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Falling Shadow Theory Applied to UP-Algebras

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Abstract : Based on the theory of fuzzy sets and falling shadows, the notion of a falling UP-subalgebra and a falling UP-ideal of a UP-algebra is introduced, and several properties are investigated. Relations between falling UP-subalgebras and falling UP-ideals are given. Relations between fuzzy UP-subalgebras (resp., fuzzy UP-ideals) and falling UP-subalgebras (resp., falling UP-ideals) are established.

Keywords : UP-subalgebra; UP-ideal; falling UP-subalgebra; falling UP-ideal.
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1 Introduction

In the study of a unified treatment of uncertainty modeled by means of combining probability and fuzzy set theory, Goodman [1] pointed out the equivalence of a fuzzy set and a class of random sets. Wang and Sanchez [2] introduced the theory of falling shadows which directly relates probability concepts with the membership

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function of fuzzy sets. Falling shadow representation theory shows us the way of selection relaid on the joint degrees distributions. It is reasonable and convenient approach for the theoretical development and the practical applications of fuzzy sets and fuzzy logics. The mathematical structure of the theory of falling shadows is formulated in [3]. Tan et al. [4, 5] established a theoretical approach to define a fuzzy inference relation and fuzzy set operations based on the theory of falling shadows. Jun and Park [6] discussed the notion of a falling fuzzy subalgebra/ideal of a BCK/BCI-algebra. Yuan and Lee [7] considered a fuzzy subgroup (subring, ideal) as the falling shadow of the cloud of the subgroup (subring, ideal).

In this paper, we establish a theoretical approach for defining a fuzzy UP-subalgebras and fuzzy UP-ideals in a UP-algebra based on the theory of falling shadows. We provide relations between falling UP-subalgebras and falling UP-ideals. We also consider relations between fuzzy UP-subalgebras (resp., fuzzy UP-ideals) and falling UP-subalgebras (resp., falling UP-ideals), and investigate several properties.

2 Preliminaries

Definition 2.1 ([8])**.** An algebra $X = (X, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* if it satisfies the following conditions.

$$(\forall x, y, z \in X)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0), \tag{2.1}$$
$$(\forall x \in X)(0 \cdot x = x), \tag{2.2}$$
$$(\forall x \in X)(x \cdot 0 = 0), \tag{2.3}$$
$$(\forall x, y \in X)(x \cdot y = 0 = y \cdot x \;\Rightarrow\; x = y). \tag{2.4}$$

Proposition 2.2 ([8, 9])**.** *In a UP-algebra X, the following assertions are valid.*

$$(\forall x \in X)(x \cdot x = 0), \tag{2.5}$$
$$(\forall x, y, z \in X)(x \cdot y = 0,\; y \cdot z = 0 \;\Rightarrow\; x \cdot z = 0), \tag{2.6}$$
$$(\forall x, y \in X)(x \cdot y = 0 \;\Rightarrow\; (z \cdot x) \cdot (z \cdot y) = 0), \tag{2.7}$$
$$(\forall x, y \in X)(x \cdot y = 0 \;\Rightarrow\; (y \cdot z) \cdot (x \cdot z) = 0), \tag{2.8}$$
$$(\forall x, y \in X)(x \cdot (y \cdot x) = 0), \tag{2.9}$$
$$(\forall x, y \in X)((y \cdot x) \cdot x = 0 \;\Leftrightarrow\; x = y \cdot x), \tag{2.10}$$
$$(\forall x, y \in X)(x \cdot (y \cdot y) = 0). \tag{2.11}$$
$$(\forall a, x, y, z \in X)((x \cdot (y \cdot z)) \cdot (x \cdot ((a \cdot y) \cdot (a \cdot z))) = 0), \tag{2.12}$$
$$(\forall a, x, y, z \in X)((((a \cdot x) \cdot (a \cdot y)) \cdot z) \cdot ((x \cdot y) \cdot z) = 0), \tag{2.13}$$
$$(\forall x, y, z \in X)(((x \cdot y) \cdot z) \cdot (y \cdot z) = 0), \tag{2.14}$$
$$(\forall x, y, z \in X)(x \cdot y = 0 \Rightarrow x \cdot (z \cdot y) = 0), \tag{2.15}$$
$$(\forall x, y, z \in X)(((x \cdot y) \cdot z) \cdot (x \cdot (y \cdot z)) = 0), \tag{2.16}$$
$$(\forall a, x, y, z \in X)(((x \cdot y) \cdot z) \cdot (y \cdot (a \cdot z)) = 0). \tag{2.17}$$

Definition 2.3 ([8]). A nonempty subset A of X is called a *UP-subalgebra* of X if it satisfies the following condition.

$$(\forall x, y \in X)(x \in A,\ y \in A\ \Rightarrow\ x \cdot y \in A). \tag{2.18}$$

Definition 2.4 ([8]). A subset A of X is called a *UP-ideal* of X if it satisfies the following conditions.

$$0 \in A, \tag{2.19}$$

$$(\forall x, y, z \in X)(x \cdot (y \cdot z) \in A,\ y \in A\ \Rightarrow\ x \cdot z \in A). \tag{2.20}$$

Definition 2.5 ([10]). A fuzzy set λ in a UP-algebra X is called a *fuzzy UP-subalgebra* of X if the following condition is valid.

$$(\forall x, y \in X)\left(\lambda(x \cdot y) \geq \min\{\lambda(x), \lambda(y)\}\right). \tag{2.21}$$

Definition 2.6 ([10]). A fuzzy set λ in a UP-algebra X is called a *fuzzy UP-ideal* of X if the following conditions are valid.

$$(\forall x \in X)\left(\lambda(0) \geq \lambda(x)\right), \tag{2.22}$$

$$(\forall x, y, z \in X)\left(\lambda(x \cdot z) \geq \min\{\lambda(x \cdot (y \cdot z)), \lambda(y)\}\right). \tag{2.23}$$

3 Falling UP-Subalgebras and Falling UP-Ideals

We first display the basic theory on falling shadows. We refer the reader to the papers [1–5] for further information about the theory of falling shadows.

Given a universe of discourse U, let $\mathscr{P}(U)$ denote the power set of U. For each $u \in U$, let

$$\ddot{u} := \{E \mid u \in E \text{ and } E \subseteq U\}, \tag{3.1}$$

and for each $E \in \mathscr{P}(U)$, let

$$\ddot{E} := \{\ddot{u} \mid u \in E\}. \tag{3.2}$$

An ordered pair $(\mathscr{P}(U), \mathscr{B})$ is said to be a *hyper-measurable structure* on U if $\mathscr{B}$ is a σ-field in $\mathscr{P}(U)$ and $\ddot{U} \subseteq \mathscr{B}$. Given a probability space $(\Omega, \mathscr{A}, P)$ and a hyper-measurable structure $(\mathscr{P}(U), \mathscr{B})$ on U, a *random set* on U is defined to be a mapping $\xi : \Omega \to \mathscr{P}(U)$ which is $\mathscr{A}$-$\mathscr{B}$ measurable, that is,

$$(\forall C \in \mathscr{B})\,(\xi^{-1}(C) = \{\omega \mid \omega \in \Omega \text{ and } \xi(\omega) \in C\} \in \mathscr{A}). \tag{3.3}$$

Suppose that ξ is a random set on U. Let

$$\tilde{\alpha}(u) := P(\omega \mid u \in \xi(\omega)) \text{ for each } u \in U.$$

Then $\tilde{\alpha}$ is a kind of fuzzy set in U. We call $\tilde{\alpha}$ a *falling shadow* of the random set ξ, and ξ is called a *cloud* of $\tilde{\alpha}$.

For example, $(\Omega, \mathscr{A}, P) = ([0,1], \mathscr{A}, m)$, where $\mathscr{A}$ is a Borel field on $[0,1]$ and m is the usual Lebesgue measure. Let $\tilde{\alpha}$ be a fuzzy set in U and $\tilde{\alpha}_t := \{u \in U \mid \tilde{\alpha}(u) \geq t\}$ be a t-cut of $\tilde{\alpha}$. Then

$$\xi : [0,1] \to \mathscr{P}(U),\ t \mapsto \tilde{\alpha}_t$$

is a random set and ξ is a cloud of $\tilde{\alpha}$. We shall call ξ defined above as the *cut-cloud* of $\tilde{\alpha}$ (see [1]).

In what follows let X denote a UP-algebra unless otherwise.

Definition 3.1. Let $(\Omega, \mathscr{A}, P)$ be a probability space, and let

$$\xi : \Omega \to \mathscr{P}(X)$$

be a random set. If $\xi(\omega)$ is a UP-ideal (resp., a UP-subalgebra) of X for any $\omega \in \Omega$, then the falling shadow $\tilde{\alpha}$ of the random set ξ, i.e.,

$$\tilde{\alpha}(x) = P(\omega \mid x \in \xi(\omega)) \tag{3.4}$$

is called a *falling UP-ideal* (resp., *falling UP-subalgebra*) of X.

Example 3.2. Consider a UP-algebra $X = \{0, a, b, c, d\}$ with the binary operation "·" which is given in the following table.

$\cdot$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	c	d
b	0	0	0	c	d
c	0	0	b	0	d
d	0	0	0	0	0

(1) Let $(\Omega, \mathscr{A}, P) = ([0,1], \mathscr{A}, m)$ and let

$$\xi : [0,1] \to \mathscr{P}(X),\ t \mapsto \begin{cases} \{0, a\} & \text{if } \ t \in [0, 0.4), \\ \{0, b, c, d\} & \text{if } \ t \in [0.4, 1]. \end{cases} \tag{3.5}$$

Then $\xi(t)$ is a UP-subalgebra of X for all $t \in [0,1]$. Hence $\tilde{\alpha}$ is a falling UP-subalgebra of X, and

$$\tilde{\alpha}(x) = \begin{cases} 0.4 & \text{if } \ x = a, \\ 0.6 & \text{if } \ x \in \{b, c, d\}, \\ 1 & \text{if } \ x = 0. \end{cases} \tag{3.6}$$

(2) Let $(\Omega, \mathscr{A}, P) = ([0,1], \mathscr{A}, m)$ and let

$$\zeta : [0,1] \to \mathscr{P}(X),\ t \mapsto \begin{cases} \{0, a, c\} & \text{if } \ t \in [0, 0.3), \\ \{0, a, b\} & \text{if } \ t \in [0.3, 0.8), \\ \{0, d\} & \text{if } \ t \in [0.8, 1]. \end{cases} \tag{3.7}$$

Then $\zeta(t)$ is a UP-subalgebra of X for all $t \in [0,1]$. Hence $\tilde{\alpha}$ is a falling UP-subalgebra of X, and

$$\tilde{\alpha}(x) = \begin{cases} 0.8 & \text{if } x = a, \\ 0.5 & \text{if } x = b, \\ 0.3 & \text{if } x = c, \\ 0.2 & \text{if } x = d, \\ 1 & \text{if } x = 0. \end{cases} \tag{3.8}$$

For any $t \in [0.8, 1]$, $\zeta(t) = \{0, d\}$ is not a UP-ideal of X since $a \cdot (d \cdot b) = a \cdot 0 = 0 \in \{0, d\}$, but $a \cdot b = b \notin \{0, d\}$. Hence $\tilde{\alpha}$ is not a falling UP-ideal of X.

Theorem 3.3. *Every falling UP-ideal is a falling UP-subalgebra.*

Proof. Let $\tilde{\alpha}$ be a falling UP-ideal of X. Then $\xi(\omega)$ is a UP-ideal of X. Let $x, y \in \xi(\omega)$. Then $x \cdot (y \cdot y) = 0 \in \xi(\omega)$ by (2.11) and (2.19). It follows from (2.20) that $x \cdot y \in \xi(\omega)$. Hence $\xi(\omega)$ is a UP-subalgebra of X, and therefore $\tilde{\alpha}$ is a falling UP-subalgebra of X. □

Example 3.2(2) shows that the converse of Theorem 3.3 is not true in general.

Let $(\Omega, \mathscr{A}, P)$ be a probability space and let

$$F(X) := \{f \mid f : \Omega \to X \text{ is a mapping}\}.$$

Define an operation $\odot$ on $F(X)$ by

$$(\forall \omega \in \Omega)\, ((f \odot g)(\omega) = f(\omega) \cdot g(\omega))$$

for all $f, g \in F(X)$. Let $\theta \in F(X)$ be defined by $\theta(\omega) = 0$ for all $\omega \in \Omega$. It can be easily checked that $(F(X); \odot, \theta)$ is a UP-algebra.

For any subset A of X and $f \in F(X)$, let

$$A_f := \{\omega \in \Omega \mid f(\omega) \in A\} \tag{3.9}$$

and

$$\xi : \Omega \to \mathscr{P}(F(X)),\ \omega \mapsto \{f \in F(X) \mid f(\omega) \in A\}. \tag{3.10}$$

Then $A_f \in \mathscr{A}$.

Theorem 3.4. *If A is a UP-subalgebra (resp., UP-ideal) of X, then*

$$\xi(\omega) = \{f \in F(X) \mid f(\omega) \in A\} \tag{3.11}$$

is a UP-subalgebra (resp., UP-ideal) of $F(X)$ for any $\omega \in \Omega$.

Proof. Let $\omega \in \Omega$. Assume that A is a UP-subalgebra of X. Let $f, g \in \xi(\omega)$. Then $f(\omega) \in A$ and $g(\omega) \in A$, and so

$$(f \odot g)(\omega) = f(\omega) \cdot g(\omega) \in A.$$

Hence $f \odot g \in \xi(\omega)$, and therefore $\xi(\omega)$ is a UP-subalgebra of X. Assume that A is a UP-ideal of X. Since $\theta(\omega) = 0 \in A$, we know that $\theta \in \xi(\omega)$. Let $f, g, h \in F(X)$ be such that $f \odot (g \odot h) \in \xi(\omega)$ and $g \in \xi(\omega)$. Then $g(\omega) \in A$ and

$$f(\omega) \cdot (g(\omega) \cdot h(\omega)) = (f \odot (g \odot h))(\omega) \in A.$$

It follows from (2.20) that $(f \odot h)(\omega) = f(\omega) \cdot h(\omega) \in A$ and so that $f \odot h \in \xi(\omega)$. Therefore $\xi(\omega)$ is a UP-ideal of $F(X)$. □

Since

$$\xi^{-1}(\ddot{f}) = \{\omega \in \Omega \mid f \in \xi(\omega)\} = \{\omega \in \Omega \mid f(\omega) \in A\} = A_f \in \mathscr{A}, \tag{3.12}$$

we can see that ξ is a random set on $F(X)$. Let

$$\tilde{\alpha}(f) = P(\omega \mid f(\omega) \in A). \tag{3.13}$$

Then $\tilde{\alpha}$ is a falling UP-subalgebra (resp., falling UP-ideal) of $F(X)$.

Theorem 3.5. *Every fuzzy UP-subalgebra (resp., fuzzy UP-ideal) of X is a falling UP-subalgebra (resp., falling UP-ideal) of X.*

Proof. Consider the probability space $(\Omega, \mathscr{A}, P) = ([0,1], \mathscr{A}, m)$ where $\mathscr{A}$ is a Borel field on $[0,1]$ and m is the usual Lebesgue measure. Let λ be a fuzzy UP-subalgebra (resp., fuzzy UP-ideal) of X. Then λ_t is a UP-subalgebra (resp., UP-ideal) of X for all $t \in [0,1]$. Let

$$\xi : [0,1] \to \mathscr{P}(X),\ t \mapsto \lambda_t$$

be a random set. Then λ is a falling UP-subalgebra (resp., falling UP-ideal) of X. □

Let $(\Omega, \mathscr{A}, P)$ be a probability space and let $\tilde{\alpha}$ be a falling shadow of a random set $\xi : \Omega \to \mathscr{P}(X)$. For any $x \in X$, let

$$\Omega_\xi(x) = \{\omega \in \Omega \mid x \in \xi(\omega)\}.$$

Then $\Omega_\xi(x) \in \mathscr{A}$.

Proposition 3.6. *Let $\tilde{\alpha}$ be a falling shadow of a random set $\xi : \Omega \to \mathscr{P}(X)$. If $\tilde{\alpha}$ is a falling UP-subalgebra of X, then*

$$(\forall x, y \in X)\left(\Omega_\xi(x) \cap \Omega_\xi(y) \subseteq \Omega_\xi(x \cdot y)\right). \tag{3.14}$$

Proof. Let $\omega \in \Omega_\xi(x) \cap \Omega_\xi(y)$. Then $x \in \xi(\omega)$ and $y \in \xi(\omega)$. Since $\xi(\omega)$ is a UP-subalgebra of X, it follows from (2.18) that $x \cdot y \in \xi(\omega)$ and so $\omega \in \Omega_\xi(x \cdot y)$. Therefore (3.14) is valid. □

Proposition 3.7. *Let $\tilde{\alpha}$ be a falling shadow of a random set $\xi : \Omega \to \mathscr{P}(X)$. If $\tilde{\alpha}$ is a falling UP-ideal of X, then*

$$(\forall x, y, z \in X)\,(\Omega_\xi(x \cdot (y \cdot z)) \cap \Omega_\xi(y) \subseteq \Omega_\xi(x \cdot z))\,. \tag{3.15}$$

Proof. If $\omega \in \Omega_\xi(x \cdot (y \cdot z)) \cap \Omega_\xi(y)$, then $x \cdot (y \cdot z) \in \xi(\omega)$ and $y \in \xi(\omega)$. Since $\xi(\omega)$ is a UP-ideal of X, we have $x \cdot z \in \xi(\omega)$. Hence $\omega \in \Omega_\xi(x \cdot z)$ which shows that (3.15) is valid. □

For any $s, t \in [0, 1]$, let $T_m(s, t) := \max\{0, s + t - 1\}$.

Theorem 3.8. *If $\tilde{\alpha}$ is a falling UP-subalgebra of X, then*

$$(\forall x, y \in X)\,(\tilde{\alpha}(x \cdot y) \geq T_m(\tilde{\alpha}(x), \tilde{\alpha}(y)))\,. \tag{3.16}$$

Proof. If $\tilde{\alpha}$ is a falling UP-subalgebra of X, then $\xi(\omega)$ is a UP-subalgebra of X for any $\omega \in \Omega$. Hence

$$\{\omega \in \Omega \mid x \in \xi(\omega)\} \cap \{\omega \in \Omega \mid y \in \xi(\omega)\} \subseteq \{\omega \in \Omega \mid x \cdot y \in \xi(\omega)\},$$

and so

$$\begin{aligned}
\tilde{\alpha}(x \cdot y) &= P(\omega \mid x \cdot y \in \xi(\omega)) \\
&\geq P(\{\omega \mid x \in \xi(\omega)\} \cap \{\omega \mid y \in \xi(\omega)\}) \\
&\geq P(\omega \mid x \in \xi(\omega)) + P(\omega \mid y \in \xi(\omega)) - P(\omega \mid x \in \xi(\omega) \text{ or } y \in \xi(\omega)) \\
&\geq \tilde{\alpha}(x) + \tilde{\alpha}(y) - 1.
\end{aligned}$$

Hence

$$\tilde{\alpha}(x \cdot y) \geq \max\{0, \tilde{\alpha}(x) + \tilde{\alpha}(y) - 1\} = T_m(\tilde{\alpha}(x), \tilde{\alpha}(y)),$$

and the proof is completed. □

Theorem 3.9. *Every falling UP-ideal $\tilde{\alpha}$ of X satisfies the following condition.*

$$(\forall x, y, z \in X)\,(\tilde{\alpha}(x \cdot z) \geq T_m(\tilde{\alpha}(x \cdot (y \cdot z)), \tilde{\alpha}(y)))\,. \tag{3.17}$$

Proof. Let $\tilde{\alpha}$ be a falling UP-ideal of X. Then $\xi(\omega)$ is a UP-ideal of X for any $\omega \in \Omega$, and thus

$$\{\omega \in \Omega \mid x \cdot (y \cdot z) \in \xi(\omega)\} \cap \{\omega \in \Omega \mid y \in \xi(\omega)\} \subseteq \{\omega \in \Omega \mid x \cdot z \in \xi(\omega)\}.$$

It follows that

$$\begin{aligned}
\tilde{\alpha}(x \cdot z) &= P(\omega \mid x \cdot z \in \xi(\omega)) \\
&\geq P(\{\omega \mid x \cdot (y \cdot z) \in \xi(\omega)\} \cap \{\omega \mid y \in \xi(\omega)\}) \\
&\geq P(\omega \mid x \cdot (y \cdot z) \in \xi(\omega)) + P(\omega \mid y \in \xi(\omega)) \\
&\quad - P(\omega \mid x \cdot (y \cdot z) \in \xi(\omega) \text{ or } y \in \xi(\omega)) \\
&\geq \tilde{\alpha}(x \cdot (y \cdot z)) + \tilde{\alpha}(y) - 1.
\end{aligned}$$

Therefore

$$\tilde{\alpha}(x \cdot z) \geq \max\{0, \tilde{\alpha}(x \cdot (y \cdot z)) + \tilde{\alpha}(y) - 1\}$$
$$= T_m(\tilde{\alpha}(x \cdot (y \cdot z)), \tilde{\alpha}(y)).$$

This completes the proof. □

4 Conclusion

We have established some connections between fuzzy mathematics and probability theory via the fuzzy UP-ideals and the fuzzy UP-ideals of UP-algebras. As an algebraic approach of the theory of falling shadows, we have introduced the notion of falling UP-subalgebras and falling UP-ideal in UP-algebras. We have discussed relations between falling UP-subalgebras and falling UP-ideals. We have also provided relations between fuzzy UP-subalgebras (resp., fuzzy UP-ideals) and falling UP-subalgebras (resp., falling UP-ideals). On the basis of these results, we will apply the theory of falling shadows to the another type of ideals, filters and deductive systems in BCK/BCI-algebras, KU-algebras and SU-algebras, etc., in future study.

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