



# Common Tripled Fixed point Theorems for $\psi$ -Geraghty-Type Contraction Mappings Endowed with a Directed Graph

Tanadon Chaobankoh and Phakdi Charoensawan<sup>1</sup>

Department of Mathematics, Faculty of Science, Chiang Mai University  
Chiang Mai 50200, Thailand

e-mail : chaobankoh@gmail.com (T. Chaobankoh)

phakdi@hotmail.com (P. Charoensawan)

**Abstract :** The purpose of this paper is to present some existence and uniqueness results for tripled coincidence point and common tripled fixed point of  $\psi$ -Geraghty-type contraction mappings in complete metric spaces endowed with a directed graph. Some applications supported our main results are also given.

**Keywords :** fixed point; tripled coincidence point; graph; edge preserving.

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## 1 Introduction

In 1987, Guo and Lakshmikantham [1] introduced the concept of coupled fixed point. Bhaskar and Lakshmikantham [2] then gave the concept of mixed monotone property for contractive operators in partially ordered metric spaces and proved the coupled fixed point theorems for mappings which satisfy the mixed monotone property. The results in [2] were extended by Lakshmikantham and Ćirić in [3] by defining the mixed  $g$ -monotone and using it to study the existence and uniqueness of coupled coincidence point for such mapping which satisfy the mixed monotone property in partially ordered metric spaces. Consequently, several coupled fixed point and coupled coincidence point results have appeared in the recent literature. Works noted in [4–11] are some examples of these works.

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<sup>1</sup>Corresponding author email.

The fixed point theorem using the context of metric spaces endowed with a graph was initiated by Jachymski [12]. Later, Chifu and Petrusel [13] generalized the result in [2] by using the context of metric spaces endowed with a graph. The result generalizes and extends some coupled fixed points theorems in partially ordered complete metric spaces with mixed monotone property.

Recently, Kadelburg and et al. [14] established common coupled fixed point theorems by using monotone and  $g$ -monotone properties instead of mixed monotone and  $g$ -mixed monotone properties. They also obtained coupled coincidence point theorems and gave some sufficient conditions for the uniqueness of a common coupled fixed point under a Garaghty-type condition.

The notion of tripled fixed point which is a fixed point of order  $N = 3$  was introduced by Samet and Vetro [15]. Later, in 2011, Berinde and Borcut [16] defined the concept of tripled fixed point in the case of ordered sets in order to keep the mixed monotone property for nonlinear mappings in partially ordered complete metric spaces. They also proved existence and uniqueness theorems for contractive type mappings. In 2012, Berinde and Borcut [17] introduced the concept of tripled coincidence point for a pair of nonlinear contractive mappings  $F : X^3 \rightarrow X$  and  $g : X \rightarrow X$  and obtained tripled coincidence point theorems which generalized results in [16]. Recently, Aydi et al. [18] introduced the concept of  $W$ -compatible for mappings  $F : X^3 \rightarrow X$  and  $g : X \rightarrow X$  in an abstract metric space and defined a notion of a tripled point of coincidence. They also established tripled and common point of coincidence theorems in an abstract metric space.

A wide discussion on a tripled coincidence point in partially ordered metric spaces, using mixed  $g$ -monotone property, has been studied to the improvement and generalization. Borcut [19] gave tripled coincidence point theorems for a pair of mappings  $F : X^3 \rightarrow X$  and  $g : X \rightarrow X$  satisfying a nonlinear contractive condition and mixed  $g$ -monotone property in partially ordered metric spaces. The presented theorems extended existing results in literature. Some authors have studied tripled fixed point and tripled coincidence point theory in different directions in several spaces with applications (see [18–32]).

The aim of this work is to prove some common tripled fixed point theorems for  $\psi$ -Geraghty-type contraction mappings by using the context of metric spaces endowed with a directed graph.

## 2 Preliminaries

In this section, we first give some useful notations, definitions and backgrounds.

Let  $(X, d)$  be a metric space and  $\Delta$  be a diagonal of  $X \times X$ . Let  $G$  be a directed graph, such that the set  $V(G)$  of its vertices coincides with  $X$  and  $\Delta \subseteq E(G)$ , where  $E(G)$  is the set of the edges of the graph. Assume also that  $G$  has no parallel edges and, thus, one can identify  $G$  with the pair  $(V(G), E(G))$ .

Throughout the paper we shall say that  $G$  with the above-mentioned properties satisfies *standard conditions*.

Let  $G^{-1}$  be the graph obtained from  $G$  by reversing the direction of edges. Thus,

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

A pair  $(X, \preceq)$  denotes a partially ordered set. By  $x \succeq y$ , we mean  $y \preceq x$ . Let  $f, g : X \rightarrow X$  be mappings. A mapping  $f$  is said to be  $g$ -non-decreasing (resp.,  $g$ -non-increasing) if, for all  $x, y \in X$ ,  $gx \preceq gy$  implies  $fx \preceq fy$  (resp.,  $fy \preceq fx$ ). If  $g$  is the identity mapping, then  $f$  is said to be non-decreasing (resp., non-increasing).

**Definition 2.1** ([14]). Let  $(X, \preceq)$  be a partially ordered set and let  $F : X^3 \rightarrow X$  and  $g : X \rightarrow X$  be two mappings. The mapping  $F$  is said to have a *g-monotone property* if  $F$  is monotone  $g$ -non-decreasing in both of its arguments, that is, for any  $x, y, z \in X$ ,

$$x_1, x_2 \in X, \quad gx_1 \preceq gx_2 \Rightarrow F(x_1, y, z) \preceq F(x_2, y, z),$$

$$y_1, y_2 \in X, \quad gy_1 \preceq gy_2 \Rightarrow F(x, y_1, z) \preceq F(x, y_2, z)$$

and

$$z_1, z_2 \in X, \quad gz_1 \preceq gz_2 \Rightarrow F(x, y, z_1) \preceq F(x, y, z_2)$$

hold.

In particular, if in the previous relations  $g$  is the identity map, then  $F$  is said to have a *monotone property*.

Aydi and et al. [18] introduced the following definitions.

**Definition 2.2** ([18]). Let  $X$  be a nonempty set and  $F : X^3 \rightarrow X$ ,  $g : X \rightarrow X$  be two mappings. An element  $(x, y, z) \in X^3$  is called:

(C<sub>1</sub>) a *triple fixed point* of  $F$  if  $x = F(x, y, z)$ ,  $y = F(y, z, x)$  and  $z = F(z, x, y)$ ;

(C<sub>2</sub>) a *triple coincidence point* of mappings  $g$  and  $F$  if

$$gx = F(x, y, z), \quad gy = F(y, z, x) \quad \text{and} \quad gz = F(z, x, y),$$

and in this case  $(gx, gy, gz)$  is called a *triple point of coincidence*;

(C<sub>3</sub>) a *common triple fixed point* of mappings  $g$  and  $F$  if

$$x = gx = F(x, y, z), \quad y = gy = F(y, z, x) \quad \text{and} \quad z = gz = F(z, x, y).$$

**Definition 2.3** ([18]). Let  $(X, d)$  be a metric space and let  $g : X \rightarrow X$ ,  $F : X^3 \rightarrow X$ . The mappings  $g$  and  $F$  are said to be *compatible* if

$$\lim_{n \rightarrow \infty} d(gF(x_n, y_n, z_n), F(gx_n, gy_n, gz_n)) = 0,$$

$$\lim_{n \rightarrow \infty} d(gF(y_n, z_n, x_n), F(gy_n, gz_n, gx_n)) = 0,$$

$$\text{and} \quad \lim_{n \rightarrow \infty} d(gF(z_n, x_n, y_n), F(gz_n, gx_n, gy_n)) = 0,$$

hold whenever  $\{x_n\}$ ,  $\{y_n\}$  and  $\{z_n\}$  are sequences in  $X$  such that

$$\lim_{n \rightarrow \infty} F(x_n, y_n, z_n) = \lim_{n \rightarrow \infty} gx_n, \quad \lim_{n \rightarrow \infty} F(y_n, z_n, x_n) = \lim_{n \rightarrow \infty} gy_n$$

$$\text{and} \quad \lim_{n \rightarrow \infty} F(z_n, x_n, y_n) = \lim_{n \rightarrow \infty} gz_n.$$

### 3 Common Tripled Fixed Point

We denote the set of all tripled coincidence points of mapping  $F : X^3 \rightarrow X$  and  $g : X \rightarrow X$  by  $TcFix(F)$ . In other words,

$$TcFix(F) = \{(x, y, z) \in X^3 : F(x, y, z) = gx, F(y, z, x) = gy \text{ and } F(z, x, y) = gz\}.$$

**Definition 3.1.** We say that a pair of mappings  $F : X^3 \rightarrow X$  and  $g : X \rightarrow X$  is  $g$ -edge preserving if

$$\begin{aligned} & [(gx, gu), (gy, gv), (gz, gw) \in E(G)] \\ & \Rightarrow [(F(x, y, z), F(u, v, w)), (F(y, z, x), F(v, w, u)), (F(z, x, y), F(w, u, v)) \in E(G)]. \end{aligned}$$

**Definition 3.2.** Let  $(X, d)$  be a complete metric space, and let  $E(G)$  be the set of the edges of the graph. We say that  $E(G)$  satisfies the transitive property if for all  $x, y, a \in X$

$$(x, a), (a, y) \in E(G) \rightarrow (x, y) \in E(G).$$

**Definition 3.3.** The operator  $F : X^3 \rightarrow X$  is called  $G$ -continuous if for all  $(x^*, y^*, z^*) \in X^3$  and for any sequence  $(n_i)_i \in \mathbb{N}$  of positive integers, with  $F(x_{n_i}, y_{n_i}, z_{n_i}) \rightarrow x^*$ ,  $F(y_{n_i}, z_{n_i}, x_{n_i}) \rightarrow y^*$  and  $F(z_{n_i}, x_{n_i}, y_{n_i}) \rightarrow z^*$ , as  $i \rightarrow \infty$ , and

$$\begin{aligned} & (F(x_{n_i}, y_{n_i}, z_{n_i}), F(x_{n_i+1}, y_{n_i+1}, z_{n_i+1})), (F(y_{n_i}, z_{n_i}, x_{n_i}), F(y_{n_i+1}, z_{n_i+1}, x_{n_i+1})), \\ & (F(z_{n_i}, x_{n_i}, y_{n_i}), F(z_{n_i+1}, x_{n_i+1}, y_{n_i+1})) \in E(G), \end{aligned}$$

we have that

$$\begin{aligned} & F(F(x_{n_i}, y_{n_i}, z_{n_i}), F(y_{n_i}, z_{n_i}, x_{n_i}), F(z_{n_i}, x_{n_i}, y_{n_i})) \rightarrow F(x^*, y^*, z^*) \\ & F(F(y_{n_i}, z_{n_i}, x_{n_i}), F(z_{n_i}, x_{n_i}, y_{n_i}), F(x_{n_i}, y_{n_i}, z_{n_i})) \rightarrow F(y^*, z^*, x^*) \\ & F(F(z_{n_i}, x_{n_i}, y_{n_i}), F(x_{n_i}, y_{n_i}, z_{n_i}), F(y_{n_i}, z_{n_i}, x_{n_i})) \rightarrow F(z^*, x^*, y^*) \\ & \text{as } i \rightarrow \infty. \end{aligned}$$

**Definition 3.4.** Let  $(X, d)$  be a complete metric space and  $G$  be a directed graph. We say that the triple  $(X, d, G)$  has the property  $A$ , if for any sequence  $(x_n)_{n \in \mathbb{N}} \subset X$  with  $x_n \rightarrow x$ , as  $n \rightarrow \infty$ , and  $(x_n, x_{n+1}) \in E(G)$ , for  $n \in \mathbb{N}$ , we have  $(x_n, x) \in E(G)$ .

Let  $(X, d)$  be a metric space endowed with a directed graph  $G$  satisfying the standard conditions.

We consider the set denoted by  $(X^3)_g^F$  and defined as

$$(X^3)_g^F = \{(x, y, z) \in X^3 : (gx, F(x, y, z)), (gy, F(y, z, x)), (gz, F(z, x, y)) \in E(G)\}.$$

Let  $\Psi$  denote the class of all functions  $\psi : [0, \infty) \rightarrow [0, \infty)$  which satisfy the following conditions:

- ( $\psi_1$ )  $\psi$  is nondecreasing;
- ( $\psi_2$ )  $\psi(s+t) \leq \psi(s) + \psi(t)$ ;
- ( $\psi_3$ )  $\psi$  is continuous;
- ( $\psi_4$ )  $\psi(t) = 0 \Leftrightarrow t = 0$ .

Let  $\Theta$  denote the class of all functions  $\theta : [0, \infty) \times [0, \infty) \times [0, \infty) \rightarrow [0, 1)$  which satisfy the following conditions:

- ( $\theta_1$ )  $\theta(s, t, r) = \theta(t, r, s) = \theta(r, s, t)$  for all  $s, t, r \in [0, \infty)$ ;
- ( $\theta_2$ ) For any three sequences  $\{s_n\}$ ,  $\{t_n\}$  and  $\{r_n\}$  of nonnegative real numbers,

$$\theta(s_n, t_n, r_n) \rightarrow 1 \implies s_n, t_n, r_n \rightarrow 0.$$

**Definition 3.5.** A pair of  $F : X^3 \rightarrow X$  and  $g : X \rightarrow X$  is called a  $\theta$ - $\psi$ -contraction if :

- (1)  $F$  and  $g$  are  $g$ -edge preserving;
- (2) there exists  $\theta \in \Theta$  and  $\psi \in \Psi$  such that for all  $x, y, z, u, v, w \in X$  satisfying  $(gx, gu), (gy, gv), (gz, gw) \in E(G)$ ,

$$\begin{aligned} & \psi(d(F(x, y, z), F(u, v, w))) \\ & \leq \theta(d(gx, gu), d(gy, gv), d(gz, gw))\psi(M(gx, gu, gy, gv, gz, gw)) \end{aligned} \quad (3.1)$$

where  $M(gx, gu, gy, gv, gz, gw) = \max\{d(gx, gu), d(gy, gv), d(gz, gw)\}$ .

**Theorem 3.6.** Let  $(X, d)$  be a complete metric space endowed with a directed graph  $G$ , and let a pair of  $F : X^3 \rightarrow X$  and  $g : X \rightarrow X$  be a  $\theta$ - $\psi$ -contraction. Suppose that:

- (i)  $g$  is continuous and  $g(X)$  is closed;
- (ii)  $F(X \times X \times X) \subset g(X)$  and  $g$  and  $F$  are compatible;
- (iii)  $F$  is  $G$ -continuous or the tripled  $(X, d, G)$  has the properties  $A$ ;
- (iv)  $E(G)$  satisfies the transitive property.

Under these conditions,  $T\text{Fix}(F) \neq \emptyset$  if and only if  $(X^3)_g^F \neq \emptyset$ .

*Proof.* Suppose the  $T\text{Fix}(F) \neq \emptyset$ . Let  $(u, v, w) \in T\text{Fix}(F)$ . We have

$$\begin{aligned} (gu, F(u, v, w)) &= (gu, gu), \quad (gv, F(v, w, u)) = (gv, gv) \\ \text{and } (gw, F(w, u, v)) &= (gw, gw) \in \Delta \subset E(G). \end{aligned}$$

Hence  $(gu, F(u, v, w))$ ,  $(gv, F(v, w, u))$  and  $(gw, F(w, u, v)) \in E(G)$  which mean that  $(u, v, w) \in (X^3)_g^F$  and thus  $(X^3)_g^F \neq \emptyset$ .

Suppose now  $(X^3)_g^F \neq \emptyset$ . Let  $x_0, y_0, z_0 \in X$  such that  $(x_0, y_0, z_0) \in (X^3)_g^F$ , we have  $(gx_0, F(x_0, y_0, z_0))$ ,  $(gy_0, F(y_0, z_0, x_0))$  and  $(gz_0, F(z_0, x_0, y_0)) \in E(G)$ .

Using that  $F(X \times X \times X) \subset g(X)$ , we can construct sequences  $\{x_n\}$ ,  $\{y_n\}$  and  $\{z_n\}$  in  $X$  such that

$$\begin{aligned} gx_n &= F(x_{n-1}, y_{n-1}, z_{n-1}), gy_n = F(y_{n-1}, z_{n-1}, x_{n-1}) \\ \text{and } gz_n &= F(z_{n-1}, x_{n-1}, y_{n-1}) \quad \text{for } n = 1, 2, \dots \end{aligned}$$

If  $gx_{n_0} = gx_{n_0-1}$ ,  $gy_{n_0} = gy_{n_0-1}$  and  $gz_{n_0} = gz_{n_0-1}$  for some  $n_0 \in \mathbb{N}$ , then  $(gx_{n_0-1}, gy_{n_0-1}, gz_{n_0-1})$  is a tripled point of coincidence for  $g$  and  $F$ . Therefore, in what follows, we will assume that for each  $n \in \mathbb{N}$ ,  $gx_n \neq gx_{n-1}$  or  $gy_n \neq gy_{n-1}$  or  $gz_n \neq gz_{n-1}$  holds.

Since  $(gx_0, F(x_0, y_0, z_0)) = (gx_0, gx_1)$ ,  $(gy_0, F(y_0, z_0, x_0)) = (gy_0, gy_1)$  and  $(gz_0, F(z_0, x_0, y_0)) = (gz_0, gz_1) \in E(G)$  and  $F$  and  $g$  is  $g$ -edge preserving, we have  $(F(x_0, y_0, z_0), F(x_1, y_1, z_1)) = (gx_1, gx_2)$ ,  $(F(y_0, z_0, x_0), F(y_1, z_1, x_1)) = (gy_1, gy_2)$  and  $(F(z_0, x_0, y_0), F(z_1, x_1, y_1)) = (gz_1, gz_2) \in E(G)$ . By induction we shall obtain  $(gx_{n-1}, gx_n), (gy_{n-1}, gy_n)$  and  $(gz_{n-1}, gz_n) \in E(G)$  for each  $n \in \mathbb{N}$ .

Hence, the  $\theta$ - $\psi$ -contraction (3.1) can be used to conclude that

$$\begin{aligned} \psi(d(gx_n, gx_{n+1})) &= \psi(d(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_n, y_n, z_n))) \\ &\leq \theta(d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n))\psi(M), \quad (3.2) \end{aligned}$$

$$\begin{aligned} \psi(d(gy_n, gy_{n+1})) &= \psi(d(F(y_{n-1}, z_{n-1}, x_{n-1}), F(y_n, z_n, x_n))) \\ &\leq \theta(d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), d(gx_{n-1}, gx_n))\psi(M) \\ &= \theta(d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n))\psi(M), \quad (3.3) \end{aligned}$$

and

$$\begin{aligned} \psi(d(gz_n, gz_{n+1})) &= \psi(d(F(z_{n-1}, x_{n-1}, y_{n-1}), F(z_n, x_n, y_n))) \\ &\leq \theta(d(gz_{n-1}, gz_n), d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n))\psi(M) \\ &= \theta(d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n))\psi(M), \quad (3.4) \end{aligned}$$

for all  $n \in \mathbb{N}$ , where  $M = M(gx_{n-1}, gx_n, gy_{n-1}, gy_n, gz_{n-1}, gz_n)$ . From (3.2), (3.3) and (3.4), we get

$$\begin{aligned} &\psi(M(gx_n, gx_{n+1}, gy_n, gy_{n+1}, gz_n, gz_{n+1})) \\ &= \psi(\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1})\}) \\ &\leq \theta(d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n))\psi(M(gx_{n-1}, gx_n, \\ &\quad gy_{n-1}, gy_n, gz_{n-1}, gz_n)) \\ &< \psi(M(gx_{n-1}, gx_n, gy_{n-1}, gy_n, gz_{n-1}, gz_n)) \quad (3.5) \end{aligned}$$

for all  $n \in \mathbb{N}$ . From (3.5), we get

$$\begin{aligned} &\psi(M(gx_n, gx_{n+1}, gy_n, gy_{n+1}, gz_n, gz_{n+1})) \\ &< \psi(M(gx_{n-1}, gx_n, gy_{n-1}, gy_n, gz_{n-1}, gz_n)). \end{aligned}$$

Regarding the properties of  $\psi$ , we conclude that

$$M(gx_n, gx_{n+1}, gy_n, gy_{n+1}, gz_n, gz_{n+1}) < M(gx_{n-1}, gx_n, gy_{n-1}, gy_n, gz_{n-1}, gz_n).$$

Thus the sequence  $d_n := M(gx_{n-1}, gx_n, gy_{n-1}, gy_n, gz_{n-1}, gz_n)$  is decreasing. It follows that  $d_n \rightarrow d$  as  $n \rightarrow \infty$  for some  $d \geq 0$ . Next, we claim that  $d = 0$ . Assume on the contrary that  $d > 0$ ; then from (3.5), we obtain that

$$\begin{aligned} & \frac{\psi(M(gx_n, gx_{n+1}, gy_n, gy_{n+1}, gz_n, gz_{n+1}))}{\psi(M(gx_{n-1}, gx_n, gy_{n-1}, gy_n, gz_{n-1}, gz_n))} \\ & \leq \theta(d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)) < 1. \end{aligned}$$

On taking limit as  $n \rightarrow \infty$ , we get

$$\theta(d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)) \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Since  $\theta \in \Theta$ , we have

$$d(gx_{n-1}, gx_n) \rightarrow 0, d(gy_{n-1}, gy_n) \rightarrow 0 \quad \text{and} \quad d(gz_{n-1}, gz_n) \rightarrow 0$$

as  $n \rightarrow \infty$  and hence

$$\lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} M(gx_{n-1}, gx_n, gy_{n-1}, gy_n, gz_{n-1}, gz_n) = 0, \quad (3.6)$$

which contradicts the assumption  $d > 0$ . Therefore, we can conclude that  $d_n := M(gx_{n-1}, gx_n, gy_{n-1}, gy_n, gz_{n-1}, gz_n) \rightarrow 0$  as  $n \rightarrow \infty$ .

Next, we show that  $\{gx_n\}$ ,  $\{gy_n\}$  and  $\{gz_n\}$  are Cauchy sequences. On the contrary, assume that at least one of  $\{gx_n\}$  or  $\{gy_n\}$  or  $\{gz_n\}$  is not a Cauchy sequence. Then there is an  $\epsilon > 0$  for which we can find subsequences  $\{gx_{n_k}\}$ ,  $\{gx_{m_k}\}$  of  $\{gx_n\}$ ,  $\{gy_{n_k}\}$ ,  $\{gy_{m_k}\}$  of  $\{gy_n\}$  and  $\{gz_{n_k}\}$ ,  $\{gz_{m_k}\}$  of  $\{gz_n\}$  with  $n_k > m_k \geq k$  such that

$$M(gx_{n_k}, gx_{m_k}, gy_{n_k}, gy_{m_k}, gz_{n_k}, gz_{m_k}) \geq \epsilon \quad (3.7)$$

and

$$M(gx_{n_k-1}, gx_{m_k}, gy_{n_k-1}, gy_{m_k}, gz_{n_k-1}, gz_{m_k}) < \epsilon. \quad (3.8)$$

Using (3.7), (3.8), and the triangle inequality, we have

$$\begin{aligned} \epsilon & \leq r_k := M(gx_{n_k}, gx_{m_k}, gy_{n_k}, y_{m_k}, gz_{n_k}, z_{m_k}) \\ & \leq M(gx_{n_k}, gx_{n_k-1}, gy_{n_k}, gy_{n_k-1}, gz_{n_k}, gz_{n_k-1}) \\ & \quad + M(gx_{n_k-1}, gx_{m_k}, gy_{n_k-1}, gy_{m_k}, gz_{n_k-1}, gz_{m_k}) \\ & < M(gx_{n_k}, gx_{n_k-1}, gy_{n_k}, gy_{n_k-1}, gz_{n_k}, gz_{n_k-1}) + \epsilon. \end{aligned}$$

On taking limit as  $k \rightarrow \infty$ , we have

$$r_k = M(gx_{n_k}, gx_{m_k}, gy_{n_k}, y_{m_k}, gz_{n_k}, z_{m_k}) \rightarrow \epsilon. \quad (3.9)$$

Since  $(gx_{n-1}, gx_n)$ ,  $(gy_{n-1}, gy_n)$  and  $(gz_{n-1}, gz_n) \in E(G)$  for each  $n \in \mathbb{N}$  and  $E(G)$  satisfies the transitive property and the triangle inequality, we get

$$\begin{aligned}
 \psi(r_k) &= \psi(M(gx_{n_k}, gx_{m_k}, gy_{n_k}, y_{m_k}, gz_{n_k}, z_{m_k})) \\
 &\leq \psi(M(gx_{n_k}, gx_{n_k+1}, gy_{n_k}, gy_{n_k+1}, gz_{n_k}, gz_{n_k+1})) \\
 &\quad + \psi(M(gx_{n_k+1}, gx_{m_k+1}, gy_{n_k+1}, gy_{m_k+1}, gz_{n_k+1}, gz_{m_k+1})) \\
 &\quad + \psi(M(gx_{m_k+1}, gx_{m_k}, gy_{m_k+1}, gy_{m_k}, gz_{m_k+1}, gz_{m_k})) \\
 &= \psi(M(gx_{n_k}, gx_{n_k+1}, gy_{n_k}, gy_{n_k+1}, gz_{n_k}, gz_{n_k+1})) \\
 &\quad + \psi(M(gx_{m_k+1}, gx_{m_k}, gy_{m_k+1}, gy_{m_k}, gz_{m_k+1}, gz_{m_k})) \\
 &\quad + \psi(M(gx_{n_k+1}, gx_{m_k+1}, gy_{n_k+1}, gy_{m_k+1}, gz_{n_k+1}, gz_{m_k+1})) \\
 &\leq \psi(M(gx_{n_k}, gx_{n_k+1}, gy_{n_k}, gy_{n_k+1}, gz_{n_k}, gz_{n_k+1})) \\
 &\quad + \psi(M(gx_{m_k+1}, gx_{m_k}, gy_{m_k+1}, gy_{m_k}, gz_{m_k+1}, gz_{m_k})) \\
 &\quad + \theta(d(gx_{n_k}, gx_{m_k}), d(gy_{n_k}, gy_{m_k}), d(gz_{n_k}, gz_{m_k}))\psi(M(gx_{n_k}, gx_{m_k}, \\
 &\quad \quad \quad gy_{n_k}, gy_{m_k}, gz_{n_k}, gz_{m_k})) \\
 &= \psi(d_{n_k+1}) + \psi(d_{m_k+1}) + \theta(d(gx_{n_k}, gx_{m_k}), d(gy_{n_k}, gy_{m_k}), d(gz_{n_k}, gz_{m_k}))\psi(r_k) \\
 &< \psi(d_{n_k+1}) + \psi(d_{m_k+1}) + \psi(r_k).
 \end{aligned}$$

Now, we have

$$\begin{aligned}
 \psi(r_k) &\leq \psi(d_{n_k+1}) + \psi(d_{m_k+1}) \\
 &\quad + \theta(d(gx_{n_k}, gx_{m_k}), d(gy_{n_k}, gy_{m_k}), d(gz_{n_k}, gz_{m_k}))\psi(r_k) \\
 &< \psi(d_{n_k+1}) + \psi(d_{m_k+1}) + \psi(r_k).
 \end{aligned}$$

On taking limit as  $k \rightarrow \infty$  and using (3.6), (3.9) and properties of  $\psi$ , we get

$$\theta(d(gx_{n_k}, gx_{m_k}), d(gy_{n_k}, gy_{m_k}), d(gz_{n_k}, gz_{m_k})) \rightarrow 1.$$

Using the properties of function  $\theta$ , we obtain

$$d(gx_{n_k}, gx_{m_k}) \rightarrow 0, \quad d(gy_{n_k}, gy_{m_k}) \rightarrow 0 \quad \text{and} \quad d(gz_{n_k}, gz_{m_k}) \rightarrow 0$$

as  $k \rightarrow \infty$  which imply that

$$\lim_{k \rightarrow \infty} r_k = \lim_{k \rightarrow \infty} M(gx_{n_k}, gx_{m_k}, gy_{n_k}, gy_{m_k}, gz_{n_k}, gz_{m_k}) = 0$$

which contradicts with  $\epsilon > 0$ .

Therefore, we get that  $\{gx_n\}$ ,  $\{gy_n\}$  and  $\{gz_n\}$  are Cauchy sequences. Since  $g(X)$  is a closed subset of a complete metric space, there exist  $u, v, w \in g(X)$  such that

$$\lim_{n \rightarrow \infty} gx_n = \lim_{n \rightarrow \infty} F(x_n, y_n, z_n) = u,$$

$$\lim_{n \rightarrow \infty} gy_n = \lim_{n \rightarrow \infty} F(y_n, z_n, x_n) = v,$$

and

$$\lim_{n \rightarrow \infty} gz_n = \lim_{n \rightarrow \infty} F(z_n, x_n, y_n) = w.$$



By condition (ii), compatibility of  $g$  and  $F$  implies that

$$\lim_{n \rightarrow \infty} d(gF(x_n, y_n, z_n), F(gx_n, gy_n, gz_n)) = 0 \quad (3.10)$$

$$\lim_{n \rightarrow \infty} d(gF(y_n, z_n, x_n), F(gy_n, gz_n, gx_n)) = 0$$

and

$$\lim_{n \rightarrow \infty} d(gF(z_n, x_n, y_n), F(gz_n, gx_n, gy_n)) = 0.$$

Consider the two possibilities given in condition (iii).

(a) Suppose that  $F$  is G-continuous. Using the triangle inequality, we obtain that

$$d(gu, F(gx_n, gy_n, gz_n)) \leq d(gu, gF(x_n, y_n, z_n)) + d(gF(x_n, y_n, z_n), F(gx_n, gy_n, gz_n)).$$

Passing to the limit as  $n \rightarrow \infty$  and using (3.10) and continuity of  $g$  and  $F$  is G-continuous, we get that  $d(gu, F(u, v, w)) = 0$ , i.e.,  $gu = F(u, v, w)$ . In a similar way,  $gv = F(v, w, u)$  and  $gw = F(w, u, v)$  are obtained. Thus  $(u, v, w)$  is a tripled coincidence of the mapping  $F$  and  $g$ ; therefore,  $TFix(F) \neq \emptyset$ .

Suppose now that we have the tripled  $(X, d, G)$  has the properties  $A$ . In this case  $gx = u$ ,  $gy = v$  and  $gz = w$  for some  $x, y, z \in X$ , we have  $(gx_n, gx)$ ,  $(gy_n, gy)$  and  $(gz_n, gz) \in E(G)$  for each  $n \in \mathbb{N}$ . Using (3.1) we get

$$\begin{aligned} & \psi(d(gx, F(x, y, z)) + d(gy, F(y, z, x)) + d(gz, F(z, x, y))) \\ & \leq \psi(d(gx, gx_{n+1}) + d(gx_{n+1}, F(x, y, z))) \\ & \quad + d(gy, gy_{n+1}) + d(gy_{n+1}, F(y, z, x)) \\ & \quad + d(gz, gz_{n+1}) + d(gz_{n+1}, F(z, x, y))) \\ & \leq \psi(d(F(x_n, y_n, z_n), F(x, y, z))) \\ & \quad + \psi(d(F(y_n, z_n, x_n), F(y, z, x))) \\ & \quad + \psi(d(F(z_n, x_n, y_n), F(z, x, y))) \\ & \quad + \psi(d(gx, gx_{n+1}) + \psi(d(gy, gy_{n+1})) + \psi(d(gz, gz_{n+1}))) \\ & \leq 3\theta(d(gx_n, gx), d(gy_n, gy), d(gz_n, gz))\psi(M(gx_n, gx, gy_n, gy, gz_n, gz)) \\ & \quad + \psi(d(gx, gx_{n+1}) + \psi(d(gy, gy_{n+1})) + \psi(d(gz, gz_{n+1}))) \\ & \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ . We have

$$\psi(d(gx, F(x, y, z)) + d(gy, F(y, z, x)) + d(gz, F(z, x, y))) = 0.$$

By properties of  $\psi$  we have

$$d(gx, F(x, y, z)) + d(gy, F(y, z, x)) + d(gz, F(z, x, y)) = 0.$$

Hence,  $gx = F(x, y, z)$ ,  $gy = F(y, z, x)$  and  $gz = F(z, x, y)$ .

□

**Corollary 3.7.** Let  $(X, d, \preceq)$  be a partially ordered complete metric space and let  $F : X^3 \rightarrow X$  have the monotone  $g$ -non-decreasing property and  $g : X \rightarrow X$  is continuous. Suppose that the following hold:

- (i) there exist  $x_0, y_0, z_0 \in X$  such that  $gx_0 \preceq F(x_0, y_0, z_0), gy_0 \preceq F(y_0, z_0, x_0)$  and  $gz_0 \preceq F(z_0, x_0, y_0)$  ;
- (ii) there exists  $\theta \in \Theta$  and  $\psi \in \Psi$  such that for all  $x, y, z, u, v, w \in X$  satisfying  $(gx \preceq gu, gy \preceq gv$  and  $gz \preceq gw)$  or  $(gu \preceq gx, gv \preceq gy$  and  $gz \preceq gw)$ ,

$$\begin{aligned} & \psi(d(F(x, y, z), F(u, v, w))) \\ & \leq \theta(d(gx, gu), d(gy, gv), d(gz, gw))\psi(M(gx, gu, gy, gv, gz, gw)) \end{aligned}$$

where  $M(gx, gu, gy, gv, gz, gw) = \max\{d(gx, gu), d(gy, gv), d(gz, gw)\}$

- (iii) (a)  $F$  is continuous or (b) if  $\{x_n\}$  is an increasing sequence in  $X$  and  $x_n \rightarrow x$  as  $n \rightarrow \infty$ , then  $x_n \preceq x$  for all  $n$ .

Then there exist  $x, y, z \in X$  such that  $gx = F(x, y, z)$ ,  $y = F(y, z, x)$  and  $z = F(z, x, y)$ , i.e.,  $F$  has a tripled coincidence point.

*Proof.* Take  $E(G) = \{(x, y) \in X \times X : x \leq y\}$ , we obtain the corollary.  $\square$

We denote the set of all common tripled fixed points of mappings  $F : X^3 \rightarrow X$  and  $g : X \rightarrow X$  by  $CTFix(F)$ , i.e.,

$$\begin{aligned} CTFix(F) = \{(x, y) \in X^3 : & F(x, y, z) = gx = x, F(y, z, x) = gy = y \\ & \text{and } F(z, x, y) = gz = z\}. \end{aligned}$$

**Theorem 3.8.** In addition to hypotheses of Theorem 3.6 assume that

- (v) for any two elements  $(x, y, z), (u, v, w) \in X^3$  there exists  $(x^*, y^*, z^*) \in X^3$  such that

$$(gx, gx^*), (gu, gx^*), (gy, gy^*), (gv, gy^*), (gz, gz^*), (gw, gz^*) \in E(G)$$

Then  $CTFix(F) \neq \emptyset$  if and only if  $(X^3)_g^F \neq \emptyset$ .

*Proof.* Theorem 3.6 implies that there exists a tripled coincidence point  $(x, y, z) \in X^3$ , that is  $gx = F(x, y, z)$ ,  $gy = F(y, z, x)$  and  $gz = F(z, x, y)$ . Suppose that there exists another tripled coincidence point  $(u, v, w) \in X \times X \times X$  and hence  $gu = F(u, v, w)$ ,  $gv = F(v, w, u)$  and  $gw = F(w, u, v)$ . We will prove that  $gx = gu$ ,  $gy = gv$  and  $gz = gw$ .

From condition (v), we get that there exists  $(x^*, y^*, z^*) \in X \times X \times X$  such that

$$(gx, gx^*), (gu, gx^*), (gy, gy^*), (gv, gy^*), (gz, gz^*), (gw, gz^*) \in E(G).$$

Put  $x_0^* = x^*$ ,  $y_0^* = y^*$ ,  $z_0^* = z^*$  and, analogously to the proof of Theorem 3.6, choose sequences  $\{x_n^*\}$ ,  $\{y_n^*\}$  and  $\{z_n^*\}$  in  $X$  satisfying

$$\begin{aligned} gx_n^* &= F(x_{n-1}^*, y_{n-1}^*, z_{n-1}^*), & gy_n^* &= F(y_{n-1}^*, z_{n-1}^*, x_{n-1}^*) \\ \text{and } gz_n^* &= F(z_{n-1}^*, x_{n-1}^*, y_{n-1}^*) & \text{for } n \in \mathbb{N}. \end{aligned}$$

Starting from  $x_0 = x$ ,  $y_0 = y$ ,  $z_0 = z$  and  $u_0 = u$ ,  $v_0 = v$ ,  $w_0 = w$ , choose sequences  $\{x_n\}$ ,  $\{y_n\}$ ,  $\{z_n\}$  and  $\{u_n\}$ ,  $\{v_n\}$ ,  $\{w_n\}$ , satisfying  $gx_n = F(x_{n-1}, y_{n-1}, z_{n-1})$ ,  $gy_n = F(y_{n-1}, z_{n-1}, x_{n-1})$ ,  $gz_n = F(z_{n-1}, x_{n-1}, y_{n-1})$  and  $gu_n = F(u_{n-1}, v_{n-1}, w_{n-1})$ ,  $gv_n = F(v_{n-1}, w_{n-1}, u_{n-1})$ ,  $gw_n = F(w_{n-1}, u_{n-1}, v_{n-1})$  for  $n \in \mathbb{N}$ . Taking into account the properties of coincidence points, it is easy to see that it can be done so that  $x_n = x$ ,  $y_n = y$ ,  $z_n = z$  and  $u_n = u$ ,  $v_n = v$ ,  $w_n = w$ , i.e.,

$$\begin{aligned} gx_n &= F(x, y, z), & gy_n &= F(y, z, x), & gz_n &= F(z, x, y) \\ \text{and } gu_n &= F(u, v, w), & gv_n &= F(v, w, u), & gw_n &= F(w, u, v) \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

Since  $(x, y, z)$  and  $(x_0^*, y_0^*, z_0^*) = (x^*, y^*, z^*) \in X \times X \times X$ , then,  $(gx, gx_0^*)$ ,  $(gy, gy_0^*)$  and  $(gz, gz_0^*) \in E(G)$ . By  $F$  and  $g$  is  $g$ -edge preserving, we have  $(F(x, y, z), F(x_0^*, y_0^*, z_0^*)) = (gx, gx_1^*)$ ,  $(F(y, z, x), F(y_0^*, z_0^*, x_0^*)) = (gy, gy_1^*)$  and  $(F(z, x, y), F(z_0^*, x_0^*, y_0^*)) = (gz, gz_1^*) \in E(G)$  and in a similar way,  $(gx, gx_n^*)$ ,  $(gy, gy_n^*)$  and  $(gz, gz_n^*) \in E(G)$ . Thus, we can apply the contractive condition (3.1) to obtain

$$\begin{aligned} \psi(d(gx, gx_{n+1}^*)) &= \psi(d(F(x, y, z), F(x_n^*, y_n^*, z_n^*))) \\ &\leq \theta(d(gx, gx_n^*), d(gy, gy_n^*), d(gz, gz_n^*))\psi(M(gx, gx_n^*, gy, gy_n^*, gz, gz_n^*)), \end{aligned}$$

$$\begin{aligned} \psi(d(gy, gy_{n+1}^*)) &= \psi(d(F(y, z, x), F(y_n^*, z_n^*, x_n^*))) \\ &\leq \theta(d(gy, gy_n^*), d(gz, gz_n^*), d(gx, gx_n^*))\psi(M(gx, gx_n^*, gy, gy_n^*, gz, gz_n^*)) \\ &= \theta(d(gx, gx_n^*), d(gy, gy_n^*), d(gz, gz_n^*))\psi(M(gx, gx_n^*, gy, gy_n^*, gz, gz_n^*)), \end{aligned}$$

and

$$\begin{aligned} \psi(d(gz, gz_{n+1}^*)) &= \psi(d(F(z, x, y), F(z_n^*, x_n^*, y_n^*))) \\ &\leq \theta(d(gz, gz_n^*), d(gx, gx_n^*), d(gy, gy_n^*))\psi(M(gx, gx_n^*, gy, gy_n^*, gz, gz_n^*)) \\ &= \theta(d(gx, gx_n^*), d(gy, gy_n^*), d(gz, gz_n^*))\psi(M(gx, gx_n^*, gy, gy_n^*, gz, gz_n^*)). \end{aligned}$$

This implies that

$$\begin{aligned} \psi(M(gx, gx_{n+1}^*, gy, gy_{n+1}^*, gz, gz_{n+1}^*)) & & (3.11) \\ &\leq \theta(d(gx, gx_n^*), d(gy, gy_n^*), d(gz, gz_n^*))\psi(M(gx, gx_n^*, gy, gy_n^*, gz, gz_n^*)) \\ &< \psi(M(gx, gx_n^*, gy, gy_n^*, gz, gz_n^*)). \end{aligned}$$

Therefore, we get

$$\psi(M(gx, gx_{n+1}^*, gy, gy_{n+1}^*, gz, gz_{n+1}^*)) < \psi(M(gx, gx_n^*, gy, gy_n^*, gz, gz_n^*)).$$

By properties of  $\psi$ , we get

$$M(gx, gx_{n+1}^*, gy, gy_{n+1}^*, gz, gz_{n+1}^*) < M(gx, gx_n^*, gy, gy_n^*, gz, gz_n^*).$$

Thus the sequence  $d_n := M(gx, gx_n^*, gy, gy_n^*, gz, gz_n^*)$  is decreasing and hence  $d_n \rightarrow d$  as  $n \rightarrow \infty$  for some  $d \geq 0$ . Now, we prove that  $d = 0$ . Assume to the contrary that  $d > 0$ ; then from (3.11), we have

$$\frac{\psi(M(gx, gx_{n+1}^*, gy, gy_{n+1}^*), gz, gz_{n+1}^*)}{\psi(M(gx, gx_n^*, gy, gy_n^*), gz, gz_n^*)} \leq \theta(d(gx, gx_n^*), d(gy, gy_n^*), d(gz, gz_n^*)) < 1.$$

Taking the limit as  $n \rightarrow \infty$  in the above inequality, we have

$$\theta(d(gx, gx_n^*), d(gy, gy_n^*), d(gz, gz_n^*)) \rightarrow 1 \text{ as } n \rightarrow \infty.$$

By the property  $(\theta_2)$  of  $\theta \in \Theta$ , we get

$$d(gx, gx_n^*) \rightarrow 0, d(gy, gy_n^*) \rightarrow 0 \quad \text{and} \quad d(gz, gz_n^*) \rightarrow 0$$

as  $n \rightarrow \infty$ . Now we have

$$\lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} M(gx, gx_n^*, gy, gy_n^*, gz, gz_n^*) = 0$$

which contradicts with  $d > 0$ . Therefore, we conclude that

$$\lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} M(gx, gx_n^*, gy, gy_n^*, gz, gz_n^*) = 0,$$

we get

$$\lim_{n \rightarrow \infty} d(gx, gx_n^*) = 0, \lim_{n \rightarrow \infty} d(gy, gy_n^*) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} d(gz, gz_n^*) = 0.$$

In a similar way, we have

$$\lim_{n \rightarrow \infty} d(gu, gx_n^*) = 0, \lim_{n \rightarrow \infty} d(gv, gy_n^*) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} d(gw, gz_n^*) = 0.$$

By the triangle inequality, we have

$$\begin{aligned} d(gx, gu) &\leq d(gx, gx_n^*) + d(gx_n^*, gu), \\ d(gy, gv) &\leq d(gy, gy_n^*) + d(gy_n^*, gv), \\ \text{and } d(gz, gw) &\leq d(gz, gz_n^*) + d(gz_n^*, gw) \end{aligned}$$

for all  $n \in \mathbb{N}$ . Taking  $n \rightarrow \infty$  in the above three inequalities, we get that  $d(gx, gu) = 0$ ,  $d(gy, gv) = 0$  and  $d(gz, gw) = 0$ . Therefore, we have  $gx = gu$ ,  $gy = gv$  and  $gz = gw$ .

Now we let  $p := gx$ ,  $q := gy$  and  $r := gz$ . Hence we have

$$gp = g(gx) = gF(x, y, z), gq = g(gy) = gF(y, z, x) \quad \text{and} \quad gr = g(gz) = gF(z, x, y).$$

By the definition of sequences  $\{x_n\}$ ,  $\{y_n\}$  and  $\{z_n\}$  we have

$$\begin{aligned} gx_n &= F(x, y, z) = F(x_{n-1}, y_{n-1}, z_{n-1}), \\ gy_n &= F(y, z, x) = F(y_{n-1}, z_{n-1}, x_{n-1}), \\ \text{and } gz_n &= F(z, x, y) = F(z_{n-1}, x_{n-1}, y_{n-1}), \end{aligned}$$

for all  $n \in \mathbb{N}$ . So we have

$$\begin{aligned} \lim_{n \rightarrow \infty} F(x_n, y_n, z_n) &= \lim_{n \rightarrow \infty} gx_n = F(x, y, z), \\ \lim_{n \rightarrow \infty} F(y_n, z_n, x_n) &= \lim_{n \rightarrow \infty} gy_n = F(y, z, x), \\ \text{and } \lim_{n \rightarrow \infty} F(z_n, x_n, y_n) &= \lim_{n \rightarrow \infty} gz_n = F(z, x, y). \end{aligned}$$

Since  $g$  and  $F$  are compatible, we have

$$\lim_{n \rightarrow \infty} d(gF(x_n, y_n, z_n), F(gx_n, gy_n, gz_n)) = 0,$$

that is  $gF(x, y, z) = F(gx, gy, gz)$ . Therefore, we get

$$gp = gF(x, y, z) = F(gx, gy, gz) = F(p, q, r),$$

in a similar way,

$$gq = gF(y, z, x) = F(gy, gz, gx) = F(q, r, p),$$

and

$$gr = gF(z, x, y) = F(gz, gx, gy) = F(r, p, q).$$

This implies that  $(p, q, r)$  is also a tripled coincidence point. By the property we have just proved, it follows that  $gp = gx = p$ ,  $gq = gy = q$  and  $gr = gz = r$ . So,

$$p = gp = F(p, q, r), q = gq = F(q, r, p) \quad \text{and} \quad r = gr = F(r, p, q),$$

and  $(p, q, r)$  is a common coupled fixed point of  $g$  and  $F$ . We can easily show that it is unique.  $\square$

## 4 Applications

In this section, we apply our theorem to the existence theorem for a solution of the following integral system:

$$\begin{aligned} x(t) &= \int_0^T f(t, s, x(s), y(s), z(s))ds + h(t); \\ y(t) &= \int_0^T f(t, s, y(s), z(s), x(s))ds + h(t); \\ z(t) &= \int_0^T f(t, s, z(s), x(s), y(s))ds + h(t), \end{aligned} \tag{4.1}$$

where  $t \in [0, T]$  and  $T$  is a positive number.

Consider  $X := C([0, T], \mathbb{R}^n)$  with the usual supremum norm,  $\|x\| = \max_{t \in [0, T]} |x(t)|$ , for  $x \in C([0, T], \mathbb{R}^n)$ .

Consider also the graph  $G$  defined by the partial order relation, i.e.

$$x, y \in X, \quad x \leq y \Leftrightarrow x(t) \leq y(t), \quad \text{for } t \in [0, T].$$

Let  $d$  be the metric induced by the norm. It follows that  $(X, d)$  is a complete metric space endowed with a directed graph  $G$ .

If we consider  $E(G) = \{(x, y) \in X \times X : x \leq y\}$ , then  $E(G)$  satisfies the transitive property and the diagonal  $\Delta$  of  $X \times X$  is included in  $E(G)$ . Moreover,  $(X, d, G)$  has the property A.

In this case

$$(X^3)_g^F = \{(x, y, z) \in X^3 : (gx, F(x, y, z)), (gy, F(y, z, x)), (gz, F(z, x, y)) \in E(G)\}.$$

For  $x = (x_1, x_2, x_3, \dots, x_n)$  and  $y = (y_1, y_2, y_3, \dots, y_n) \in \mathbb{R}^n$ ,

$$x \leq y \iff x_i \leq y_i, \quad \text{for all } i = 1, 2, \dots, n.$$

**Theorem 4.1.** *Consider the system (4.1). Suppose*

- (i)  $f : [0, T] \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $h : [0, T] \rightarrow \mathbb{R}^n$  are continuous;
- (ii) for  $x, y, z, u, v, w \in \mathbb{R}^n$  with  $x \leq u, y \leq v, z \leq w$ ,  $f(t, s, x, y, z) \leq f(t, s, u, v, w)$  for all  $t, s \in [0, T]$ ;
- (iii) there exists  $0 \leq k < 1$  and  $T > 0$  such that

$$|f(t, s, x, y, z) - f(t, s, u, v, w)| \leq \frac{k}{3T}(|x - u| + |y - v| + |z - w|),$$

for each  $t, s \in [0, T], x, y, z, u, v, w \in \mathbb{R}^n$  and  $x \leq u, y \leq v, z \leq w$ ;

- (iv) there exists  $(x_0, y_0, z_0) \in X^3$  such that

$$\begin{aligned} x_0(t) &\leq \int_0^T f(t, s, x_0(s), y_0(s), z_0(s)) ds + h(t); \\ y_0(t) &\leq \int_0^T f(t, s, y_0(s), z_0(s), x_0(s)) ds + h(t); \\ z_0(t) &\leq \int_0^T f(t, s, z_0(s), x_0(s), y_0(s)) ds + h(t), \end{aligned}$$

where  $t \in [0, T]$ .

Then there exists at least one solution of the integral system (4.1).

*Proof.* Let  $F : X^3 \rightarrow X$ ,  $(x, y, z) \mapsto F(x, y, z)$ , where

$$F(x, y, z)(t) = \int_0^T f(t, s, x(s), y(s), z(s))ds + h(t), \quad t \in [0, T],$$

and let  $g : X \rightarrow X$  be defined by  $gx(t) = x(t)$ . Then, the system (4.1) can be written as

$$x = F(x, y, z), \quad y = F(y, z, x), \quad z = F(z, x, y).$$

Let  $x, y, z, u, v, w \in X$  such that  $gx \leq gu$ ,  $gy \leq gv$  and  $gz \leq gw$ . We have that for  $x \leq u$ ,  $y \leq v$  and  $z \leq w$ ,

$$\begin{aligned} F(x, y, z)(t) &= \int_0^T f(t, s, x(s), y(s), z(s))ds + h(t) \\ &\leq \int_0^T f(t, s, u(s), v(s), w(s))ds + h(t) = F(u, v, w)(t), \\ &\text{for each } t \in [0, T], \end{aligned}$$

$$\begin{aligned} F(y, z, x)(t) &= \int_0^T f(t, s, y(s), z(s), x(s))ds + h(t) \\ &\leq \int_0^T f(t, s, v(s), w(s), u(s))ds + h(t) = F(v, w, u)(t), \\ &\text{for each } t \in [0, T] \end{aligned}$$

and

$$\begin{aligned} F(z, x, y)(t) &= \int_0^T f(t, s, z(s), x(s), y(s))ds + h(t) \\ &\leq \int_0^T f(t, s, w(s), u(s), v(s))ds + h(t) = F(w, u, v)(t), \\ &\text{for each } t \in [0, T]. \end{aligned}$$

Thus, if  $gx \leq gu$ ,  $gy \leq gv$  and  $gz \leq gw$ , then  $F(x, y, z) \leq F(u, v, w)$ ,  $F(y, z, x) \leq F(v, w, u)$  and  $F(z, x, y) \leq F(w, u, v)$ . According to the definition of  $E(G)$ , we obtain that the pair of  $F$  and  $g$  is  $g$ -edge preserving.

On the other hand,

$$\begin{aligned}
& |F(x, y, z)(t) - F(u, v, w)(t)| \\
& \leq \int_0^T |f(t, s, x(s), y(s), z(s)) - f(t, s, u(s), v(s), w(s))| ds \\
& = \int_0^T |f(t, s, x(s), y(s), z(s)) - f(t, s, u(s), v(s), w(s))| ds \\
& \leq \frac{k}{3T} \int_0^T (|x(s) - u(s)| + |y(s) - v(s)| + |z(s) - w(s)|) ds \\
& \leq k \left( \frac{\|gx - gu\| + \|gy - gv\| + \|gz - gw\|}{3} \right) \\
& \leq kM(gx, gu, gy, gv, gz, gw), \quad \text{for } t \in [0, T].
\end{aligned}$$

Therefore, there exists  $\psi(t) = t$  and  $\theta \in \Theta$ , where  $\theta(s, t, r) = k$  for  $s, t, r \in [0, \infty)$  and  $k \in [0, 1)$  such that

$$\begin{aligned}
\psi(\|F(x, y, z) - F(u, v, w)\|) \\
\leq \theta(\|gx - gu\|, \|gy - gv\|, \|gz - gw\|) \psi(M(gx, gu, gy, gv, gz, gw))
\end{aligned}$$

where  $M(gx, gu, gy, gv, gz, gw) = \max\{\|gx - gu\|, \|gy - gv\|, \|gz - gw\|\}$ . As a result, the pair of  $F$  and  $g$  is a  $\theta$ - $\psi$ -contraction.

Finally, from condition (iv),

$$(X^3)^F = \{(x, y, z) \in X^3 : (gx, F(x, y, z)), (gy, F(y, z, x)), (gz, F(z, x, y))\} \neq \emptyset.$$

Thus, there exists  $(x^*, y^*, z^*) \in X^3$  is a tripled common fixed point of the mappings  $F$  and  $g$ , which is the solution for the integral (4.1).  $\square$

**Theorem 4.2.** Consider the system (4.1). Suppose

- (i)  $f : [0, T] \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $h : [0, T] \rightarrow \mathbb{R}^n$  are continuous;
- (ii) for all  $x, y, z, u, v, w \in \mathbb{R}^n$  with  $x \leq u, y \leq v, z \leq w$  we have  $f(t, s, x, y, z) \leq f(t, s, u, v, w)$  for all  $t, s \in [0, T]$ ;
- (iii)

$$\begin{aligned}
& |f(t, s, x, y, z) - f(t, s, u, v, w)| \\
& \leq \frac{1}{T} \ln(1 + \max\{|x - u|, |y - v|, |z - w|\}),
\end{aligned}$$

for each  $t, s \in [0, T], x, y, z, u, v, w \in \mathbb{R}^n, x \leq u, y \leq v, z \leq w$ ;



(iv) there exists  $(x_0, y_0, z_0) \in X \times X \times X$  such that

$$\begin{aligned} x_0(t) &\leq \int_0^T f(t, s, x_0(s), y_0(s), z_0(s))ds + h(t); \\ y_0(t) &\leq \int_0^T f(t, s, y_0(s), z_0(s), x_0(s))ds + h(t); \\ z_0(t) &\leq \int_0^T f(t, s, z_0(s), x_0(s), y_0(s))ds + h(t), \end{aligned}$$

where  $t \in [0, T]$ .

Then there exists at least one solution of the integral system (4.1).

*Proof.* Let  $F : X \times X \times X \rightarrow X$ ,  $(x, y, z) \mapsto F(x, y, z)$ , where

$$F(x, y, z)(t) = \int_0^T f(t, s, x(s), y(s), z(s))ds + h(t), \quad t \in [0, T],$$

and  $g : X \rightarrow X$  by  $gx(t) = x(t)$ . As in Theorem 4.1, we have  $F$  and  $g$  are  $g$ -edge preserving.

On the other hand,

$$\begin{aligned} &|F(x, y, z)(t) - F(u, v, w)(t)| \\ &\leq \int_0^T |f(t, s, x(s), y(s), z(s)) - f(t, s, u(s), v(s), w(s))|ds \\ &= \int_0^T |f(t, s, x(s), y(s), z(s)) - f(t, s, u(s), v(s), w(s))|ds \\ &\leq \frac{1}{T} \int_0^T \ln(1 + \max\{|x(s) - u(s)|, |y(s) - v(s)|, |z(s) - w(s)|\})ds \\ &\leq \ln(1 + \max\{\|x - u\|, \|y - v\|, \|z - w\|\}) \quad \text{for each } t \in [0, T] \\ &= \ln(1 + M(gx, gu, gy, gv, gz, gw)) \end{aligned}$$

where  $M(gx, gu, gy, gv, gz, gw) = \max\{\|gx - gu\|, \|gy - gv\|, \|gz - gw\|\}$ .

Consequently,

$$\begin{aligned} &\ln(|F(x, y, z)(t) - F(u, v, w)(t)| + 1) \\ &\leq \ln(\ln(1 + M(gx, gu, gy, gv, gz, gw)) + 1) \\ &= \frac{\ln(\ln(1 + M(gx, gu, gy, gv, gz, gw)) + 1)}{\ln(1 + M(gx, gu, gy, gv, gz, gw))} \ln(1 + M(gx, gu, gy, gv, gz, gw)). \end{aligned}$$

Hence, there exists  $\psi(x) = \ln(x + 1)$  and  $\theta \in \Theta$  where

$$\theta(s, t, r) = \begin{cases} \frac{\ln(\ln(1 + \max\{s, t, r\}))}{\ln(1 + \max\{s, t, r\})}, & s > 0 \text{ or } t > 0, \\ r \in [0, 1), & s = 0, t = 0, r = 0 \end{cases}$$

such that

$$\begin{aligned} \psi(d(F(x, y, z), F(u, v, w))) &= \psi(\|F(x, y, z) - F(u, v, w)\|) \\ &\leq \theta(d(gx, gu), d(gy, gv), d(gz, gw))\psi(M(gx, gu, gy, gv, gz, gw)). \end{aligned}$$

We have that the pair of  $F$  and  $g$  is a  $\theta$ - $\psi$ -contraction.

Condition (iv) shows that

$$(X^3)_g^F = \{(x, y, z) \in X^3 : (gx, F(x, y, z)), (gy, F(y, z, x)), (gz, F(z, x, y))\} \neq \emptyset.$$

Thus, there exists  $(x^*, y^*, z^*) \in X^3$  is a tripled common fixed point of  $F$  and  $g$ , which is a solution for the integral (4.1).  $\square$

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## References

- [1] D. Guo, V. Lakshmikantham, Coupled fixed points of nonlinear operators with applications, *Nonlinear Anal., Theory Methods Appl.* 11 (1987) 623-632.
- [2] T. Gnana Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Anal. TMA* 65 (2006) 1379-1393.
- [3] V. Lakshmikantham, L. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, *Nonlinear Anal. TMA* 70 (2009) 4341-4349.
- [4] B.S. Choudhury, A. Kundu, A coupled coincidence point result in partially ordered metric spaces for compatible mappings, *Nonlinear Anal. TMA* 73 (2010) 2524-2531.
- [5] V. Berinde, Coupled fixed point theorems for  $\varphi$ -contractive mixed monotone mappings in partially ordered metric spaces, *Nonlinear Analysis, Theory, Methods and Applications* 75 (6) 3218-3228.
- [6] V. Berinde, Coupled coincidence point theorems for mixed monotone nonlinear operators, *Computers Mathematics with Applications* 64 (6) (2012) 1770-1777.
- [7] W. Sintunavarat, P. Kumam, YJ. Cho, Coupled fixed point theorems for nonlinear contractions without mixed monotone property, *Fixed Point Theory Appl.* (2012) <https://doi.org/10.1186/1687-1812-2012-170>.
- [8] W. Sintunavarat, S. Radenović, Z. Golubović, P. Kuman, Coupled fixed point theorems for  $F$ -invariant set, *Appl. Math. Inf. Sci.* 7 (1) (2013) 247-255.

- [9] E. Karapinar, N.V. Luong, N.X. Thuan, T.T. Hai, Coupled coincidence points for mixed monotone operators in partially ordered metric spaces, *Arabian Journal of Mathematics* 1 (2012) 329-339.
- [10] M. Borcut, Tripled fixed point theorems for monotone mappings in partially ordered metric spaces, *Carpathian J. Math.* 28 (2) (2012) 207-214.
- [11] M. Borcut, Tripled coincidence theorems for monotone mappings in partially ordered metric spaces, *Creat. Math. Inform.* 21 (2) (2012) 135-142.
- [12] J. Jachymski, The contraction principle for mappings on a metric space with a graph, *Proc. Am. Math. Soc.* 136 (2008) 1359-1373.
- [13] C. Chifu, G. Petrusel, New results on coupled fixed point theory in metric spaces endowed with a directed graph, *Fixed Point Theory and Applications* (2014) <https://doi.org/10.1186/1687-1812-2014-151>.
- [14] Z. Kadelburg, P. Kumam, S. Radenović, W. Sintunavarat, Common coupled fixed point theorems for Geraghty-type contraction mappings using monotone property, *Fixed Point Theory and Applications* (2015) <https://doi.org/10.1186/s13663-015-0278-5>.
- [15] B. Samet, C. Vetro, Coupled fixed point  $F$ -invariant set and fixed point of  $N$ -order, *Ann. Funct. Anal.* 1 (2010) 46-56.
- [16] V. Berinde, M. Borcut, Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, *Nonlinear Anal.* 74 (2011) 4889-4897.
- [17] V. Berinde, M. Borcut, Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces, *Applied Mathematics and Computation* 218 (10) (2012) 5929-5936.
- [18] H. Aydi, M. Abbas, W. Sintunavarat, P. Kumam, Tripled fixed point of  $W$ -compatible mappings in abstract metric spaces, *Fixed Point Theory and Applications* (2012) <https://doi.org/10.1186/1687-1812-2012-134>.
- [19] M. Borcut, Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces, *Applied Mathematical and Computation* 218 (2012) 7339-7346.
- [20] H. Aydi, E. Karapinar, W. Shatanawi, Tripled coincidence point results for generalized contractions in ordered generalized metric spaces, *Fixed Point Theory and Applications* (2012) <https://doi.org/10.1186/1687-1812-2012-101>.
- [21] S.A. Mohiuddine, A. Alotaibi, Some results on a tripled fixed point for nonlinear contractions in partially ordered  $G$ -metric spaces, *Fixed Point Theory Appl.* (2012) <https://doi.org/10.1186/1687-1812-2012-179>.
- [22] H. Aydi, E. Karapinar, Tripled fixed points in ordered metric spaces, *Bulletin of Mathematical Analysis and Applications* 4 (1) (2012) 197-207.

- [23] M. Abbas, H. Aydi, E. Karapinar, Tripled fixed points of multi-valued non-linear contraction mappings in partially ordered metric spaces, *Abstr. Appl. Anal.*, Volume 2011 (2011) Article ID 812690.
- [24] M. Abbas, B. Ali, W. Sintunavarat, P. Kumam, Tripled fixed point and tripled coincidence point theorems in intuitionistic fuzzy normed spaces, *Fixed Point Theory and Applications* (2012) <https://doi.org/10.1186/1687-1812-2012-187>.
- [25] A. Amini-Harandi, Coupled and tripled fixed point theory in partially ordered metric spaces with application to initial value problem, *Mathematical and Computer Modelling* 57 (2013) 2343-2348.
- [26] H. Aydi, E. Karapinar, M. Postolache, Tripled coincidence points for weak  $\varphi$ -contractions in partially ordered metric spaces, *Fixed point Theory and Applications* (2012) <https://doi.org/10.1186/1687-1812-2012-44>.
- [27] H. Aydi, E. Karapinar, C. Vetro, Meir-Keeler type contractions for tripled fixed points, *Acta Mathematica Scientia* 32 (6) (2012) 2119-2130.
- [28] H. Aydi, E. Karapinar, New Meir-Keeler type tripled fixed point theorems on ordered partial metric spaces, *Math. Probl. Eng.* 2012 (2012) Article ID 409872.
- [29] H. Aydi, M. Abbas, Tripled coincidence and fixed point results in partial metric spaces, *Applied General Topology* 13 (2) (2012) 193-206.
- [30] Y.J. Cho, A. Gupta, E. Karapinar, P. Kumam, W. Sintunavarat, Tripled best proximity point theorem in metric space, *Mathematical Inequalities and Applications* 16 (4) (2013) 1197-1216.
- [31] P. Kumam, J.M. Moreno, A.F.L. Hierro, C.R.L. Hierro, Berinde-Borcut, Tripled fixed point theorem in partially ordered (intuitionistic) fuzzy normed spaces, *Journal of Inequalities and Application* (2014) <https://doi.org/10.1186/1029-242X-2014-47>.
- [32] N. Wairojjana, W. Sintunavarat, P. Kumam, Common tripled fixed points for  $W$ -compatible mappings along with  $CLR_g$  property in abstract metric spaces, *Journal of Inequalities and Applications* (2014) <https://doi.org/10.1186/1029-242X-2014-133>.

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