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On a Spectral Subdivision of the Operator Δ_i^2 over the Sequence Spaces c_0 and ℓ_1

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Abstract : The main objective of this paper is to determine the spectrum and the fine spectrum of the difference operator Δ_i^2 over the sequence spaces c_0 and ℓ_1. For any sequence $x = (x_k)$ in c_0 or ℓ_1, the generalized difference operator Δ_i^2 over c_0 or ℓ_1 is defined by $(\Delta_i^2(x))_k = \sum_{i=0}^{2} \frac{(-1)^i}{i+1}\binom{2}{i} x_{k-i} = x_k - x_{k-1} + \frac{1}{3}x_{k-2}$, with $x_k = 0$ for $k < 0$. Moreover, we compute the spectrum, the point spectrum, the residual spectrum and the continuous spectrum of the difference operator Δ_i^2 over the basic sequence spaces c_0 and ℓ_1.

Keywords : Difference operator Δ_i^2; spectrum of an operator; sequence spaces.
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1 Introduction, Preliminaries and Definitions

Let ω be the space of all sequences of real or complex numbers. Any subspace of ω is called a sequence space. By ℓ_∞, c, c_0 and ℓ_1, we denote the spaces of all bounded, convergent, null and absolutely summable sequences, respectively. Now, we define a difference operator $\Delta_i^2 : c_0(\text{or } \ell_1) \to c_0(\text{or } \ell_1)$ by

$$(\Delta_i^2)_k = \sum_{i=0}^{2} \frac{(-1)^i}{i+1}\binom{2}{i} x_{k-i} = x_k - x_{k-1} + \frac{1}{3}x_{k-2},$$

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with $x_k = 0$ for $k < 0$, where $x \in c_0$ or ℓ_1 and $k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$. It is natural to express the operator Δ_i^2 as a lower triangular matrix (a_{nk}), where

$$a_{nk} = \begin{cases} 1, & (k = n), \\ -1, & (k = n-1), \\ \frac{1}{3}, & (k = n-2), \\ 0, & \text{otherwise.} \end{cases} \quad \text{for all } n, k \in \mathbb{N}_0.$$

Equivalently, one may write

$$\Delta_i^2 = (a_{nk}) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & \dots \\ \frac{1}{3} & -1 & 1 & 0 & \dots \\ 0 & \frac{1}{3} & -1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. By $\mathcal{R}(T)$, we denote the range of T, i.e.

$$\mathcal{R}(T) = \{y \in Y : y = Tx \; ; \; x \in X\}.$$

By $B(X)$, we denote the space all bounded linear operators on X into itself. If X is any Banach space and $T \in B(X)$ then the *adjoint* T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*\phi)(x) = \phi(Tx)$ for all $\phi \in X^*$ and $x \in X$ with $\|T\| = \|T^*\|$.

Let $X \neq \{\mathbf{0}\}$ be a normed linear space over the complex field and $T : D(T) \to X$ be a linear operator, where $D(T)$ denotes the domain of T. With T, for a complex number λ, we associate an operator $T_\lambda = T - \lambda I$, where I is called the identity operator on $D(T)$ and if T_λ has an inverse, we denote it by T_λ^{-1} i.e.

$$T_\lambda^{-1} = (T - \lambda I)^{-1}$$

and is called the *resolvent* operator of T. Many properties of T_λ and T_λ^{-1} depend on λ and the spectral theory is concerned with those properties. We are interested in the set of all λ's in the complex plane such that T_λ^{-1} exists/ T_λ^{-1} is bounded/ domain of T_λ^{-1} is dense in X. For our investigation, we need some basic concepts in spectral theory which are given as some definitions and lemmas.

Definition 1.1. ([1], pp. 371). Let X and T be defined as above. A *regular value* of T is a complex number λ such that
(R1) T_λ^{-1} exists;
(R2) T_λ^{-1} is bounded;
(R3) T_λ^{-1} is defined on a set which is dense in X.

The *resolvent* set $\rho(T, X)$ of T is the set of all regular values of T. Its complement $\sigma(T, X) = \mathbb{C} \setminus \rho(T, X)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of T. Furthermore, the spectrum $\sigma(T, X)$ is partitioned into three disjoint sets as

follows:
(I)*Point spectrum* $\sigma_p(T,X)$**:** It is the set of all $\lambda \in \mathbb{C}$ such that (R1) does not hold. The elements of $\sigma_p(T,X)$ are called eigenvalues of T.
(II)*Continuous spectrum* $\sigma_c(T,X)$**:** It is the set of all $\lambda \in \mathbb{C}$ such that (R1) holds and satisfies (R3) but does not satisfy (R2).
(III)*Residual spectrum* $\sigma_r(T,X)$**:** It is the set of all $\lambda \in \mathbb{C}$ such that (R1) holds but does not satisfy (R3). The condition (R2) may or may not hold.

Goldberg's classification of operator T_λ **:**([2], pp. 58-71). Let X be a Banach space and $T_\lambda = (T - \lambda I) \in B(X)$, where λ is a complex number. Again, let $R(T_\lambda)$ and T_λ^{-1} denote the range and inverse of the operator T_λ respectively. Then the following possibilities may occur:
(A) $R(T_\lambda) = X$;
(B) $R(T_\lambda) \neq \overline{R(T_\lambda)} = X$;
(C) $\overline{R(T_\lambda)} \neq X$;
and
(1) T_λ is injective and T_λ^{-1} is continuous;
(2) T_λ is injective and T_λ^{-1} is discontinuous;
(3) T_λ is not injective.
Taking the permutations (A), (B), (C) and (1), (2), (3), we get nine different states. These are labeled by $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2$ and C_3. If λ is a complex number such that $T_\lambda \in A_1$ or $T_\lambda \in B_1$, then λ is in the resolvent set $\rho(T,X)$ of T on X. The other classifications give rise to the fine spectrum of T . We use $\lambda \in B_2\sigma(T,X)$ means the operator $T_\lambda \in B_2$, i.e. $R(T_\lambda) \neq \overline{R(T_\lambda)} = X$ and T_λ is injective but $T_\lambda^{-1}$is discontinuous. Similarly others.

Lemma 1.2. ([2], pp. 59). *A linear operator T has a dense range if and only if the adjoint T^* is one to one.*

Lemma 1.3. ([2], pp. 60). *The adjoint operator T^* is onto if and and only if T has a bounded inverse.*

Let P, Q be two nonempty subsets of the space w of all real or complex sequences and $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $n, k \in \mathbb{N}_0$. For every $x = (x_k) \in P$ and every positive integer n, we write

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k.$$

The sequence $Ax = (A_n(x))$, if it exists, is called the transformation of x by the matrix A. Infinite matrix $A \in (P,Q)$ if and only if $Ax \in Q$ whenever $x \in P$.

Lemma 1.4. ([3], pp. 129). *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(c_0)$ from c_0 to itself if and only if*

(i) *the rows of A are in ℓ_1 and their ℓ_1 norms are bounded,*

(ii) *the columns of A are in c_0 and*

The operator norm of T is the supremum of ℓ_1 norms of the rows.

Lemma 1.5. ([3], pp. 126). *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(\ell_1)$ from ℓ_1 to itself if and only if the supremum of ℓ_1 norms of the columns of A is bounded.*

In analysis the notion of eigenvalues of a matrix is generalized by the spectrum of the operator corresponds to that matrix. Therefore, the study of spectrum and fine spectrum of different operators carries a prominent position in different branches of mathematics like summabilitry theory, schauder basis theory, matrix theory and operator theory. In the existing literature, several authors have devoted their knowledge in achieving new results and theorems concerning the spectrum and the fine spectra of an operator over different sequence spaces. For instance; Reade [4] studied the spectrum of the Cesàro operator over the sequence space c_0. The fine spectra of the Cesàro operator over the sequence spaces c_0 and bv_p have been determined by Akhmedov and Başar [5,6]. Akhmedov and Başar [7,8] have studied the fine spectrum of the difference operator Δ over the sequence spaces c_0 and bv_p where $1 < p < \infty$. Altay and Başar [9,10] have determined the fine spectrum of the difference operator Δ over the sequence spaces c_0, c and ℓ_p, for $0 < p < 1$. Srivastava and Kumar [11,12] have examined the fine spectrum of the generalized difference operator Δ_ν over the sequence spaces c_0 and ℓ_1. Panigrahi and Srivastava [13] have studied the spectrum and fine spectrum of the generalized second order difference operator Δ^2_{uv} on c_0 . Recently, the spectrum of the generalized rth difference operator Δ^r_ν and 2nd order difference operator Δ^2 have been determined by Dutta and Baliarsingh [14,15] over ℓ_1 and c_0, respectively and also, for more investigations one may refer [16,17].

2 The Spectrum of the Operator Δ^2_i over c_0

In this section, we compute the spectrum, the point spectrum, the continuous spectrum, the residual spectrum and the fine spectrum of the operator Δ^2_i on the sequence space c_0.

Theorem 2.1. *The operator $\Delta^2_i : c_0 \to c_0$ is a linear operator and*

$$\| \Delta^2_i \|_{(c_0:c_0)} = \frac{7}{3}. \tag{2.1}$$

Proof. Proof of this theorem follows from Lemma 1.4 and with the fact that

$$1 + |-1| + \frac{1}{3} = \frac{7}{3}.$$ □

Theorem 2.2. *The spectrum of Δ^2_i on the sequence space c_0 is given by*

$$\sigma(\Delta^2_i, c_0) = \Big\{ \lambda \in \mathbb{C} : |1-\lambda| \le \frac{4}{3} \Big\}. \tag{2.2}$$

Proof. We divide the entire proof into two sections.

In the first part, we have to show that

$$\sigma(\Delta_i^2, c_0) \subseteq \Big\{\lambda \in \mathbb{C} : |1-\lambda| \leq \frac{4}{3}\Big\}.$$

Equivalently, we need to show that if $\lambda \in \mathbb{C}$ with $|1-\lambda| > \frac{4}{3} \Rightarrow \lambda \notin \sigma(\Delta_i^2, c_0)$. Suppose $\lambda \in \mathbb{C}$ with $|1-\lambda| > \frac{4}{3}$. Now, $(\Delta_i^2 - \lambda I)$ is a triangle and hence has an inverse $(\Delta_i^2 - \lambda I)^{-1} = (b_{nk})$ where

$$(b_{nk}) = \begin{pmatrix} \frac{1}{(1-\lambda)} & 0 & 0 & 0 & \cdots \\ \frac{1}{(1-\lambda)^2} & \frac{1}{(1-\lambda)} & 0 & 0 & \cdots \\ b_{20} & \frac{1}{(1-\lambda)^2} & \frac{1}{(1-\lambda)} & 0 & \cdots \\ b_{30} & b_{31} & \frac{1}{(1-\lambda)^2} & \frac{1}{(1-\lambda)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $b_{20}, b_{30}, b_{31}....$ etc. are as follows:

$$b_{20} = \frac{1}{(1-\lambda)^3} - \frac{1}{3(1-\lambda)^2}, \quad b_{31} = \frac{1}{(1-\lambda)^3} - \frac{1}{3(1-\lambda)^2},$$

Similarly, $$b_{30} = \frac{1}{(1-\lambda)^4} - \frac{2}{3(1-\lambda)^3}.$$

In fact, for $n \in \mathbb{N}_0$ one can calculate

$$\begin{aligned} b_{nn} &= \frac{1}{1-\lambda}, \quad b_{n,n-1} = \frac{1}{(1-\lambda)^2}, \\ b_{n,n-2} &= \frac{1}{(1-\lambda)^3} - \frac{1}{3(1-\lambda)^2}, \\ b_{n,n-3} &= \frac{1}{(1-\lambda)^4} - \frac{2}{3(1-\lambda)^3}, \\ b_{n,n-4} &= \frac{1}{(1-\lambda)^5} - \frac{1}{(1-\lambda)^4} - \frac{1}{9(1-\lambda)^3}, \\ &\vdots \end{aligned}$$

and so on.

By using Lemma 1.4, we need to show that $(\Delta_i^2 - \lambda I)^{-1} \in B(c_0)$, i.e.,

(i) The series $\sum_{k=0}^{\infty} |b_{nk}|$ is convergent for each $n \in \mathbb{N}_0$ and $\sup_n \sum_{k=0}^{\infty} |b_{nk}| < \infty$,

(ii) $\lim_{n\to\infty} |b_{nk}| = 0$ for each $k \in \mathbb{N}_0$.

Therefore, first we prove that the series $\sum_{k=0}^{\infty} |b_{nk}|$ is convergent for each $n \in \mathbb{N}_0$.

Let

$$\begin{aligned}
S_n &= \sum_{k=0}^{\infty} |b_{nk}| = |b_{n,0}| + |b_{n,1}| + |b_{n,2}| + ... + |b_{nn}| \\
&= |b_{nn}| + |b_{n,n-1}| + |b_{n,n-2}| + \dots \\
&= \Big|\frac{1}{1-\lambda}\Big| + \Big|\frac{1}{(1-\lambda)^2}\Big| + \Big|\frac{1}{(1-\lambda)^3} - \frac{1}{3(1-\lambda)^2}\Big| + \Big|\frac{1}{(1-\lambda)^4} - \frac{2}{3(1-\lambda)^3}\Big| \\
&\quad + \Big|\frac{1}{(1-\lambda)^5} - \frac{1}{(1-\lambda)^4} - \frac{1}{9(1-\lambda)^3}\Big| + \dots
\end{aligned}$$

Now,

$$\begin{aligned}
\lim_{n\to\infty} S_n &= \Big|\frac{1}{1-\lambda}\Big| + \Big|\frac{1}{(1-\lambda)^2}\Big| + \Big|\frac{1}{(1-\lambda)^3} - \frac{1}{3(1-\lambda)^2}\Big| \\
&\quad + \Big|\frac{1}{(1-\lambda)^4} - \frac{2}{3(1-\lambda)^3}\Big| + \dots \\
&\leq \Big|\frac{1}{1-\lambda}\Big| + \Big|\frac{1}{1-\lambda}\Big|^2 + \Big|\frac{1}{1-\lambda}\Big|^3 + \frac{1}{3}\Big|\frac{1}{1-\lambda}\Big|^2 + \Big|\frac{1}{1-\lambda}\Big|^4 + \frac{2}{3}\Big|\frac{1}{1-\lambda}\Big|^3 \\
&\quad + \Big|\frac{1}{1-\lambda}\Big|^5 + \Big|\frac{1}{1-\lambda}\Big|^4 + \frac{1}{9}\Big|\frac{1}{1-\lambda}\Big|^3 + \dots \\
&= \Big|\frac{1}{1-\lambda}\Big| + \left(\sum_{i\geq 1} n_{2(i)}\right)\Big|\frac{1}{1-\lambda}\Big|^2 + \left(\sum_{i\geq 1} n_{3(i)}\right)\Big|\frac{1}{1-\lambda}\Big|^3 \\
&\quad + \left(\sum_{i\geq 1} n_{4(i)}\right)\Big|\frac{1}{1-\lambda}\Big|^4 + \dots \\
&= \Big|\frac{1}{1-\lambda}\Big| + \left(1+\frac{1}{3}\right)\Big|\frac{1}{1-\lambda}\Big|^2 + \left(1+\frac{1}{3}+\frac{1}{3}+\frac{1}{9}\right)\Big|\frac{1}{1-\lambda}\Big|^3 \\
&\quad + \left(1+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{27}\right)\Big|\frac{1}{1-\lambda}\Big|^4 + \dots \\
&= \frac{1}{|1-\lambda|}\left\{1 + \left(\frac{4}{3}\right)\Big|\frac{1}{1-\lambda}\Big| + \left(\frac{16}{9}\right)\Big|\frac{1}{1-\lambda}\Big|^2 + \left(\frac{64}{27}\right)\Big|\frac{1}{1-\lambda}\Big|^3 + \dots\right\} \\
&= \frac{1}{|1-\lambda|}\left\{1 + \Big|\frac{4}{3(1-\lambda)}\Big| + \Big|\frac{4}{3(1-\lambda)}\Big|^2 + \Big|\frac{4}{3(1-\lambda)}\Big|^3 + \dots\right\} \\
&= \frac{3}{3|1-\lambda| - 4} < \infty,
\end{aligned}$$

where $n_{k(i)}$ denote the coefficients of $\left|\frac{1}{1-\lambda}\right|^k$ for $k \geq 2$. As per the assumption $\left|\frac{4}{3(1-\lambda)}\right| < 1$, thus $\lim_n S_n < \infty$. Now, (S_n) is a sequence of positive real numbers and is convergent, this implies the boundedness of (S_n).

Secondly, $\lim_{n\to\infty}|b_{nk}| = 0$ for each $k \in \mathbb{N}_0$, this follows from the fact that $\left|\frac{1}{(1-\lambda)}\right| < \left|\frac{4}{3(1-\lambda)}\right| < 1$. As a result, $(\Delta_i^2 - \lambda I)^{-1} \in B(c_0)$ with $|1-\lambda| > \frac{4}{3}$.

Now, we can show that domain of the operator $(\Delta_i^2 - \lambda I)^{-1}$ is dense in c_0 equivalently, the range of $(\Delta_i^2 - \lambda I)$ is dense in c_0, which implies the operator $(\Delta_i^2 - \lambda I)^{-1}$ is onto. Thus,

$$\sigma(\Delta_i^2, c_0) \subseteq \left\{\lambda \in \mathbb{C} : |1-\lambda| \leq \frac{4}{3}\right\}. \tag{2.3}$$

Conversely, consider $\lambda \neq 1$ and $|1-\lambda| \leq \frac{4}{3}$, clearly $(\Delta_i^2 - \lambda I)$ is a triangle and hence $(\Delta_i^2 - \lambda I)^{-1}$ exists, but the sequence (S_n) is unbounded.

$$\Rightarrow \qquad (\Delta_i^2 - \lambda I)^{-1} \notin B(c_0) \quad \text{with} \quad |1-\lambda| < \frac{4}{3} \quad \text{and} \quad |1-\lambda| = \frac{4}{3}.$$

Finally, we prove the result under the assumption $\lambda = 1$. If $\lambda = 1$, then we have

$$(\Delta_i^2 - \lambda I) = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ -1 & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & -1 & 0 & 0 & \cdots \\ 0 & \frac{1}{3} & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which is not invertible. Thus,

$$\left\{\lambda \in \mathbb{C} : |1-\lambda| \leq \frac{4}{3}\right\} \subseteq \sigma(\Delta_i^2, c_0). \tag{2.4}$$

Combining (2.3) and (2.4), we conclude the proof. □

Theorem 2.3. *The point spectrum of the operator Δ_i^2 over c_0 is given by*

$$\sigma_p(\Delta_i^2, c_0) = \emptyset.$$

Proof. Suppose $x \in c_0$ and consider $\Delta_i^2 x = \lambda x$ for $x \neq \theta$ in c_0, which gives a system of linear equations:

$$\left.\begin{array}{c} x_0 = \lambda x_0 \\ -x_0 + x_1 = \lambda x_1 \\ \frac{1}{3}x_0 - x_1 + x_2 = \lambda x_2 \\ \frac{1}{3}x_1 - x_2 + x_3 = \lambda x_3 \\ \ldots\ldots\ldots \\ \frac{1}{3}x_{k-2} - x_{k-1} + x_k = \lambda x_k \\ \ldots\ldots\ldots \end{array}\right\} \tag{2.5}$$

On solving above system of equations, it is clear that

$$x_1 = \frac{x_0}{1-\lambda}, x_2 = \left(\frac{1}{(1-\lambda)^2} - \frac{1}{3(1-\lambda)}\right)x_0, x_3 = \left(\frac{1}{(1-\lambda)^3} - \frac{2}{3(1-\lambda)^2}\right)x_0 \ldots$$

and so on
If $x_0 \neq 0$, then we obtain $\lambda = 1$ and hence $x_0 = 0, x_1 = 0, x_2 = 0\ldots$ which is a contradiction.
If $x_0 = 0$, then also $x_k = 0$ for all $k \geq 1$ which contradicts our assumption.

Again suppose x_k is the first non zero entry of $x = (x_k)$ and from the above system of equations we obtain $\lambda = 1$ and $x_{k-1} \neq 0$, which is a contradiction. Thus $\sigma_p(\Delta_i^2, c_0) = \emptyset$. □

Theorem 2.4. *The point spectrum of the adjoint operator* $(\Delta_i^2)^*$ *of* Δ_i^2 *over* $c_0^* \cong \ell_1$ *is given by*

$$\sigma_p((\Delta_i^2)^*, \ell_1) = \Big\{\lambda \in \mathbb{C} : |1-\lambda| < \frac{4}{3}\Big\}.$$

Proof. Suppose $(\Delta_i^2)^* f = \lambda f$ for $\mathbf{0} \neq f \in \ell_1$, where

$$(\Delta_i^2)^* = \begin{pmatrix} 1 & -1 & \frac{1}{3} & 0 & \cdots \\ 0 & 1 & -1 & \frac{1}{3} & \cdots \\ 0 & 0 & 1 & -1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } f = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{pmatrix}.$$

Consider the system of linear equations

$$\left.\begin{array}{c} f_0 - f_1 + \frac{1}{3} f_2 = \lambda f_0 \\ f_1 - f_2 + \frac{1}{3} f_3 = \lambda f_1 \\ f_2 - f_3 + \frac{1}{3} f_4 = \lambda f_2 \\ \ldots\ldots\ldots \\ f_k - f_{k+1} + \frac{1}{3} f_{k+2} = \lambda f_k \\ \ldots\ldots\ldots \end{array}\right\} \qquad (2.6)$$

Therefore, we have

$$\begin{aligned} |f_k| &= \frac{1}{|1-\lambda|}\left|\Big(f_{k+1} - \frac{1}{3} f_{k+2}\Big)\right| \\ &\leq \frac{1}{|1-\lambda|}\left[|f_{k+1}| + \left|\frac{1}{3} f_{k+2}\right|\right]. \end{aligned}$$

It is clear that $f = (f_k)$, defined by $f_k = r^k$, with $|r| < 1$ is an eigenvector corresponding to the eigenvalue λ satisfying $|1-\lambda| < \frac{4}{3}$. It can also be shown that $|f_0| \geq |f_1| \geq |f_2| \ldots \geq |f_k| \ldots$ i.e. $|f_0| \geq |f_k|$, for all $k \in \mathbb{N}_0$, this implies that $\sup_k |f_k| < \infty$ and hence $f \in \ell_1$.

Conversely, it is trivial to show that if $\sup_k |f_k| < \infty$, then $|1-\lambda| < \frac{4}{3}$. □

Theorem 2.5. *The residual spectrum of the operator* Δ_i^2 *over* c_0 *is given by*

$$\sigma_r(\Delta_i^2, c_0) = \Big\{\lambda \in \mathbb{C} : |1-\lambda| < \frac{4}{3}\Big\}.$$

Proof. For $|1-\lambda| < \frac{4}{3}$, the operator $\Delta_i^2 - \lambda I$ has an inverse. By Theorem 2.4 the operator $(\Delta_i^2)^* - \lambda I$ is not one to one for $\lambda \in \mathbb{C}$ with $|1-\lambda| < \frac{4}{3}$. By using Lemma 1.2, we have $\overline{R(\Delta_i^2 - \lambda I)} \neq c_0$. Hence,

$$\sigma_r(\Delta_i^2, c_0) = \Big\{\lambda \in \mathbb{C} : |1-\lambda| < \frac{4}{3}\Big\}.$$

□

Theorem 2.6. *The continuous spectrum of the operator Δ_i^2 over c_0 is given by*

$$\sigma_c(\Delta_i^2, c_0) = \Big\{\lambda \in \mathbb{C} : |1-\lambda| = \frac{4}{3}\Big\}. \tag{2.7}$$

Proof. The proof follows from Theorems 2.2, 2.3, 2.5 and the fact that

$$\sigma(\Delta_i^2, c_0) = \sigma_p(\Delta_i^2, c_0) \cup \sigma_r(\Delta_i^2, c_0) \cup \sigma_c(\Delta_i^2, c_0).$$ □

3 The Spectrum of the Operator Δ_i^2 over ℓ_1

In this section, we determine the fine spectrum of the operator Δ_i^2 over the sequence space ℓ_1. For our investigation, we need certain results which follow from Lemma 1.5. The results that obtained for the sequence space ℓ_1 are very similar to that of the sequence space c_0. In order to avoid the similar statements as discussed in previous sections, we omit some detail explanations and give certain results without proofs.

Theorem 3.1. *The operator $\Delta_i^2 : \ell_1 \to \ell_1$ is a linear operator and*

$$\| \Delta_i^2 \|_{(\ell_1:\ell_1)} = \frac{7}{3}. \tag{3.1}$$

Proof. The proof follows from Theorem 2.1 and Lemma 1.5. □

Theorem 3.2. *The spectrum of Δ_i^2 on the sequence space ℓ_1 is given by*

$$\sigma(\Delta_i^2, \ell_1) = \Big\{\lambda \in \mathbb{C} : |1-\lambda| \leq \frac{4}{3}\Big\}. \tag{3.2}$$

Proof. In view of Theorem 2.2, the proof of this theorem is almost similar. By using Lemma 1.5, only we need to show that the supremum of ℓ_1 norms of the columns of $(\Delta_i^2 - \lambda I)^{-1}$ is bounded, i.e, for all $n \in \mathbb{N}_0$

$$\sup_k \sum_{n=0}^{\infty} |b_{nk}| < \infty.$$

Therefore, first we prove that the series$\sum_{n=0}^{\infty}|b_{nk}|$ is convergent for each $k \in \mathbb{N}_0$. This is clear and this follows from the fact that for all $k \in \mathbb{N}_0$, we have

$$\begin{aligned}
b_{kk} &= \frac{1}{1-\lambda}, \quad b_{k,k-1} = b_{k+1,k} = \frac{1}{(1-\lambda)^2},\\
b_{k,k-2} = b_{k+2,k} &= \frac{1}{(1-\lambda)^3} - \frac{1}{3(1-\lambda)^2},\\
b_{k,k-3} = b_{k+3,k} &= \frac{1}{(1-\lambda)^4} - \frac{2}{3(1-\lambda)^3},\\
b_{k,k-4} = b_{k+4,k} &= \frac{1}{(1-\lambda)^5} - \frac{1}{(1-\lambda)^4} - \frac{1}{9(1-\lambda)^3},\\
&\vdots
\end{aligned}$$

Now, using Lemma 1.5, we obtain

$$\begin{aligned}
\lim_k S_k = \lim_k \sum_{n=0}^{\infty} |b_{nk}| &= \lim_k (|b_{kk}| + |b_{k+1,k}| + |b_{k+2,k}| + \dots)\\
&= \lim_n (|b_{nn}| + |b_{n,n-1}| + |b_{n,n-2}| + \dots) < \frac{3}{3|1-\lambda|-4} < \infty,
\end{aligned}$$

by Theorem 2.2. Therefore, (S_k) is sequence of real numbers and $\sup_k S_k$ is finite, which implies that (S_k) is bounded. □

Theorem 3.3. **(i)** $\sigma_p(\Delta_i^2, \ell_1) = \emptyset$.

(ii) $\sigma_r(\Delta_i^2, \ell_1) = \left\{\lambda \in \mathbb{C} : |1-\lambda| < \frac{4}{3}\right\}$.

(iii) $\sigma_c(\Delta_i^2, \ell_1) = \left\{\lambda \in \mathbb{C} : |1-\lambda| = \frac{4}{3}\right\}$.

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