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Buchdahl-Like Transformations in General Relativity

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Abstract : Based on earlier work, we develop "algorithmic" techniques that permit one (in a purely mechanical way) to generate large classes of general relativistic static perfect fluid spheres. Working in Schwarzschild curvature coordinates, we used these algorithmic ideas to prove several "solution-generating theorems" of varying levels of complexity. In addition, we now consider the situation in other coordinate systems: In particular, in isotropic coordinates we shall encounter a variant of the so-called "Buchdahl transformation", while in other coordinate systems we shall find a number of more complex "Buchdahl-like transformations" and "solution-generating theorems" that may be used to investigate and classify the general relativistic static perfect fluid sphere.
Keywords : Fluid spheres; Buchdahl-like transformations

1 Introduction

Understanding perfect fluid spheres in general relativity is an important topic, recently deemed worthy of a full chapter in an updated edition of the premier book summarizing and surveying "exact solutions" in general relativity [1]. Physically, perfect fluid spheres are a first approximation to realistic models for a general relativistic star. The study of perfect fluid spheres is a long-standing topic with a venerable history [1, 2, 3, 4, 5, 6, 7], and continuing interest [8, 9, 10, 11]. In particular, as derived in references [12] and [13], and as further described in references [14, 15, 16], we have developed several "algorithmic" techniques that permit one to generate large classes of perfect fluid spheres from first principles in a purely mechanical way. We now generalize these algorithmic ideas, originally derived for Schwarzschild curvature coordinates, (and to a lesser extent isotoropic coordinates), to a number of other coordinate systems [17]. This sometimes leads to much simpler results, and sometimes more general results. In this current article we shall:

1. Report a number of "transformation theorems" and "solution gener-

ating theorems" that allow us to map perfect fluid spheres into perfect fluid spheres.

2. Seek to understand how these various theorems and various coordinate systems inter-relate to one another.

2 Strategy

We start the discussion by considering static spherically symmetric distributions of matter, which implies (purely by symmetry), that in orthonormal components the stress energy tensor takes the specific form

$$T_{\hat{a}\hat{b}} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p_r & 0 & 0 \\ 0 & 0 & p_t & 0 \\ 0 & 0 & 0 & p_t \end{bmatrix}. \tag{2.1}$$

If the matter is a perfect fluid, then in addition we must have

$$p_r = p_t. \tag{2.2}$$

Invoking the Einstein equations, this then implies a differential relation on the spacetime geometry arising from equating the appropriate orthonormal components of the Einstein tensor

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{r}\hat{r}} = G_{\hat{\phi}\hat{\phi}}. \tag{2.3}$$

Equivalently, one could work with the appropriate orthonormal components of the Ricci tensor

$$R_{\hat{\theta}\hat{\theta}} = R_{\hat{r}\hat{r}} = R_{\hat{\phi}\hat{\phi}}. \tag{2.4}$$

In terms of the metric components, this now leads to an ordinary differential equation [ODE], which constrains the spacetime geometry for *any* general relativistic static perfect fluid sphere.

This equation constraining the spacetime geometry of static perfect fluid spheres is now analyzed in several different coordinate systems: general diagonal coordinates, Schwarzschild curvature coordinates, isotropic coordinates, and lesser-known coordinate systems such as Buchdahl coordinates.

To place the use of "unusual" coordinate systems in perspective, one might observe that Finch and Skea [3] estimate that about 55% of all work on fluid spheres is carried out in Schwarzschild curvature coordinates, that isotropic coordinates account for about 35% of related research, and that the remaining 10% is spread over multiple specialized coordinate systems. We take the viewpoint that the "unusual" coordinate systems are useful *only* insofar as they enable us to obtain particularly simple analytic results, and the central point of this article is to see just how much we can do in this regard.

3 General diagonal coordinates

We begin by setting the notation. Choose coordinates to put the metric into the form:

$$ds^2 = -\zeta(r)^2\, dt^2 + \frac{dr^2}{B(r)} + R(r)^2 d\Omega^2. \tag{3.1}$$

A brief calculation yields [17]

$$G_{\hat{r}\hat{r}} = -\frac{1 - B(R')^2}{R^2} + \frac{2B\zeta' R'}{R\zeta}, \tag{3.2}$$

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = \frac{1}{2}\,\frac{2B\zeta' R' + 2\zeta R'' B + \zeta B' R' + 2R\zeta'' B + R\zeta' B'}{R\zeta}, \tag{3.3}$$

and

$$G_{\hat{t}\hat{t}} = \frac{1 - B(R')^2}{R^2} + \frac{2BR'' - B'R'}{R}. \tag{3.4}$$

3.1 ODEs

If this spacetime geometry is to be a perfect fluid sphere, then we must have $G_{\hat{r}\hat{r}} = G_{\hat{\theta}\hat{\theta}}$. This isotropy constraint supplies us with an ODE, which we can write in the form:

$$[R(R\zeta)']B' + 2[RR''\zeta + R^2\zeta'' - RR'\zeta' - (R')^2\zeta]B + 2\zeta = 0. \tag{3.5}$$

In this form the ODE is a first-order linear non-homogeneous ODE in $B(r)$, and hence *explicitly* solvable. (Though it must be admitted that the explicit solution is sometimes rather messy.) The physical impact of this ODE (3.5) is that it reduces the freedom to choose the three *a priori* arbitrary functions in the general diagonal spacetime metric (3.1) to two arbitrary functions, (and we still have some remaining coordinate freedom in the r–t plane, which then allows us to specialize the ODE even further).

The *same* ODE (3.5) can also be rearranged as:

$$[2R^2B]\zeta'' + [R^2B' - 2BRR']\zeta' + [2B(RR'' - [R']^2) + RB'R' + 2]\zeta = 0 \quad (3.6)$$

In this form the ODE is a second-order linear and homogeneous ODE in $\zeta(r)$. While this ODE is not explicitly solvable in closed form, it is certainly true that a lot is known about its generic behaviour.

We can also view the two equivalent ODEs (3.5) and (3.6) as an ODE for $R(r)$:

$$[2B\zeta]RR'' + [B'\zeta - 2B\zeta']RR' - [2B\zeta](R')^2 + [2B\zeta'' + B'\zeta']R^2 + 2\zeta = 0 \quad (3.7)$$

Viewed in this manner it is a second-order nonlinear ODE of no discernible special form — and this approach does not seem to lead to any useful insights.

3.2 Solution generating theorems

Two rather general solution generating theorems can be derived for the general diagonal line element [17].

Theorem 1 (General diagonal 1) *Suppose we adopt general diagonal coordinates and suppose that $\{\zeta(r), B(r), R(r)\}$ represents a perfect fluid sphere. Define*

$$\Delta(r) = \lambda \left(\frac{R(r)\zeta(r)}{(R(r)\zeta(r))'} \right)^2 \exp\left(2 \int \frac{\zeta'(r)}{\zeta(r)} \cdot \frac{(R'(r)\zeta(r) - R(r)\zeta'(r))}{(R'(r)\zeta(r) + R(r)\zeta'(r))}\, \mathrm{d}r \right). \quad (3.8)$$

Then for all λ, the geometry defined by holding $\zeta(r)$ fixed and setting

$$\mathrm{d}s^2 = -\zeta(r)^2\, \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{B(r) + \Delta(r)} + R(r)^2 \mathrm{d}\Omega^2 \quad (3.9)$$

is also a perfect fluid sphere. That is, the mapping

$$\mathsf{T}_{gen1}(\lambda) : \{\zeta, B, R\} \mapsto \{\zeta, B + \Delta, R\} \quad (3.10)$$

takes perfect fluid spheres into perfect fluid spheres.

Proof [Proof for Theorem 1] The proof is based on the techniques used in [12, 14, 17]. No new principles are involved and we quickly sketch the argument. Assume that $\{\zeta(r), B(r), R(r)\}$ is a solution for equation (3.5).

Under what conditions does $\{\zeta(r), \tilde{B}(r), R(r)\}$ also satisfy equation (3.5)? Without loss of generality, we write

$$\tilde{B}(r) = B(r) + \Delta(r). \tag{3.11}$$

Substitute $\tilde{B}(r)$ in equation (3.5)

$$[R(R\zeta)'](B+\Delta)' + 2[RR''\zeta + R^2\zeta'' - RR'\zeta' - (R')^2\zeta](B+\Delta) + 2\zeta = 0 \tag{3.12}$$

Rearranging we see

$$\begin{aligned} &[R(R\zeta)']B' + 2[RR''\zeta + R^2\zeta'' - RR'\zeta' - (R')^2\zeta]B + 2\zeta \\ &\quad +[R(R\zeta)']\Delta' + 2[RR''\zeta + R^2\zeta'' - RR'\zeta' - (R')^2\zeta]\Delta = 0. \end{aligned} \tag{3.13}$$

But we know that the first line in equation (3.13) is zero, because $\{R,\, \zeta,\, B\}$ corresponds by hypothesis to a perfect fluid sphere. Therefore

$$[R(R\zeta)']\Delta' + 2[RR''\zeta + R^2\zeta'' - RR'\zeta' - (R')^2\zeta]\Delta = 0, \tag{3.14}$$

which is an ordinary homogeneous first order differential equation in Δ. A straightforward calculation, including an integration by parts, now leads to

$$\Delta = \lambda \left(\frac{R\zeta}{(R\zeta)'}\right)^2 \exp\left(2\int \frac{\zeta'}{\zeta} \cdot \frac{(R'\zeta - R\zeta')}{(R'\zeta + R\zeta')}\, \mathrm{d}r\right) \tag{3.15}$$

as required.

Theorem 2 (General diagonal 2) *Suppose we adopt general diagonal coordinates, and suppose that $\{\zeta(r), B(r), R(r)\}$ represents a perfect fluid sphere. Define*

$$Z(r) = \sigma + \epsilon \int \frac{R(r)}{\sqrt{B(r)}\,\zeta(r)^2}\, \mathrm{d}r. \tag{3.16}$$

Then for all σ and ϵ, the geometry defined by holding $B(r)$ and $R(r)$ fixed and setting

$$\mathrm{d}s^2 = -\zeta(r)^2\, Z(r)^2\, \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{B(r)} + R(r)^2 \mathrm{d}\Omega^2 \tag{3.17}$$

is also a perfect fluid sphere. That is, the mapping

$$\mathsf{T}_{gen2}(\sigma, \epsilon) : \{\zeta, B, R\} \mapsto \{\zeta\, Z(\zeta, B, R), B, R\} \tag{3.18}$$

takes perfect fluid spheres into perfect fluid spheres.

Proof The proof is based on the technique of "reduction in order" as used in [12, 14, 17]. No new principles are involved and we quickly sketch the argument. Assuming that $\{\zeta(r), B(r), R(r)\}$ solves equation (3.6), write

$$\zeta(r) \to \zeta(r)\; Z(r)\,. \tag{3.19}$$

and demand that $\{\zeta(r)\, Z(r), B(r), R(r)\}$ also solves equation (3.6). Then

$$[2R^2\, B]\,(\zeta\, Z)''+[R^2\, B'-2\, B\, R\, R'](\zeta\, Z)'+[2B\,(R\, R''-[R']^2)+R\, B' R'+2](\zeta\, Z) = 0. \tag{3.20}$$

We expand the above equation to

$$\begin{aligned}[2R^2\, B](\zeta''\, Z + 2\zeta'\, Z' + \zeta\, Z'') + [R^2\, B' - 2B\, R\, R'](\zeta'\, Z + \zeta\, Z')\\ +[2B\,(R\, R'' - [R']^2) + R\, B'\, R' + 2](\zeta\, Z) = 0,\end{aligned} \tag{3.21}$$

and then re-group to obtain

$$\begin{aligned}\big\{[2R^2\, B]\,\zeta'' + [R^2\, B' - 2\, B\, R\, R']\zeta' + [2B\,(R\, R'' - [R']^2) + R\, B'\, R' + 2]\zeta\big\}\, Z\\ [2R^2\, B](2\zeta'\, Z' + \zeta\, Z'') + [R^2\, B' - 2B\, R\, R']\zeta\, Z' = 0.\end{aligned} \tag{3.22}$$

This is a linear homogeneous 2nd order ODE for Z. But under the current hypotheses the entire first line simplifies to zero — so the ODE now simplifies to

$$[R^2\, B'\, \zeta + 4\, R^2\, B\, \zeta' - 2B\, R\, R'\, \zeta]\, Z' + (2R^2\, B\, \zeta)\, Z'' = 0. \tag{3.23}$$

This is a first-order linear ODE in the dependent quantity $Z'(r)$. Rearrange the above equation into

$$\frac{Z''}{Z'} \;=\; \frac{-[R^2\, B'\, \zeta + 4\, R^2\, B\, \zeta' - 2B\, R\, R'\, \zeta]}{(2R^2\, B\, \zeta).} \tag{3.24}$$

Simplifying

$$\frac{Z''}{Z'} = -\frac{1}{2}\,\frac{B'}{B} - 2\,\frac{\zeta'}{\zeta} + \frac{R'}{R}. \tag{3.25}$$

Re-write $Z''/Z' = \mathrm{d}\ln(Z')/\mathrm{d}r$, and integrate twice over both sides of equation (3.25), to obtain

$$Z(r) = \sigma + \epsilon \int \frac{R}{\sqrt{B}\,\zeta^2}\, \mathrm{d}r, \tag{3.26}$$

depending on the old solution $\{\zeta, B, R\}$, and two arbitrary integration constants σ and ϵ.

4 Schwarzschild curvature coordinates

Schwarzchild curvature coordinates is the most popular coordinate system used in the study of perfect fluid spheres, and this coordinate choice (corresponding to $R(r) \to r$) accounts for approximately 55% of the research reported on perfect fluid spheres [3]. The line element of a spherically symmetric spacetime in Schwarzschild curvature coordinates is

$$\mathrm{d}s^2 = -\zeta(r)^2 \, \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{B(r)} + r^2 \mathrm{d}\Omega^2. \tag{4.1}$$

To begin with, we calculate [17]

$$G_{\hat{r}\hat{r}} = \frac{2B\zeta' r - \zeta + \zeta B}{r^2 \zeta}, \tag{4.2}$$

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = \frac{1}{2} \, \frac{B'\zeta + 2B\zeta' + 2r\zeta'' B + r\zeta' B'}{r\zeta}. \tag{4.3}$$

and

$$G_{\hat{t}\hat{t}} = -\frac{B' r - 1 + B}{r^2}. \tag{4.4}$$

4.1 ODEs

Pressure isotrropy leads to the ODE

$$[r(r\zeta)']B' + [2r^2\zeta'' - 2(r\zeta)']B + 2\zeta = 0. \tag{4.5}$$

This is first-order linear non-homogeneous in $B(r)$. Solving for $B(r)$ in terms of $\zeta(r)$ is the basis of the analysis in reference [9], and is also the basis for Theorem **1** in reference [12]. (See also [14] and the reports in [15, 16]. After rephrasing in terms of the TOV equation this is also related to Theorem **P2** in reference [13]). If we re-group in terms of $\zeta(r)$ we find

$$2r^2\zeta'' + (r^2 B' - 2rB)\zeta' + (rB' - 2B + 2)\zeta = 0, \tag{4.6}$$

which is a linear homogeneous second-order ODE. This is the basis of Theorem **2** in [12], and after being recast as a Riccati equation is also the basis of Theorem **P1** in reference [13].

5 Isotropic coordinates

Isotropic coordinates are again commonly used coordinates when investigating perfect fluid spheres. The spacetime metric for an arbitrary static spherically symmetric spacetime in isotropic coordinates is conveniently given by

$$\mathrm{d}s^2 = -\zeta(r)^2\, \mathrm{d}t^2 + \frac{1}{\zeta(r)^2\, B(r)^2}\{\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2\}. \tag{5.1}$$

We calculate [17]

$$G_{\hat{r}\hat{r}} = -2BB'\,\zeta^2/r + (B')^2\zeta^2 - \zeta'^2 B^2, \tag{5.2}$$

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = -BB'\zeta^2/r + (B')^2\,\zeta^2 - BB''\zeta^2 + B^2\zeta'^2, \tag{5.3}$$

and

$$G_{\hat{t}\hat{t}} = 2B^2\zeta\zeta'' + 4B^2\zeta\zeta'/r - 3B^2\zeta'^2 - 2BB'\zeta\zeta' + 2BB''\zeta^2 - 3B'^2\zeta^2 + 4BB'\zeta^2/r. \tag{5.4}$$

5.1 ODEs

The pressure isotropy condition leads to the very simple looking ODE [10]:

$$\left(\frac{\zeta'}{\zeta}\right)^2 = \frac{B'' - B'/r}{2B}. \tag{5.5}$$

There are several ways of improving this. For instance, if we write $\zeta(r) = \exp(\int g(r)\mathrm{d}r)$ then we have an algebraic equation for $g(r)$ [10]:

$$g(r) = \pm\sqrt{\frac{B'' - B'/r}{2B}}. \tag{5.6}$$

Conversely, the isotropy condition can be written in terms of $B(r)$ as:

$$B'' - B'/r - 2g^2 B = 0. \tag{5.7}$$

There is also an improvement obtained by writing $B(r) = \exp(2\int h(r)\mathrm{d}r)$ so that

$$g(r)^2 = 2h(r)^2 + h'(r) - h(r)/r \tag{5.8}$$

which is the basis of the analysis in [10].

5.2 Solution generating theorems

Let us now report two transformation theorems appropriate to isotropic coordinates [17].

Theorem 3 (Isotropic 1 — Buchdahl transformation) *In Isotropic coordinates, if* $\{\zeta(r), B(r)\}$ *describes a perfect fluid then so does* $\{\zeta(r)^{-1}, B(r)\}$*. This is the well-known Buchdahl transformation in disguise. That is, if*

$$\mathrm{d}s^2 = -\zeta(r)^2\,\mathrm{d}t^2 + \frac{1}{\zeta(r)^2\,B(r)^2}\{\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\} \tag{5.9}$$

represents a perfect fluid sphere, then the geometry defined by

$$\mathrm{d}s^2 \to -\frac{1}{\zeta(r)^2}\,\mathrm{d}t^2 + \frac{\zeta(r)^2}{B(r)^2}\{\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\} \tag{5.10}$$

is also a perfect fluid sphere. Alternatively, the mapping

$$\mathsf{T}_{\mathit{Iso}\ 1} : \{\zeta, B\} \mapsto \{\zeta^{-1}, B\} \tag{5.11}$$

takes perfect fluid spheres into perfect fluid spheres, and furthermore is a "square root of unity" in the sense that:

$$\mathsf{T}_{\mathit{Iso}\ 1} \circ \mathsf{T}_{\mathit{Iso}\ 1} = I. \tag{5.12}$$

Proof By inspection. *Vide* equation (5.5).

Theorem 4 (Isotropic 2) *Let* $\{\zeta(r), B(r)\}$ *describe a perfect fluid sphere. Define*

$$Z = \left\{\sigma + \epsilon\int\frac{r\mathrm{d}r}{B(r)^2}\right\}. \tag{5.13}$$

Then for all σ *and* ϵ*, the geometry defined by holding* $\zeta(r)$ *fixed and setting*

$$\mathrm{d}s^2 = -\zeta(r)^2\,\mathrm{d}t^2 + \frac{1}{\zeta(r)^2\,B(r)^2\,Z(r)^2}\{\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\} \tag{5.14}$$

is also a perfect fluid sphere. That is, the mapping

$$\mathsf{T}_{\mathit{Iso}\ 2}(\sigma,\epsilon) : \{\zeta, B\} \mapsto \{\zeta, B\ Z(B)\} \tag{5.15}$$

takes perfect fluid spheres into perfect fluid spheres.

Proof The proof is based on the technique of "reduction in order", and is a simple variant on the discussion in [12]. (See also [17].) Assuming that $\{\zeta(r), B(r)\}$ solves equation (5.7), write

$$B(r) \to B(r)\ Z(r)\,. \tag{5.16}$$

and demand that $\{\zeta(r), B(r)Z(r)\}$ also solves equations (5.5) and (5.7). We find

$$(B\ Z)'' - (B\ Z)'/r - 2g^2(B\ Z) = 0. \tag{5.17}$$

Re-grouping

$$\left\{B'' - B'/r - 2g^2 B\right\} Z + (2B' - B/r)Z' + BZ'' = 0. \tag{5.18}$$

This is a linear homogeneous second-order ODE for Z which now simplifies [in view of (5.7)] to

$$(2B' - B/r)Z' + BZ'' = 0\,, \tag{5.19}$$

which is an ordinary homogeneous second-order differential equation, depending only on Z' and Z''. (So it can be viewed as a first-order homogeneous order differential equation in Z', which is solvable.) Separating the unknown variable to one side,

$$\frac{Z''}{Z'} = -2\frac{B'}{B} + \frac{1}{r}\,. \tag{5.20}$$

Re-write $Z''/Z' = \mathrm{d}\ln(Z')/\mathrm{d}r$, and integrate twice over both sides of equation (5.20), to obtain

$$Z(r) = \left\{\sigma + \epsilon \int \frac{r\ \mathrm{d}r}{B(r)^2}\right\}. \tag{5.21}$$

depending on the old solution $\{\zeta(r), B(r)\}$, and two arbitrary integration constants σ and ϵ.

Theorem 5 (Isotropic 3) *The transformations $T_{Iso\ 1}$ and $T_{Iso\ 2}$ commute.*

Proof By inspection.

Note that the fact that these two transformation theorems commute is specific to isotropic coordinates. Such behaviour certainly does not occur in Schwarzschild coordinates or in general diagonal coordinates.

6 Buchdahl coordinates

Without loss of generality we can put the metric in "Buchdahl coordinates" and choose the coefficients to be

$$\mathrm{d}s^2 = -\zeta(r)^2\,\mathrm{d}t^2 + \zeta(r)^{-2}\left\{\mathrm{d}r^2 + R(r)^2\mathrm{d}\Omega^2\right\}. \tag{6.1}$$

This coordinate system is a sort of cross between Synge isothermal (tortoise) coordinates and Gaussian polar (proper radius) coordinates. We calculate [17]

$$G_{\hat{r}\hat{r}} = -(\zeta')^2 - \frac{\zeta^2\,[1-(R')^2]}{R^2}, \tag{6.2}$$

$$G_{\hat{\theta}\hat{\theta}} = +(\zeta')^2 + \frac{\zeta^2\,R''}{R}, \tag{6.3}$$

and

$$G_{\hat{t}\hat{t}} = -3(\zeta')^2 + 2\zeta\zeta'' - \frac{2\zeta(\zeta R'' - 2R'\zeta')}{R} + \frac{\zeta^2\,[1-(R')^2]}{R^2}. \tag{6.4}$$

Imposing pressure isotropy supplies us with a first-order homogeneous ODE for $\zeta(r)$:

$$\left(\frac{\zeta'}{\zeta}\right)^2 = -\frac{[1-(R')^2+RR'']}{2R^2}. \tag{6.5}$$

This is very similar to the equation we obtained in isotropic coordinates. (Rearranging this into an ODE for $R(r)$ yields a second-order nonlinear differential equation which does not seem to be particularly useful.)

Theorem 6 (Buchdahl) *If $\{\zeta(r), R(r)\}$ describes a perfect fluid then so does $\{\zeta(r)^{-1}, R(r)\}$. This is the Buchdahl transformation in yet another disguise. The geometry defined by holding $R(r)$ fixed and setting*

$$\mathrm{d}s^2 = -\zeta(r)^{-2}\,\mathrm{d}t^2 + \zeta(r)^2\left\{\mathrm{d}r^2 + R(r)^2\mathrm{d}\Omega^2\right\} \tag{6.6}$$

is also a perfect fluid sphere. That is, the mapping

$$\mathsf{T}_{\mathsf{Buchdahl}} : \{\zeta, R\} \mapsto \left\{\zeta^{-1}, R\right\} \tag{6.7}$$

takes perfect fluid spheres into perfect fluid spheres.

Proof By inspection. (Note strong similarities to the discussion for isotropic coordinates.)

7 Generalized Buchdahl ansatz

We now return to general diagonal coordinates and, based on our insight from dealing with isotropic coordinates and Buchdahl coordinates, make a specific *ansatz* for the functional form of the metric components [17]. Without loss of generality we choose the metric coefficients to be

$$\mathrm{d}s^2 = -\zeta(r)^2\, \mathrm{d}t^2 + \zeta(r)^{-2}\left\{\frac{\mathrm{d}r^2}{E(r)} + R(r)^2 \mathrm{d}\Omega^2\right\}. \tag{7.1}$$

We calculate [17]

$$G_{\hat{r}\hat{r}} = -E(\zeta')^2 - \frac{\zeta^2\,[1 - E\,(R')^2]}{R^2}, \tag{7.2}$$

$$G_{\hat{\theta}\hat{\theta}} = +E(\zeta')^2 + \frac{\zeta^2\,[2ER'' + E'R']}{2R}, \tag{7.3}$$

and

$$G_{\hat{t}\hat{t}} = -3E(\zeta')^2 + 2E\zeta\zeta'' + \zeta E'\zeta' - \frac{2E\zeta(\zeta R'' - 2R'\zeta')}{R} + \frac{\zeta^2\,[1 - E(R')^2]}{R^2} - \frac{\zeta^2 E' R'}{R}. \tag{7.4}$$

Imposing pressure isotropy supplies us with a first-order homogeneous ODE for $\zeta(r)$:

$$\left(\frac{\zeta'}{\zeta}\right)^2 = -\frac{[2 - 2E(R')^2 + 2ERR'' + RE'R']}{4ER^2}. \tag{7.5}$$

This is very similar to the equation we obtained in isotropic coordinates and Buchdahl coordinates, but now in general diagonal coordinates — the key point is that we have carefully chosen the functional form of the metric components. Rearranging this into an ODE for $E(r)$ yields a first-order linear differential equation

$$[\zeta^2 RR']E' + [4R^2(\zeta')^2 + 2R\zeta^2 R'' - 2\zeta^2(R')^2]E + 2\zeta^2 = 0. \tag{7.6}$$

(In contrast, rearranging this into an ODE for $R(r)$ yields a second-order nonlinear differential equation which does not seem to be particularly useful.)

Theorem 7 (Generalized Buchdahl) *If $\{\zeta(r), E(r), R(r)\}$ describes a perfect fluid then so does $\{\zeta(r)^{-1}, E(r), R(r)\}$. This is the Buchdahl transformation in yet another disguise. The geometry defined by holding $E(r)$ and $R(r)$ fixed and transforming*

$$\mathrm{d}s^2 = -\zeta(r)^2\, \mathrm{d}t^2 + \zeta(r)^{-2}\left\{\frac{\mathrm{d}r^2}{E(r)} + R(r)^2 \mathrm{d}\Omega^2\right\}. \tag{7.7}$$

into

$$\mathrm{d}s^2 = -\zeta(r)^{-2}\,\mathrm{d}t^2 + \zeta(r)^2 \left\{ \frac{\mathrm{d}r^2}{E(r)} + R(r)^2 \mathrm{d}\Omega^2 \right\}. \tag{7.8}$$

is also a perfect fluid sphere. That is, the mapping

$$\mathsf{T}_{\textsf{Generalized Buchdahl}} : \{\zeta, E, R\} \mapsto \{\zeta^{-1}, E, R\} \tag{7.9}$$

takes perfect fluid spheres into perfect fluid spheres.

Proof By inspection. (Note very strong similarities to the discussion for isotropic coordinates and Buchdahl coordinates.)

In addition, we could use the fact that the isotropy condition, when viewed as a differential equation in $E(r)$, is first-order linear to develop yet another variant on the theorem **General diagonal 1**. As no new significant insight is gained we suppress the details.

8 Discussion

In this article, we have reported several new transformation theorems that map perfect fluid spheres to perfect fluid spheres using both "usual" and "unusual" coordinate systems — such as Schwarzschild (curvature), isotropic, and Buchdahl coordinates. In each case we developed at least one such transformation theorem, while in several cases we have been able to develop multiple transformation theorems. We have also investigated regularity conditions at the centre of the fluid sphere in all these coordinate systems.

In summary, we have now extended the algorithmic technique originally introduced in [10] to many other coordinate systems and many other functional forms for the metric.

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