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Convergence of the Ishikawa Iteration Scheme With Errors for I-Asymptotically Quasi-Nonexpansive Mappings

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Abstract : In this paper, we construct an iterative scheme with errors involving I-asymptotically quasi-nonexpansive mappings and show that the iterative scheme converges strongly to a common fixed point of I-asymptotically quasi-nonexpansive mappings in a uniformly convex Banach space under some appropriate conditions.

Keywords : Common fixed point; I-asymptotically nonexpansive mapping; Uniformly convex Banach space.
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1 Introduction.

Throughout this paper, we denote the set of all fixed points of a mapping T by $F(T)$, $T^0 = E$, where E denotes the mapping $E : C \to C$ defined by $Ex = x$, respectively.

Let C be a nonempty subset of a real normed linear space X. Let T be a self-mapping of C. T is said to be asymptotically nonexpansive if there exists a real sequence $\{\lambda_n\} \subset [0, +\infty)$, with $lim_{n\to\infty}\lambda_n = 0$, such that $\|T^n x - T^n y\| \leq (1 + \lambda_n)\|x - y\|$, for all $x, y \in C$. T is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$.

It was proved in [3] that if X is uniformly convex and if C is bounded closed convex subset of X, then every asymptotically nonexpansive mapping has a fixed point.

Let $T, I : C \to C$, then T is called I-quasi nonexpansive on C if $\|Tu - f\| \leq \|Iu - f\|$ for all $u \in C$ and $f \in F(T) \cap F(I)$. T is called I-asymptotically quasi-nonexpansive if there exists a sequence $\{\lambda'_k\} \subset [0, \infty)$ with $lim_{k\to\infty}\lambda'_k = 0$ such that $\|T^k u - f\| \leq (\lambda'_k + 1)\|I^k u - f\|$ for all $u \in C$ and $f \in F(T) \cap F(I)$ and $k \geq 1$.

In 2005, Khan and Hafiz [5] introduced the following iterative scheme with errors for a pair of nonexpansive mappings as follows: for any given $x_1 \in C$,

$$\begin{aligned} x_{n+1} &= a_n S y_n + b_n x_n + c_n u_n, \\ y_n &= a'_n T x_n + b'_n x_n + c'_n v_n, \end{aligned} \quad n \geq 1, \tag{1.1}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ are real sequences in $[0,1)$ with $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C.

In 2006, Jeong and Kim [4] generalize the scheme (1.1) for a pair of asymptotically nonexpansive mappings.

In the past few decades, many results on fixed points of asymptotically nonexpansive, quasi-nonexpansive and asymptotically quasi-nonexpansive mappings have been obtained in Banach space and metric spaces (see, e.g., [2,6,7]). Very recently, Temir and Gul [9] obtained the weakly almost convergence theorems for I-asymptotically quasi-nonexpansive mapping in Hilbert space. Tian, Chang and Huang [10] got the strong convergence for a finite family of non-self asymptotically quasi-nonexpansive-type mappings in Banach spaces.

Definition 1.1. Let $T : C \to C$ be an I-asymptotically quasi-nonexpansive mapping, I be a asymptotically nonexpansive mapping on C, where C is nonempty closed convex subset of a Banach space X. Then Ishikawa iterative scheme with errors is the sequences of $\{x_n\}$ defined by, for given $x_0 \in C$,

$$\begin{array}{rcl} y_n & = & \alpha'_n T^n x_n + \beta'_n x_n + \gamma'_n v_n, \\ x_{n+1} & = & \alpha_n I^n y_n + \beta_n x_n + \gamma_n u_n, \end{array} \quad n \geq 0, \tag{1.2}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}$ are real sequences in (0,1) with $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C.

In this paper, we show that scheme (1.2) converges strongly to a common fixed point of $I-$asymptotically quasi-nonexpansive mappings in uniformly convex Banach spaces (in section 3) or in Banach spaces (in section 4).

2 Preliminaries.

Definition 2.1. Let X be a Banach space, C a nonempty subset of X. Let $T : C \to C$. Then T is said to be

(1) demiclosed at y if whenever $\{x_n\} \subset C$ such that $x_n \rightharpoonup x \in C$ and $Tx_n \to y$ then $Tx = y$.

(2) semi-compact if for any bounded sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to some x^* in K.

(3) completely continuous if the sequence $\{x_n\}$ in C converges weakly to x_0 implies that $\{Tx_n\}$ converges strongly to Tx_0.

(4) uniformly L-Lipschitzian mapping if for arbitrary $x, y \in C$, we have $\|T^n x - T^n y\| \leq L\|x - y\|$, where $n = 1, 2, \cdots$ and L is a positive constant.

We restate the following lemmas which play important roles in our proofs.

Lemma 2.1 [5]. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\mu_n\}$ be four nonnegative real sequences satisfying $\alpha_{n+1} \leq (1+\gamma_n)(1+\mu_n)\alpha_n + \beta_n$, for all $n \geq 1$. If $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, then $lim_{n\to\infty}\alpha_n$ exists.

Lemma 2.2 [1]. Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X. If T is asymptotically nonexpansive mapping of C into itself, then $E - T$ is demiclosed at *zero*.

Lemma 2.3 [8]. Let C be a nonempty closed bounded and convex subset of a uniformly convex Banach space X. Let $0 < b \leq l_n \leq c < 1$, $\forall n \geq 1$, and let $\{x_n\}$ and $\{y_n\}$ be two sequences in C with $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$, and $\lim_{n\to\infty} \|l_n x_n + (1 - l_n)y_n\| = a$, where $a \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

3 Strong convergence for I-asymptotically quasi-nonexpansive mappings in uniformly convex Banach spaces

Lemma 3.1. Let C be a nonempty closed convex subset of uniformly convex Banach space X. Let $T : C \to C$ be an I-asymptotically quasi-nonexpansive mapping with sequence $\{l_n\} \subseteq [1, \infty)$ such that $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, I be an asymptotically nonexpansive mapping on C with sequence $\{k_n\} \subseteq [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Suppose the sequence $\{x_n\}$ is generated by (1.2), where $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$. If $F_1 = F(T) \cap F(I) \neq \emptyset$, then $lim_{n\to\infty}\|x_n - x^*\|$ exists for any $x^* \in F_1$.

Proof. Since $\{u_n\}, \{v_n\}$ are bounded sequences in C, there exists $M > 0$ such that $\|u_n - x^*\| \leq M, \|v_n - x^*\| \leq M$, for all $n \in \mathbb{N}$ and $x^* \in F_1$. Setting $k_n = 1 + r_n,\ \ l_n = 1 + s_n$. Since $\sum_{n=1}^{\infty}(k_n - 1) < \infty, \sum_{n=1}^{\infty}(l_n - 1) < \infty$, so $\sum_{n=1}^{\infty} r_n < \infty, \sum_{n=1}^{\infty} s_n < \infty$. For any $x^* \in F_1$, we have

$$
\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n I^n y_n + \beta_n x_n + \gamma_n u_n - x^*\| \\
&\leq \alpha_n \|I^n y_n - x^*\| + (1 - \alpha_n - \gamma_n)\|x_n - x^*\| + \gamma_n \|u_n - x^*\| \\
&\leq \alpha_n (1 + r_n)\|y_n - x^*\| + (1 - \alpha_n - \gamma_n)\|x_n - x^*\| + \gamma_n M \\
&\leq \alpha_n (1 + r_n)\|\alpha'_n (T^n x_n - x^*) + (1 - \alpha'_n - \gamma'_n)(x_n - x^*) + \gamma'_n (v_n - x^*)\| \\
&\quad + (1 - \alpha_n - \gamma_n)\|x_n - x^*\| + \gamma_n M \\
&\leq \alpha_n \alpha'_n (1 + r_n)(1 + s_n)\|I^n x_n - x^*\| + \alpha_n (1 + r_n)\|(1 - \alpha'_n - \gamma'_n)(x_n - x^*) \\
&\quad + \gamma'_n (v_n - x^*)\| + (1 - \alpha_n - \gamma_n)\|x_n - x^*\| + \gamma_n M \\
&\leq \alpha_n \alpha'_n (1 + s_n)(1 + r_n)^2 \|x_n - x^*\| + \alpha_n \gamma'_n (1 + r_n) M \\
&\quad + \alpha_n (1 - \alpha'_n - \gamma'_n)(1 + r_n)\|x_n - x^*\| + (1 - \alpha_n - \gamma_n)\|x_n - x^*\| + \gamma_n M \\
&\leq \alpha_n \alpha'_n (1 + s_n)(1 + r_n)^2 \|x_n - x^*\| + \alpha_n (1 - \alpha'_n)(1 + r_n)^2 (1 + s_n)\|x_n - x^*\| \\
&\quad + \alpha_n \gamma'_n (1 + r_n) M + (1 - \alpha_n)(1 + r_n)^2 (1 + s_n)\|x_n - x^*\| + \gamma_n M \\
&\leq (1 + r_n)^2 (1 + s_n)\|x_n - x^*\| + (1 + r_n)\alpha_n \gamma'_n M + \gamma_n M.
\end{aligned}
$$

Since $\sum_n^{\infty} \gamma_n < \infty, \sum_n^{\infty} \gamma'_n < \infty$, thus it follows from Lemma 2.1 that $\lim_{n\to\infty} \|x_n - x^*\|$ exists. This completes the proof.

Lemma 3.2. Let X be a uniformly convex Banach space and C be a nonempty bounded closed convex subset of X. Let $T, I, \{x_n\}$ be same as Lemma 3.1, where

$a \leq \alpha_n \leq b$, $a \leq \alpha'_n \leq b$ for all $n \geq 0$ and some $a, b \in (0,1)$. If T is an uniformly L-Lipschizian mapping and $F_1 \neq \emptyset$, then $\lim_{n\to\infty} \|Tx_n - x_n\| = \lim_{n\to\infty} \|Ix_n - x_n\| = 0$.

Proof. By Lemma 3.1, for any $x^* \in F_1$, $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Assume $\lim_{n\to\infty} \|x_n - x^*\| = c \geq 0$. Since

$$\begin{aligned}
\|y_n - x^*\| &= \|\alpha'_n T^n x_n + \gamma'_n(v_n - x_n) + (1-\alpha'_n)x_n - x^*\| \\
&\leq \alpha'_n\|T^n x_n - x^*\| + \gamma'_n\|v_n - x^*\| + (1-\alpha'_n - \gamma'_n)\|x_n - x^*\| \\
&\leq \alpha'_n(1+s_n)\|I^n x_n - x^*\| + \gamma'_n\|v_n - x^*\| + (1-\alpha'_n - \gamma'_n)\|x_n - x^*\| \\
&\leq \alpha'_n(1+s_n)(1+r_n)\|x_n - x^*\| + (1-\alpha'_n - \gamma'_n)\|x_n - x^*\| + \gamma'_n\|v_n - x^*\| \\
&\leq \alpha'_n(1+s_n)(1+r_n)\|x_n - x^*\| + (1-\alpha'_n)(1+s_n)(1+r_n)\|x_n - x^*\| + \gamma'_n\|v_n - x_n\| \\
&\leq (1+s_n)(1+r_n)\|x_n - x^*\| + \gamma'_n M.
\end{aligned}$$

Taking limsup on both sides in above inequality, we obtain

$$\limsup_{n\to\infty} \|y_n - x^*\| \leq c. \tag{3.1}$$

Since $\|I^n y_n - x^* + \gamma_n(u_n - x^*)\| \leq (1+r_n)\|y_n - x^*\| + \gamma_n M$. By (3.1), we have $\limsup_{n\to\infty} \|I^n y_n - x^* + \gamma_n(u_n - x^*)\| \leq c$. And $\|x_n - x^* + \gamma_n(u_n - x^*)\| \leq \|x_n - x^*\| + \gamma_n M$, which implies $\limsup_{n\to\infty} \|x_n - x^* + \gamma_n(u_n - x^*)\| \leq c$. Further, $lim_{n\to\infty}\|x_{n+1} - x^*\| = c$ means that

$$\lim_{n\to\infty} \|\alpha_n(I^n y_n - x^* + \gamma_n(u_n - x^*)) + (1-\alpha_n)(x_n - x^* + \gamma_n(u_n - x^*))\| = c.$$

It follows from Lemma 2.3 that

$$\lim_{n\to\infty} \|I^n y_n - x_n\| = 0. \tag{3.2}$$

Next,

$$\begin{aligned}
\|x_n - x^*\| &\leq \|x_n - I^n y_n\| + \|I^n y_n - x^*\| \\
&\leq \|x_n - I^n y_n\| + (1+r_n)\|y_n - x^*\|
\end{aligned}$$

gives that $c = \lim_{n\to\infty} \|x_n - x^*\| \leq \liminf_{n\to\infty} \|y_n - x^*\|$. By (3.1) $\lim_{n\to\infty} \|y_n - x^*\| = c$.

Since $\|T^n x_n - x^* + \gamma'_n(v_n - x^*)\| \leq (1+s_n)(1+r_n)\|x_n - x^*\| + \gamma'_n M$ and $\|x_n - x^* + \gamma'_n(v_n - x^*)\| \leq \|x_n - x^*)\| + \gamma'_n M$, we have $\limsup_{n\to\infty} \|T^n x_n - x^* + \gamma'_n(v_n - x^*)\| \leq c$ and $\limsup_{n\to\infty} \|x_n - x^* + \gamma'_n(v_n - x^*)\| \leq c$. Further, $lim_{n\to\infty}\|y_n - x^*\| = c$ means that

$$\lim_{n\to\infty} \|\alpha'_n(T^n x_n - x^* + \gamma'_n(v_n - x^*)) + (1-\alpha'_n)(x_n - x^* + \gamma'_n(v_n - x^*))\| = c$$

By Lemma 2.3, we have

$$lim_{n\to\infty}\|T^n x_n - x_n\| = 0. \tag{3.3}$$

Also,

$$\begin{aligned}\|I^n x_n - x_n\| &\le \|I^n x_n - I^n y_n\| + \|I^n y_n - x_n\| \\ &\le (1+r_n)\|x_n - y_n\| + \|I^n y_n - x_n\| \\ &= (1+r_n)\|a'_n(x_n - T^n x_n) + \gamma'_n(x^* - v_n) + \gamma'_n(x_n - x^*)\| \\ &\quad + \|I^n y_n - x_n\| \\ &\le \alpha'_n(1+r_n)\|x_n - T^n x_n\| + \gamma'_n(1+r_n)\|x^* - v_n\| \\ &\quad + \gamma'_n(1+r_n)\|x_n - x^*\| + \|I^n y_n - x_n\| \\ &\le \alpha'_n(1+r_n)\|x_n - T^n x_n\| + \gamma'_n(1+r_n)M \\ &\quad + \gamma'_n(1+r_n)\|x_n - x^*\| + \|I^n y_n - x_n\|.\end{aligned}$$

Taking limsup on both sides in the above inequality, we have

$$\limsup_{n\to\infty} \|I^n x_n - x_n\| \le 0.$$

That is

$$\lim_{n\to\infty} \|I^n x_n - x_n\| = 0. \tag{3.4}$$

In addition,

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|\alpha_n I^n y_n + (1-\alpha_n-\gamma_n)x_n + \gamma_n u_n - x_n\| \\ &= \|\alpha_n(I^n y_n - x_n) + \gamma_n(u_n - x^*) + \gamma_n(x^* - x_n)\| \\ &\le \alpha_n\|I^n y_n - x_n\| + \gamma_n(x^* - x_n)\| + \gamma_n M,\end{aligned}$$

thus it follows from (3.2) that

$$lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.5}$$

Hence,

$$\begin{aligned}\|Tx_n - x_n\| &\le \|T^n x_n - x_n\| + \|T^n x_n - Tx_n\| \\ &\le \|T^n x_n - x_n\| + L\|T^{n-1}x_n - x_n\| \\ &\le \|T^n x_n - x_n\| + L\|T^{n-1}x_n - T^{n-1}x_{n-1}\| \\ &\quad + L\|T^{n-1}x_{n-1} - x_n\| \\ &\le \|T^n x_n - x_n\| + L^2\|x_n - x_{n-1}\| \\ &\quad + L\|T^{n-1}x_{n-1} - x_{n-1}\| + L\|x_{n-1} - x_n\|,\end{aligned}$$

By (3.3) and (3.5), we have

$$lim_{n\to\infty}\|Tx_n - x_n\| = 0. \tag{3.6}$$

On the other hand,

$$\begin{aligned}\|Ix_n - x_n\| &\le \|I^n x_n - x_n\| + \|I^n x_n - Ix_n\| \\ &\le \|I^n x_n - x_n\| + k_1\|I^{n-1}x_n - x_n\| \\ &\le \|I^n x_n - x_n\| + k_1\|I^{n-1}x_n - I^{n-1}x_{n-1}\| + k_1\|I^{n-1}x_{n-1} - x_n\| \\ &\le \|I^n x_n - x_n\| + k_1 k_{n-1}\|x_n - x_{n-1}\| + k_1\|I^{n-1}x_{n-1} - x_{n-1}\| \\ &\quad + k_1\|x_{n-1} - x_n\|,\end{aligned}$$

by (3.4) and (3.5), we have

$$lim_{n\to\infty}\|Ix_n - x_n\| = 0. \tag{3.7}$$

This completes the proof.

Theorem 3.3. Let X be a uniformly convex Banach space and $C, T, I, \{x_n\}$ be same as in Lemma 3.2. If I is a semi-compact mapping and $F_1 \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of T and I.

Proof. Since I is semi-compact mapping, $\{x_n\}$ is bounded and $lim_{n\to\infty}\|x_n - Ix_n\| = 0$, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges to x^*. It follows from Lemma 2.2, $x^* \in F(I)$. In addition, since T is an uniformly L-Lipschizian mapping and $lim_{n\to\infty}\|x_n - Tx_n\| = 0$. So $\|x^* - Tx^*\| = 0$. This implies that $x^* \in F(I) \cap F(T)$. Since the subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to x^* and $lim_{n\to\infty}\|x_n - x^*\|$ exists, then $\{x_n\}$ converges strongly to the common fixed point $x^* \in F_1$. The proof is completed.

Theorem 3.4. Let X be a uniformly convex Banach space and $C, T, I, \{x_n\}$ be same as in Lemma 3.2. If I is completely continuous mapping and $F_1 \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of T and I.

Proof. By Lemma 3.1, $\{x_n\}$ is bounded. Since $lim_{n\to\infty}\|x_n - Ix_n\| = lim_{n\to\infty}\|x_n - Tx_n\| = 0$, then $\{Tx_n\}$ and $\{Ix_n\}$ are bounded. Since I is completely continuous, that exists subsequence $\{Ix_{n_j}\}$ of $\{Ix_n\}$ such that $\{Ix_{n_j}\} \to p$ as $j \to \infty$. Thus, we have $lim_{j\to\infty}\|x_{n_j} - Tx_{n_j}\| = lim_{j\to\infty}\|x_{n_j} - Ix_{n_j}\| = 0$. So, by the continuity of I and Lemma 2.2, we have $lim_{j\to\infty}\|x_{n_j} - p\| = 0$ and $p \in F(I)$. Further,

$$\|Tx_{n_j} - p\| \leq \|Tx_{n_j} - x_{n_j}\| + \|x_{n_j} - Ix_{n_j}\| + \|Ix_{n_j} - p\| \tag{3.8}$$

Thus, $lim_{n_j\to\infty}\|Tx_{n_j} - p\| = 0$. This implies that $\{Tx_{n_j}\}$ converges strongly to p. Since T is uniformly L-Lipschzian, T is continuous. So, $p = Tp$. Hence $p \in F_1$. By Lemma 3.1, $lim_{n\to\infty}\|x_n - p\|$ exists. Thus $lim_{n\to\infty}\|x_n - p\| = 0$. The proof is completed.

4 Strong convergence for I-asymptotically quasi-nonexpansive-type mappings in Banach spaces

Definition 4.1. Let X be a real Banach space, C be a nonempty closed convex subset of X. Then

(1) $T : C \to C$ is called asymptotically nonexpansive-type if

$$\limsup_{n\to\infty}\{\sup_{u,v\in C}[\|T^k u - T^k v\| - \|u - v\|]\} \leq 0$$

for all $k \geq 1$.

(2)$T : C \to C$ is called asymptotically quasi-nonexpansive-type if

$$\limsup_{n\to\infty}\{\sup_{u\in C}[\|T^k u - f\| - \|u - f\|]\} \leq 0$$

for all $f \in F(T)$ and $k \geq 1$.

(3)Let $I : C \to C$. T is called I-asymptotically quasi-nonexpansive-type if $\limsup_{n\to\infty}\{sup_{u\in C}[\|T^k u - f\| - \|I^k u - f\|]\} \leq 0$ for all $f \in F(T) \cap F(I)$ and $k \geq 1$.

Remark 4.2. It follows from Definition 4.1 that

(a) If $T : C \to C$ is an asymptotically nonexpansive mapping, then T is an asymptotically nonexpansive-type mapping;

(b) if $F(T)$ is nonempty and $T : C \to C$ is an asymptotically nonexpansive-type mapping, then T is an asymptotically quasi-nonexpansive-type mapping;

(c) if $T, I : C \to C$, T is an I-asymptotically quasi-nonexpansive mapping, then T is an I-asymptotically quasi-nonexpansive-type mapping.

Theorem 4.3. Let X be a real Banach space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be an I-asymptotically quasi-nonexpansive-type mapping, I be an asymptotically nonexpansive-type mapping on C. Suppose the sequence $\{x_n\}$ is generated by (1.2) with $\sum_{n=1}^{\infty}\alpha_n < \infty, \sum_{n=1}^{\infty}\alpha'_n < \infty, \sum_{n=1}^{\infty}\gamma_n < \infty, \sum_{n=1}^{\infty}\gamma'_n < \infty$. If the common fixed point set $F_1 = F(T)\cap F(I)$ is nonempty, then the sequence $\{x_n\}$ converges strongly to a common fixed point of T and I, if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. The necessity is obvious.

Next we prove the sufficiency. For any given $p \in F_1$, since $\{u_n\}$, $\{v_n\}$ are bounded sequences in C, we may set $M_1 = sup\{\|u_n - p\|, \|v_n - p\|\}$.

Since T is an I-asymptotically nonexpansive-type mapping, I is an asymptotically nonexpansive-type mapping and $\{x_n\}, \{y_n\} \subset C$, for any given $\varepsilon > 0$, there exists a positive integer n_0 such that for all $n \geq n_0$ and any $u \in F_1$

$$\begin{array}{rcl} \|T^n x - u\| - \|I^n x - u\| & < & \varepsilon \\ \|I^n x - u\| - \|x - u\| & < & \varepsilon, \end{array} \tag{4.1}$$

Hence for any $n \geq n_0$, it follows from (1.2) and (4.1) that

$$\begin{array}{rcl} \|x_{n+1} - p\| & = & \|\alpha_n I^n y_n + \beta_n x_n + \gamma_n u_n - p\| \\ & \leq & \alpha_n\|I^n y_n - p\| + \beta_n\|x_n - p\| + \gamma_n\|u_n - p\| \\ & \leq & \alpha_n\{\|I^n y_n - p\| - \|y_n - p\|\} + \alpha_n\|y_n - p\| + \beta_n\|x_n - p\| + \gamma_n\|u_n - p\| \\ & \leq & \alpha_n\varepsilon + \alpha_n\|y_n - p\| + \beta_n\|x_n - p\| + \gamma_n M_1. \\ & \leq & \alpha_n\varepsilon + \alpha_n\|y_n - p\| + (1 - \alpha_n)\|x_n - p\| + \gamma_n M_1. \end{array} \tag{4.2}$$

Next,

$$\begin{array}{rcl} \|y_n - p\| & = & \|\alpha'_n T^n x_n + \beta'_n x_n + \gamma'_n u_n - p\| \\ & \leq & \alpha'_n\|T^n x_n - p\| + \beta'_n\|x_n - p\| + \gamma'_n\|u_n - p\| \\ & \leq & \alpha'_n\{\|T^n x_n - p\| - \|x_n - p\|\} + \alpha'_n\|x_n - p\| + \beta'_n\|x_n - p\| + \gamma'_n\|u_n - p\| \\ & \leq & \alpha'_n\varepsilon + (\alpha'_n + \beta'_n)\|x_n - p\| + \gamma'_n M_1. \\ & \leq & \alpha'_n\varepsilon + \|x_n - p\| + \gamma'_n M_1. \end{array} \tag{4.3}$$

From (4.2) and (4.3), as $n \geq n_0$ we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n\varepsilon + \alpha_n(\alpha'_n\varepsilon + \|x_n - p\| + \gamma'_n M) + (1-\alpha_n)\|x_n - p\| + \gamma_n M_1 \\ &\leq \|x_n - p\| + A_n, \end{aligned} \tag{4.4}$$

where $A_n = \alpha_n(1+\alpha'_n)\varepsilon + \alpha_n\gamma'_n M_1 + \gamma_n M_1, n \geq 1$. Since $\sum_{n=1}^{\infty}\alpha_n < \infty$, $\sum_{n=1}^{\infty}\alpha'_n < \infty$, $\sum_{n=1}^{\infty}\gamma_n < \infty$ and $\sum_{n=1}^{\infty}\gamma'_n < \infty$, we have that $\Sigma_{n=1}^{\infty}A_n < \infty$.

By the arbitrariness of $p \in F$, from (4.4) we have

$$\inf_{p\in F_1}\|x_{n+1} - p\| \leq \inf_{p\in F_1}\|x_n - p\| + A_n, \quad \forall n \geq n_0,$$

and so we have

$$d(x_{n+1}, F_1) \leq d(x_n, F_1) + A_n, \quad \forall n \geq n_0. \tag{4.5}$$

By Lemma 2.1, the limit $\lim_{n\to\infty} d(x_n, F_1)$ exists. By the condition $\liminf_{n\to\infty} d(x_n, F_1) = 0$, we have

$$\lim_{n\to\infty} d(x_n, F_1) = 0. \tag{4.6}$$

For any $\varepsilon > 0$, since $\lim_{n\to\infty} d(x_n, F_1) = 0$, there exists natural number n_1 such that when $n \geq n_1$, $d(x_n, F_1) < \frac{\varepsilon}{3}$. Thus, there exists $x^* \in F_1$ such that for above ε there exists positive integer $n_2 \geq n_1$ such that as $n \geq n_2$

$$\|x_n - x^*\| < \frac{\varepsilon}{2}.$$

Now for arbitrary $n, m \geq n_2$, consider

$$\|x_n - x_m\| \leq \|x_n - x^*\| + \|x_m - x^*\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This implies $\{x_n\}$ is a cauchy sequence in C. Since C is a closed subset of X, there exists $p^* \in C$ such that $x_n \to p^*$.

By the routine method, it is easy to show that F_1 is a closed set. If $d(p^*, F_1) > 0$. For any $p \in F_1$, we have $\|p^* - p\| \leq \|p^* - x_n\| + \|x_n - p\|$. This implies that

$$d(p^*, F_1) \leq \|p^* - x_n\| + d(x_n, F_1). \tag{4.7}$$

Letting $n \to \infty$ in (4.7) and noting (4.6), it gets $d(p^*, F_1) \leq 0$. This is a contradiction. So $d(p^*, F_1) = 0$, hence $p^* \in F_1$. The proof is completed.

By Remark 4.2 and Theorem 4.3, we can directly obtain the following corollary.

Corollary 4.4. Let X be a real Banach space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be an I-asymptotically quasi-nonexpansive mapping with sequence $\{l_n\} \subseteq [1, \infty)$ such that $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, I be an asymptotically nonexpansive mapping on C with sequence $\{k_n\} \subseteq [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Suppose the sequence $\{x_n\}$ is generated by (1.2), where $\sum_{n=1}^{\infty}\alpha_n < \infty, \sum_{n=1}^{\infty}\alpha'_n < \infty, \sum_{n=1}^{\infty}\gamma_n < \infty, \sum_{n=1}^{\infty}\gamma'_n < \infty$. If the common fixed point set $F_1 = F(T)\cap F(I)$ is nonempty, then the sequence $\{x_n\}$ converges strongly to a common fixed point of T and I, if and only if $liminf_{n\to\infty} d(x_n, F) = 0$.

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