_2, \dots, x_{n-1}, T_2(y_n) - T_2(y_m), t)] = 0.$$

Consider

$$\begin{aligned} \beta'_n(t) &= \nu(x_1, x_2, \dots, x_{n-1}, T_2(x_n) - T_2(x_{n+1}), t) \\ &\quad \circ \nu(x_1, x_2, \dots, x_{n-1}, T_2(y_n) - T_2(y_{n+1}), t). \end{aligned} \quad (3.17)$$

From 3.4 and 3.2 we have

$$\begin{aligned} &\nu(x_1, x_2, \dots, x_{n-1}, T_2(x_n) - T_2(x_{n+1}), kt) \\ &= \nu(x_1, x_2, \dots, x_{n-1}, T_1(x_{n-1}, y_{n-1}) - T_1(x_n, y_n), kt) \\ &\leq \nu(x_1, x_2, \dots, x_{n-1}, T_2(x_{n-1}) - T_2(x_n), t) \\ &\quad \circ \nu(x_1, x_2, \dots, x_{n-1}, T_2(y_{n-1}) - T_2(y_n), t) \\ &= \beta'_{n-1}(t) \end{aligned} \quad (3.18)$$

and

$$\begin{aligned} &\nu(x_1, x_2, \dots, x_{n-1}, T_2(y_n) - T_2(y_{n+1}), kt) \\ &= \nu(x_1, x_2, \dots, x_{n-1}, T_1(y_{n-1}, x_{n-1}) - T_1(y_n, x_n), kt) \\ &\leq \nu(x_1, x_2, \dots, x_{n-1}, T_2(y_{n-1}) - T_2(y_n), t) \\ &\quad \circ \nu(x_1, x_2, \dots, x_{n-1}, T_2(x_{n-1}) - T_2(x_n), t) \\ &= \beta'_{n-1}(t). \end{aligned} \quad (3.19)$$

From the t -conorm property we have,

$$\beta'_n(kt) \leq \beta'_{n-1}(t) \circ \beta'_{n-1}(t) \leq [\beta'_{n-1}(t)]^2.$$

Repeating this process we have,

$$\beta'_n(t) \leq [\beta'_{n-1}(\frac{t}{k})]^2 \leq \dots \leq [\beta'_0(\frac{t}{k^n})]^{2^n}.$$

Implies that

$$\begin{aligned} &\nu(x_1, x_2, \dots, x_{n-1}, T_2(x_n) - T_2(x_{n+1}), kt) \\ &\quad \circ \nu(x_1, x_2, \dots, x_{n-1}, T_2(y_n) - T_2(y_{n+1}), kt) \\ &\leq [\nu(x_1, x_2, \dots, x_{n-1}, T_2(x_0) - T_2(x_1), \frac{t}{k^n})]^{2^n} \\ &\quad \circ [\nu(x_1, x_2, \dots, x_{n-1}, T_2(y_0) - T_2(y_1), \frac{t}{k^n})]^{2^n}. \end{aligned} \quad (3.20)$$

As

$$t(1 - k)(1 + k + \dots + k^{m-n-1}) < t, \text{ for all } m > n, 0 < k < 1.$$

By t -conorm property, we have

$$\begin{aligned}
& \nu(x_1, x_2, \dots, x_{n-1}, T_2(x_n) - T_2(x_m), t) \\
& \circ \nu(x_1, x_2, \dots, x_{n-1}, T_2(y_n) - T_2(y_m), t) \\
& \leq \nu(x_1, x_2, \dots, x_{n-1}, T_2(x_n) - T_2(x_m), t(1-k)(1+k+\dots+k^{m-n-1})) \\
& \quad \circ \nu(x_1, x_2, \dots, x_{n-1}, T_2(y_n) - T_2(y_m), t(1-k)(1+k+\dots+k^{m-n-1})) \\
& \leq \nu(x_1, x_2, \dots, x_{n-1}, T_2(x_{n+1}) - T_2(x_n), t(1-k)) \\
& \quad \circ \nu(x_1, x_2, \dots, x_{n-1}, T_2(y_{n+1}) - T_2(y_n), t(1-k)) \\
& \quad \circ \nu(x_1, x_2, \dots, x_{n-1}, T_2(x_{n+1}) - T_2(x_{n+2}), t(1-k)k) \\
& \quad \circ \nu(x_1, x_2, \dots, x_{n-1}, T_2(y_{n+1}) - T_2(y_{n+2}), t(1-k)k) \circ \dots \circ \\
& \nu(x_1, x_2, \dots, x_{n-1}, T_2(x_{m-1}) - T_2(x_m), t(1-k)k^{m-n-1}) \\
& \quad \circ \nu(x_1, x_2, \dots, x_{n-1}, T_2(y_{m-1}) - T_2(y_m), t(1-k)k^{m-n-1}) \\
& \leq \nu(x_1, x_2, \dots, x_{n-1}, T_2(x_0) - T_2(x_1), (1-k)\frac{t}{k^n}) \\
& \quad \circ \nu(x_1, x_2, \dots, x_{n-1}, T_2(y_0) - T_2(y_1), (1-k)\frac{t}{k^n}) \circ \dots \circ \\
& \nu(x_1, x_2, \dots, x_{n-1}, T_2(x_0) - T_2(x_1), (1-k)\frac{t}{k^n}) \\
& \quad \circ \nu(x_1, x_2, \dots, x_{n-1}, T_2(y_0) - T_2(y_1), (1-k)\frac{t}{k^n}) \\
& \leq [\nu(x_1, x_2, \dots, x_{n-1}, T_2(x_0) - T_2(x_1), (1-k)\frac{t}{k^n})]^{m-n} \\
& \quad \circ [\nu(x_1, x_2, \dots, x_{n-1}, T_2(y_0) - T_2(y_1), (1-k)\frac{t}{k^n})]^{m-n} \\
& \leq [\nu(x_1, x_2, \dots, x_{n-1}, T_2(x_0) - T_2(x_1), (1-k)\frac{t}{k^n})]^m \\
& \quad \circ [\nu(x_1, x_2, \dots, x_{n-1}, T_2(y_0) - T_2(y_1), (1-k)\frac{t}{k^n})]^m \\
& \leq [\nu(x_1, x_2, \dots, x_{n-1}, T_2(x_0) - T_2(x_1), (1-k)\frac{t}{k^n})]^{n^p} \\
& \quad \circ [\nu(x_1, x_2, \dots, x_{n-1}, T_2(y_0) - T_2(y_1), (1-k)\frac{t}{k^n})]^{n^p},
\end{aligned} \tag{3.21}$$

where $p > 0$ such that $m < n^p$. Since (μ, ν) has the n -property, we have

$$\lim_{n \rightarrow \infty} [\nu(x_1, x_2, \dots, x_{n-1}, T_2(x_0) - T_2(x_1), (1-k)\frac{t}{k^n})]^{n^p} = 0$$

and so

$$\lim_{n \rightarrow \infty} [\nu(x_1, x_2, \dots, x_{n-1}, T_2(x_n) - T_2(x_m), t) \circ \nu(x_1, x_2, \dots, x_{n-1}, T_2(y_n) - T_2(y_m), t)] = 0.$$

Hence the sequences $T_2(x_n)$ and $T_2(y_n)$ are Cauchy sequences in X . As X is complete, there exist $x, y \in X$ such that

$$\lim_{n \rightarrow \infty} T_2(x_n) = x \text{ and } \lim_{n \rightarrow \infty} T_2(y_n) = y.$$

From the continuity of T_2 , we have

$$\lim_{n \rightarrow \infty} T_2 T_2(x_n) = T_2(x) \text{ and } \lim_{n \rightarrow \infty} T_2 T_2(y_n) = T_2(y).$$

Using commutativity of T_1 and T_2 and 3.3 we have

$$\begin{aligned} T_2 T_2(x_{n+1}) &= T_2 T_1(x_n, y_n) = T_1(T_2(x_n), T_2(y_n)) \text{ and} \\ T_2 T_2(y_{n+1}) &= T_2 T_1(y_n, x_n) = T_1(T_2(y_n), T_2(x_n)). \end{aligned}$$

Next, we show that, $T_2(x) = T_1(x, y)$ and $T_2(y) = T_1(y, x)$.

First we consider (a) holds. Taking limit as $n \rightarrow \infty$ and from the continuity of T_1 we have,

$$\begin{aligned} T_2(x) &= \lim_{n \rightarrow \infty} T_2(T_2(x_{n+1})) = \lim_{n \rightarrow \infty} T_1(T_2(x_n), T_2(y_n)) \\ &= T_1(\lim_{n \rightarrow \infty} T_2(x_n), \lim_{n \rightarrow \infty} T_2(y_n)) \quad (3.22) \\ &= T_1(x, y) \end{aligned}$$

and

$$\begin{aligned} T_2(y) &= \lim_{n \rightarrow \infty} T_2(T_2(y_{n+1})) = \lim_{n \rightarrow \infty} T_1(T_2(y_n), T_2(x_n)) \\ &= T_1(\lim_{n \rightarrow \infty} T_2(y_n), \lim_{n \rightarrow \infty} T_2(x_n)) \quad (3.23) \\ &= T_1(y, x). \end{aligned}$$

Therefore $T_2(x) = T_1(x, y)$ and $T_2(y) = T_1(y, x)$, i.e. T_1 has a coupled coincidence point.

Next consider (b) holds. Since T_{x_n} is non-decreasing and $T_2(x_n) \rightarrow x$ from (i) we have $T_2 T_2(x_n) \subseteq T_2(x)$, for all $n \in \mathbb{N}$. Similarly, $T_2(y_n)$ is non-decreasing and $T_2(y_n) \rightarrow y$ from (ii) we have $T_2 T_2(y_n) \supseteq T_2(y)$, for all $n \in \mathbb{N}$. Then we have,

$$\begin{aligned} &\mu(x_1, x_2, \dots, x_{n-1}, T_2 T_2(x_{n+1}) - T_1(x, y), kt) \\ &= \mu(x_1, x_2, \dots, x_{n-1}, T_2 T_1(x_n, y_n) - T_1(x, y), kt) \\ &= \mu(x_1, x_2, \dots, x_{n-1}, T_1(T_2(x_n), T_2(y_n)) - T_1(x, y), kt) \quad (3.24) \\ &\geq \mu(x_1, x_2, \dots, x_{n-1}, T_2 T_2(x_n) - T_2(x), t) \\ &\quad * \mu(x_1, x_2, \dots, x_{n-1}, T_2 T_2(y_n) - T_2(y), t) \end{aligned}$$

and

$$\begin{aligned} &\nu(x_1, x_2, \dots, x_{n-1}, T_2 T_2(x_{n+1}) - T_1(x, y), kt) \\ &= \nu(x_1, x_2, \dots, x_{n-1}, T_2 T_1(x_n, y_n) - T_1(x, y), kt) \\ &= \nu(x_1, x_2, \dots, x_{n-1}, T_1(T_2(x_n), T_2(y_n)) - T_1(x, y), kt) \quad (3.25) \\ &\leq \nu(x_1, x_2, \dots, x_{n-1}, T_2 T_2(x_n) - T_2(x), t) \\ &\quad \circ \nu(x_1, x_2, \dots, x_{n-1}, T_2 T_2(y_n) - T_2(y), t). \end{aligned}$$

Taking limit as $n \rightarrow \infty$ in 3.24 and 3.25, we have

$$\begin{aligned}\mu(x_1, x_2, \dots, x_{n-1}, T_2(x) - T_1(x, y), kt) &\geq 1 \text{ and} \\ \nu(x_1, x_2, \dots, x_{n-1}, T_2(x) - T_1(x, y), kt) &\leq 0.\end{aligned}$$

Hence $T_2(x) = T_1(x, y)$ and similarly $T_2(y) = T_1(y, x)$.

Therefore T_1 and T_2 have a coupled coincidence point at (x, y) . \square

Next we prove the existence of coupled coincidence points by considering mappings which are non-commutative in nature.

Theorem 3.2. *Suppose $(X, \mu, \nu, *, \circ)$ is an IFnBS with (μ, ν) has n -property where (X, \preceq) is partially ordered and $a * b \geq ab$, $a \circ b \leq ab$ for all $a, b \in [0, 1]$. Let $T_1 : X \times X \rightarrow X$ and $T_2 : X \rightarrow X$ be two mapping such that T_1 has the mixed T_2 -monotone property and for all $x, y, u, v \in X$, $t \in \mathbb{R}$ and $x_1, x_2, \dots, x_{n-1} \in X$*

$$\begin{aligned}\mu(x_1, x_2, \dots, x_{n-1}, T_1(x, y) - T_1(u, v), kt) &\geq \mu(x_1, x_2, \dots, x_{n-1}, T_2(x) - T_2(u), t) \\ &\quad * \mu(x_1, x_2, \dots, x_{n-1}, T_2(y) - T_2(v), t)\end{aligned}\tag{3.26}$$

and

$$\begin{aligned}\nu(x_1, x_2, \dots, x_{n-1}, T_1(x, y) - T_1(u, v), kt) &\leq \nu(x_1, x_2, \dots, x_{n-1}, T_2(x) - T_2(u), t) \\ &\quad \circ \nu(x_1, x_2, \dots, x_{n-1}, T_2(y) - T_2(v), t),\end{aligned}\tag{3.27}$$

for which $T_2(x) \leq T_2(u)$ and $T_2(y) \geq T_2(v)$, where $0 < k < 1$, $T_1(X \times X) \subseteq T_2(X)$, $T_2(X)$ is complete and T_2 is continuous.

Suppose either

(a) T_1 is continuous or

(b) X has the following property:

- (i) if x_n is a non-decreasing sequence and $\lim_{n \rightarrow \infty} x_n = x$, then $T_2(x_n) \leq T_2(x)$ for all $n \in \mathbb{N}$.
- (ii) if y_n is a non-increasing sequence and $\lim_{n \rightarrow \infty} y_n = y$, then $T_2(y_n) \geq T_2(y)$ for all $n \in \mathbb{N}$.

If there exist $x_0, y_0 \in X$ such that

$$T_2(x_0) \leq T_1(x_0, y_0), T_2(y_0) \geq T_1(y_0, x_0)$$

Then, T_1 and T_2 have a coupled coincidence point in X .

Proof. From Lemma 2.12, there exists $S \in X$ such that $T_2(S) = T_2(X)$ and $T_2 : E \rightarrow X$ is one-one. Let us define a mapping $F : T_2(S) \times T_2(S) \rightarrow X$ such that

$$F(T_2(x), T_2(y)) = T_1(x, y), \text{ for all } T_2(x), T_2(y) \in T_2(S)\tag{3.28}$$

Since T_2 is one-one on $T_2(S)$, so F is well-defined. Thus, we have

$$\begin{aligned} \mu(x_1, x_2, \dots, x_{n-1}, F(T_2(x), T_2(y)) - F(T_2(u), T_2(v)), kt) \\ \geq \mu(x_1, x_2, \dots, x_{n-1}, T_2(x) - T_2(u), t) \\ * \mu(x_1, x_2, \dots, x_{n-1}, T_2(y) - T_2(v), t) \end{aligned} \quad (3.29)$$

and

$$\begin{aligned} \nu(x_1, x_2, \dots, x_{n-1}, F(T_2(x), T_2(y)) - F(T_2(u), T_2(v)), kt) \\ \leq \nu(x_1, x_2, \dots, x_{n-1}, T_2(x) - T_2(u), t) \\ \circ \nu(x_1, x_2, \dots, x_{n-1}, T_2(y) - T_2(v), t), \end{aligned} \quad (3.30)$$

for all $T_2(x), T_2(y), T_2(u), T_2(v) \in T_2(S)$ with $T_2(x) \leq T_2(u)$ and $T_2(y) \geq T_2(v)$. Since T_1 has the mixed T_2 -monotone property, for all $x, y \in X$ we have

$$x_1, x_2 \in X, T_2(x_1) \leq T_2(x_2) \implies T_1(x_1, y) \leq T_1(x_2, y). \quad (3.31)$$

and

$$y_1, y_2 \in X, T_2(y_1) \geq T_2(y_2) \implies T_1(x, y_1) \leq T_1(x, y_2). \quad (3.32)$$

From 3.28, 3.31 and 3.32 we have, for all $T_2(x), T_2(y) \in T_2(S)$

$$T_2(x_1), T_2(x_2) \in T_2(S), T_2(x_1) \leq T_2(x_2) \implies F(T_2(x_1), T_2(y)) \leq F(T_2(x_2), T_2(y)). \quad (3.33)$$

and

$$T_2(y_1), T_2(y_2) \in T_2(S), T_2(y_1) \geq T_2(y_2) \implies F(T_2(x), T_2(y_1)) \leq F(T_2(x), T_2(y_2)). \quad (3.34)$$

This implies that F has the mixed monotone property.

Now we consider that assumption (a) holds. Since T_1 is continuous, F is also continuous. Therefore from Theorem 3.1 and Definition 2.8 F has a coupled fixed point $(u, v) \in T_2(X) \times T_2(X)$.

Next assume that (b) holds. Then similarly from Theorem 3.1 and Definition 2.8 F has a coupled fixed point $(u, v) \in T_2(X) \times T_2(X)$.

Finally we show that T_1 and T_2 have a coupled coincidence point in X . Since (u, v) is a coupled fixed point of F , we have

$$u = F(u, v), v = F(v, u)$$

Since $(u, v) \in T_2(X) \times T_2(X)$, there exists a point $(u_1, v_1) \in T_2(X) \times T_2(X)$ such that

$$u = T_2(u_1), v = T_2(v_1).$$

Thus we have,

$$T_2(u_1) = F(T_2(u_1), T_2(v_1)), T_2(v_1) = F(T_2(v_1), T_2(u_1))$$

Also we have

$$T_2(u_1) = F(u_1, v_1), T_2(v_1) = F(v_1, u_1).$$

Therefore, (u_1, v_1) is a coupled coincidence point of T_1 and T_2 in X , where T_1 and T_2 are not commutative. \square

Corollary 3.3. *Suppose $(X, \mu, \nu, *, \circ)$ is an IFnBS with (μ, ν) has n -property where (X, \preceq) is a partially ordered set and $a * b \geq ab$, $a \circ b \leq ab$ for all $a, b \in [0, 1]$. Let $T_1 : X \times X \rightarrow X$ be a mapping having the mixed monotone property on X and for all $x, y, u, v \in X$, $t \in \mathbb{R}$ and $x_1, x_2, \dots, x_{n-1} \in X$*

$$\begin{aligned} \mu(x_1, x_2, \dots, x_{n-1}, T_1(x, y) - T_1(u, v), kt) \geq & \mu(x_1, x_2, \dots, x_{n-1}, x - u, t) \\ & * \mu(x_1, x_2, \dots, x_{n-1}, y - v, t) \end{aligned} \quad (3.35)$$

and

$$\begin{aligned} \nu(x_1, x_2, \dots, x_{n-1}, T_1(x, y) - T_1(u, v), kt) \leq & \nu(x_1, x_2, \dots, x_{n-1}, x - u, t) \\ & \circ \nu(x_1, x_2, \dots, x_{n-1}, y - v, t), \end{aligned} \quad (3.36)$$

for which $x \leq u$ and $y \geq v$, where $0 < k < 1$.

Suppose either

(a) T_1 is continuous or

(b) X has the following property:

- (i) if x_n is a non-decreasing sequence and $\lim_{n \rightarrow \infty} x_n = x$, then $x_n \leq x$ for all $n \in \mathbb{N}$.
- (ii) if y_n is a non-increasing sequence and $\lim_{n \rightarrow \infty} y_n = y$, then $y_n \geq y$ for all $n \in \mathbb{N}$.

If there exist $x_0, y_0 \in X$ such that

$$x_0 \leq T_1(x_0, y_0), y_0 \geq T_1(y_0, x_0)$$

Then there exist $x, y \in X$ such that

$$x = T_1(x, y), y = T_1(y, x)$$

Further, if x_0 and y_0 are comparable, then $x = y$, i.e. $x = T_1(x, x)$.

Proof. In Theorem 3.1 taking $T_2 = I$ (i.e. the identity mapping) the first part of the corollary is completed. Next we have to show that $x = T_1(x, x)$. Suppose $x_0 \leq y_0$. We show that

$$x_n \leq y_n, \text{ for all } n \geq 0, \quad (3.37)$$

where, $x_n = T_{x_{n-1}, y_{n-1}}$ and $y_n = T_1(y_{n-1}, x_{n-1})$, for all $n \geq 1$.

Suppose 3.37 holds for some $n \geq 0$. Then from the monotone property of T_1 , it follows that $x_{n+1} = T_1(x_n, y_n) \leq T_1(y_n, x_n) = y_{n+1}$, for all $n \geq 0$. Then 3.37 holds for some $n + 1$. Therefore 3.37 holds for all $n \geq 0$.

Next, we prove that $x = y$. Suppose $x \neq y$, then we have

$$\begin{aligned}
& \mu(x_1, x_2, \dots, x_{n-1}, x_{n+1} - y_{n+1}, kt) \\
&= \mu(x_1, x_2, \dots, T_1(x_n, y_n) - T_1(y_n, x_n), kt) \\
&\geq \mu(x_1, x_2, \dots, x_{n-1}, x_n - y_n, t) * \mu(x_1, x_2, \dots, x_{n-1}, y_n - x_n, t) \\
&\geq [\mu(x_1, x_2, \dots, x_{n-1}, x_n - y_n, t)]^2
\end{aligned} \tag{3.38}$$

and

$$\begin{aligned}
& \nu(x_1, x_2, \dots, x_{n-1}, x_{n+1} - y_{n+1}, kt) \\
&= \nu(x_1, x_2, \dots, T_1(x_n, y_n) - T_1(y_n, x_n), kt) \\
&\leq \nu(x_1, x_2, \dots, x_{n-1}, x_n - y_n, t) \circ \nu(x_1, x_2, \dots, x_{n-1}, y_n - x_n, t) \\
&\leq [\nu(x_1, x_2, \dots, x_{n-1}, x_n - y_n, t)]^2.
\end{aligned} \tag{3.39}$$

Thus we have

$$\mu(x_1, x_2, \dots, x_{n-1}, x_n - y_n, kt) \geq [\mu(x_1, x_2, \dots, x_{n-1}, x_0 - y_0, \frac{t}{k^n})]^{2^n}$$

and

$$\nu(x_1, x_2, \dots, x_{n-1}, x_n - y_n, kt) \leq [\nu(x_1, x_2, \dots, x_{n-1}, x_0 - y_0, \frac{t}{k^n})]^{2^n}.$$

Again from the triangle inequality, we have

$$\begin{aligned}
& \mu(x_1, x_2, \dots, x_{n-1}, x - y, t) \\
&\geq \mu(x_1, x_2, \dots, x_{n-1}, x - x_n, \frac{t}{3}) * \mu(x_1, x_2, \dots, x_{n-1}, x_n - y_n, \frac{t}{3}) \\
&\quad * \mu(x_1, x_2, \dots, x_{n-1}, y_n - y, \frac{t}{3}) \\
&\geq \mu(x_1, x_2, \dots, x_{n-1}, x - x_n, \frac{t}{3}) * [\mu(x_1, x_2, \dots, x_{n-1}, x_0 - y_0, \frac{t}{3k^n})]^{2^n} \\
&\quad * \mu(x_1, x_2, \dots, x_{n-1}, y_n - y, \frac{t}{3}) \\
&\longrightarrow 1.
\end{aligned} \tag{3.40}$$

and

$$\begin{aligned}
 & \nu(x_1, x_2, \dots, x_{n-1}, x - y, t) \\
 & \leq \nu(x_1, x_2, \dots, x_{n-1}, x - x_n, \frac{t}{3}) \circ \nu(x_1, x_2, \dots, x_{n-1}, x_n - y_n, \frac{t}{3}) \\
 & \quad \circ \nu(x_1, x_2, \dots, x_{n-1}, y_n - y, \frac{t}{3}) \\
 & \leq \nu(x_1, x_2, \dots, x_{n-1}, x - x_n, \frac{t}{3}) \circ [\nu(x_1, x_2, \dots, x_{n-1}, x_0 - y_0, \frac{t}{3k^n})]^{2^n} \\
 & \quad \circ \nu(x_1, x_2, \dots, x_{n-1}, y_n - y, \frac{t}{3}) \\
 & \rightarrow 0.
 \end{aligned} \tag{3.41}$$

as $n \rightarrow \infty$. Thus we have

$$\mu(x_1, x_2, \dots, x_{n-1}, x - y, t) = 1$$

and

$$\nu(x_1, x_2, \dots, x_{n-1}, x - y, t) = 0.$$

Therefore, $x = y$. □

4 Examples

In this section we are going to discuss examples regarding Theorems 3.1 and 3.2. First example shows the existence of coupled fixed point for the mappings with commutative condition and second one show the existence for the mapping having not-commutative condition.

Example 4.1. Let us consider $X = \mathbb{R}$, set of real numbers, $a * b = ab = a \circ b$ for all $a, b \in [0, 1]$ and $\varphi(t) = 1 - e^{-t}$. Then $(X, \mu, \nu, *, \circ)$ is a complete IFnNS with the norm μ, ν satisfying the n -property on $X \times (0, \infty)$ and

$\mu(x_1, x_2, \dots, x_{n-1}, x, t) = [\varphi(t)]^{|x|}$ and $\nu(x_1, x_2, \dots, x_{n-1}, x, t) = 1 - [\varphi(t)]^{|x|}$, for all $x \in X$, $t \in \mathbb{R}$ and $x_1, x_2, \dots, x_{n-1} \in X$.

If X is endowed with the usual order as $x \leq y \Rightarrow y - x \in [0, \infty)$, then (X, \preceq) is a partially ordered set. Next define mappings

$$T_1 : X \times X \rightarrow X \text{ such that } T_1(x, y) = x - y, \text{ for all } (x, y) \in X \times X.$$

and

$$T_2(x) : X \rightarrow X \text{ such that } T_2 = 5x, \text{ for all } x \in X,$$

where T_1 is a mixed T_2 -monotone mapping and $T_1(X \times X) \subseteq T_2(X)$.

Next we check commutative condition.

$$T_2(T_1(x, y)) = T_2(x - y) = 5(x - y) \text{ and } T_1(T_2(x), T_2(y)) = T_1(5x, 5y) = 5(x - y).$$

Thus, $T_2(T_1(x, y)) = T_1(T_2(x), T_2(y))$, i.e., T_1 and T_2 are commutative.

Let $x_0 = -1$ and $Y_0 = 2$ then we have

$$-5 = T_2(x_0) \leq T_1(x_0, y_0) = -3 \text{ and } 10 = T_2(y_0) \geq T_1(y_0, x_0) = 3$$

Now for any $x, y, u, v \in X$ with $T_2(x) \leq T_2(u)$ and $T_2(y) \geq T_2(v)$, we get

$$\begin{aligned} & \mu(x_1, x_2, \dots, x_{n-1}, T_1(x, y) - T_1(u, v), \frac{t}{4}) \\ &= [1 - e^{-\frac{t}{4}}]^{|x-y-(u-v)|} \\ &\geq [1 - e^{-\frac{t}{4}}]^{|x-u|+|v-y|} \\ &= [1 - e^{-\frac{t}{4}}]^{|T_2(x)-T_2(u)|} \cdot [1 - e^{-\frac{t}{4}}]^{|T_2(y)-T_2(v)|} \\ &= \mu(x_1, x_2, \dots, x_{n-1}, T_2(x) - T_2(u), t) \\ &\quad * \mu(x_1, x_2, \dots, x_{n-1}, T_2(y) - T_2(v), t). \end{aligned} \tag{4.1}$$

Similarly

$$\begin{aligned} \nu(x_1, x_2, \dots, x_{n-1}, T_1(x, y) - T_1(u, v), \frac{t}{4}) &\leq \nu(x_1, x_2, \dots, x_{n-1}, T_2(x) - T_2(u), t) \\ &\quad * \nu(x_1, x_2, \dots, x_{n-1}, T_2(y) - T_2(v), t). \end{aligned} \tag{4.2}$$

where $0 < k < 1$. Therefore from Theorem 3.1 T_1 and T_2 have a coupled fixed point in $X \times X$ and a point $(-1, 2)$ is a coupled coincidence point of T_1 and T_2 .

Example 4.2. Let us consider $X = \mathbb{R}$, set of real numbers, $a * b = ab = a \circ b$ for all $a, b \in [0, 1]$ and $\varphi(t) = 1 - e^{-t}$. Then $(X, \mu, \nu, *, \circ)$ is a complete IFnNS with the norm μ, ν satisfying the n -property on $X \times (0, \infty)$ and $\mu(x_1, x_2, \dots, x_{n-1}, x, t) = [\varphi(t)]^{|x|}$ and $\nu(x_1, x_2, \dots, x_{n-1}, x, t) = 1 - [\varphi(t)]^{|x|}$, for all $x \in X$, $t \in \mathbb{R}$ and $x_1, x_2, \dots, x_{n-1} \in X$.

If X is endowed with the usual order as $x \leq y \Rightarrow y - x \in [0, \infty]$, then (X, \preceq) is a partially ordered set. Next define mappings

$$T_1 : X \times X \rightarrow X \text{ such that } T_1(x, y) = 1, \text{ for all } (x, y) \in X \times X.$$

and

$$T_2(x) : X \rightarrow X \text{ such that } T_2 = x - 1, \text{ for all } x \in X.$$

where T_1 and T_2 are continuous, T_1 is a mixed T_2 -monotone mapping and $T_1(X \times X) \subseteq T_2(X)$.

Next we check commutative condition.

$$T_2(T_1(x, y)) = T_2(1) = 0 \text{ and } T_1(T_2(x), T_2(y)) = 1.$$

Thus, $T_2(T_1(x, y)) \neq T_1(T_2(x), T_2(y))$, i.e., T_1 and T_2 are not satisfy the commutative condition.

Let $x_0 = 1$ and $Y_0 = 5$ then we have

$$T_2(x_0) = 0 \leq T_1(x_0, y_0) = 1 \text{ and } 4 = T_2(y_0) \geq T_1(y_0, x_0) = 1.$$

Now for any $x, y, u, v \in X$ with $T_2(x) \leq T_2(u)$ and $T_2(y) \geq T_2(v)$, we get

$$\begin{aligned} \mu(x_1, x_2, \dots, x_{n-1}, T_1(x, y) - T_1(u, v), kt) &= \mu(x_1, x_2, \dots, x_{n-1}, 0, kt) \\ &= 1 \\ &\geq \mu(x_1, x_2, \dots, x_{n-1}, T_2(x) - T_2(u), t) \\ &\quad * \mu(x_1, x_2, \dots, x_{n-1}, T_2(y) - T_2(v), t) \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} \nu(x_1, x_2, \dots, x_{n-1}, T_1(x, y) - T_1(u, v), kt) &= \nu(x_1, x_2, \dots, x_{n-1}, 0, t) \\ &= 0 \\ &\leq \nu(x_1, x_2, \dots, x_{n-1}, T_2(x) - T_2(u), t) \\ &\quad \circ \nu(x_1, x_2, \dots, x_{n-1}, T_2(y) - T_2(v), t), \end{aligned} \tag{4.4}$$

where $0 < k < 1$. Therefore from Theorem 3.2 T_1 and T_2 have a coupled fixed point in $X \times X$ and a point $(2, 2)$ is a coupled coincidence point of T_1 and T_2 .

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