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Green's Relations and Regularity for Semigroups of Transformations with Restricted Range that Preserve Double Direction Equivalence Relations

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Abstract : Let $T(X)$ be the full transformation semigroup on a set X. For an equivalence E on X and a nonempty subset Y of X, let

$$T_{E^*}(X,Y) = \{\alpha \in T(X) : X\alpha \subseteq Y \text{ and } \forall x, y \in X, (x,y) \in E \Leftrightarrow (x\alpha, y\alpha) \in E\}.$$

In this article, we give a necessary and sufficient condition for $T_{E^*}(X,Y)$ to be a subsemigroup of $T(X)$ under the composition of functions and study the regularity of $T_{E^*}(X,Y)$. Finally, we characterize Green's relations on this semigroup.

Keywords : transformation semigroup; equivalence; regular element; Green's relations.
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1 Introduction

Let $T(X)$ be the set of all functions from X into itself. We have $T(X)$ under the composition of functions is a semigroup which is called the *full transformation semigroup on* X. In 1975, J. S. V. Symons [9] studied the subsemigroup $T(X,Y)$ of $T(X)$ defined by

$$T(X,Y) = \{\alpha \in T(X) : X\alpha \subseteq Y\}$$

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where Y is a nonempty subset of X. The author studied the automorphism of $T(X,Y)$ and the isomorphism between two semigroups $T(X_1,Y_1)$ and $T(X_2,Y_2)$.

In 2008, J. Sanwong and W. Sommanee [5] gave a necessary and sufficient condition for $T(X,Y)$ to be regular and found the largest regular subsemigroup of $T(X,Y)$. Then they characterized Green's relations and obtained a class of maximal inverse subsemigroups of $T(X,Y)$. Furthermore, a natural partial order on $T(X,Y)$ was studied in some detail in [4, 8].

Let E be an equivalence on X. Write

$$T_E(X) = \{\alpha \in T(X) : \forall (x,y) \in E, (x\alpha, y\alpha) \in E\},$$

then $T_E(X)$ is a subsemigroup of $T(X)$. Moreover, we see that $T_E(X)$ is $S(X)$, the semigroup of all continuous self-maps of the topological space X for which all E classes form a basis. In 2005, H. Pei [3] studied the regularity and Green's relations for $T_E(X)$. Moreover, in 2008, L. Sun, H. Pei and Z. Cheng [6] studied $T_E(X)$ with the natural partial order $\leq$ and investigated the condition under which $\alpha \leq \beta$ for two elements $\alpha, \beta \in T_E(X)$. Then they considered the compatibility of multiplication under $\leq$. Finally, the maximal, minimal and covering elements were described.

In 2010, L.-Z. Deng, J.-W. Zeng and B. Xu [1] defined a subsemigroup $T_{E^*}(X)$ of $T(X)$ by

$$T_{E^*}(X) = \{\alpha \in T(X) : \forall x, y \in X, (x,y) \in E \Leftrightarrow (x\alpha, y\alpha) \in E\}.$$

Similar to the semigroup $T_E(X)$, we obtain that $T_{E^*}(X)$ is a semigroup of continuous self-maps of the topological space X for which all E classes form a basis. In [1], the authors studied regularity and Green's relations for $T_{E^*}(X)$. In 2013, L. Sun and J. Sun [7] characterized the natural partial order on $T_{E^*}(X)$. Then they studied the compatibility and described the maximal (minimal) elements. In addition, they considered the greatest lower bound of two elements.

In this paper, we aim to generalize the results of [1] by defining a subset $T_{E^*}(X,Y)$ of $T_{E^*}(X)$ as follows. Let E be an equivalence on X and Y a nonempty subset of X. Define

$$T_{E^*}(X,Y) = \{\alpha \in T(X) : X\alpha \subseteq Y \text{ and } \forall x, y \in X, (x,y) \in E \Leftrightarrow (x\alpha, y\alpha) \in E\}.$$

Equivalently,

$$T_{E^*}(X,Y) = \{\alpha \in T_{E^*}(X) : X\alpha \subseteq Y\} = T_{E^*}(X) \cap T(X,Y).$$

In the next section, we give a necessary and sufficient condition for $T_{E^*}(X,Y)$ to be a subsemigroup of $T_{E^*}(X)$. Obviously, if $X = Y$, then $T_{E^*}(X,Y) = T_{E^*}(X)$. Hence $T_{E^*}(X)$ is a special case of $T_{E^*}(X,Y)$. Furthermore, if E is the universal relation, $E = X \times X$, then $T_{E^*}(X,Y)$ becomes $T(X,Y)$, as shown in Theorem 2.5. Moreover, it is not difficult to check that $T_{E^*}(X,Y)$ is a semigroup of all continuous self-maps of the topological space X for which all E classes form a basis carrying

X into a subspace Y, and is referred to as a semigroup of continuous functions (see [2] for details).

In section 3, we study regularity for $T_{E^*}(X,Y)$ and then the characterization of Green's relations will be considered in the last section.

2 Preliminaries

Let X/E be the quotient set where E is an equivalence on X. For each $\alpha \in T_{E^*}(X,Y)$, let

$$\pi(\alpha) = \{x\alpha^{-1} : x \in X\alpha\}$$

be the partition of X induced by α. Then $\pi(\alpha) = X/\ker(\alpha)$ where $\ker(\alpha) = \{(x,y) \in X \times X : x\alpha = y\alpha\}$. As in [1], for a subset A of X, we write

$$\pi_A(\alpha) = \{M \in \pi(\alpha) : M \cap A \neq \emptyset\}.$$

We also define

$$\tilde{\pi}_A(\alpha) = \{M \in \pi(\alpha) : M \cap A \cap Y \neq \emptyset\}.$$

It is clear that $\tilde{\pi}_A(\alpha)$ is an appropriate extension of $\pi_A(\alpha)$ in the sense that if $Y = X$, then $\tilde{\pi}_A(\alpha) = \pi_A(\alpha)$. Obviously, $\tilde{\pi}_A(\alpha) \subseteq \pi_A(\alpha)$.

We denote by $\Delta(X)$ the diagonal relation on X, that is, $\Delta(X) = \{(x,x) : x \in X\}$.

Let $\alpha \in T_{E^*}(X)$. The restriction of the equivalence E on $X\alpha$, denoted by E_α, is defined by

$$E_\alpha = \{(x,y) : x, y \in X\alpha, (x,y) \in E\} = E \cap (X\alpha \times X\alpha).$$

Then

$$X\alpha/E_\alpha = \{A \cap X\alpha : A \in X/E, A \cap X\alpha \neq \emptyset\} = \{A\alpha : A \in X/E\}.$$

Let E be an equivalence relation on a set X and let U, V be subsets of X and φ a mapping from U into V. If for any $u, v \in U$, $(u,v) \in E$ implies $(u\varphi, v\varphi) \in E$, then we say that φ is *E-preserving.* If $(u,v) \in E$ if and only if $(u\varphi, v\varphi) \in E$, then φ is said to be *E^*-preserving.*

For convenience, we state the following two lemmas appeared in [1] which will prove useful.

Lemma 2.1 ([1]). *Let E be an equivalence on X, $M \subseteq X$ and*

$$E_M = \{(x,y) \in E : x, y \in M\} = E \cap (M \times M).$$

Then the following statements hold.

(1) E_M is an equivalence on M.

(2) $M/E_M = \{A \cap M : A \in X/E, A \cap M \neq \emptyset\}$.

Lemma 2.2 ([1])**.** *Let $\alpha \in T_{E^*}(X)$ and $x, y \in X$. Then the following statements hold.*

(1) $(x, y) \notin E$ if and only if $(x\alpha, y\alpha) \notin E$.

(2) $|X/E| = |X\alpha/E_\alpha|$.

Next, we give a necessary and sufficient condition for $T_{E^*}(X,Y)$ to be a subsemigroup of $T_{E^*}(X)$. Clearly, $T_{E^*}(X,Y) = T_{E^*}(X) \cap T(X,Y)$. Therefore, we need only to find a condition that $T_{E^*}(X,Y)$ is nonempty.

Theorem 2.3. *$T_{E^*}(X,Y)$ is nonempty if and only if $|Y/E_Y| = |X/E|$.*

Proof. Assume that $\alpha \in T_{E^*}(X,Y) \neq \emptyset$. By Lemma 2.2 (2), we obtain that

$$|X/E| = |X\alpha/E_\alpha| \leq |Y/E_Y| \leq |X/E|.$$

Thus $|Y/E_Y| = |X/E|$.

Conversely, suppose that $|Y/E_Y| = |X/E|$. Then there is a bijection $\Phi : X/E \to Y/E_Y$. For each $A \in X/E$, choose $y_A \in A\Phi$ and define a function $\alpha : X \to Y$ by $x\alpha = y_A$ for each $x \in A$. It is clear that $\alpha \in T_{E^*}(X,Y) \neq \emptyset$. □

By the above theorem, since $T_{E^*}(X,Y)$ must not be empty, we will assume that Y is a subset of X such that $|Y/E_Y| = |X/E|$ in the remaining of this paper.

Lemma 2.4. *Let $\alpha \in T_{E^*}(X,Y)$, $E_{Y\alpha} = \{(x,y) : x, y \in Y\alpha, (x,y) \in E\}$ and $Y\alpha/E_{Y\alpha} = \{A \cap Y\alpha : A \in X/E, A \cap Y\alpha \neq \emptyset\}$. Then $Y\alpha/E_{Y\alpha} = \{(A \cap Y)\alpha : A \in X/E, A \cap Y \neq \emptyset\}$.*

Proof. Let $A \cap Y\alpha \in Y\alpha/E_{Y\alpha}$. Then $A \in X/E$ and $A \cap Y\alpha \neq \emptyset$. There exists $y \in A \cap Y\alpha$ which implies that $y \in A$ and $y = x\alpha$ for some $x \in Y$. Then there is $B \in X/E$ such that $x \in B \cap Y$ and hence $y = x\alpha \in (B \cap Y)\alpha$.

Next, we claim that $A \cap Y\alpha = (B \cap Y)\alpha$. Indeed, let $a \in A \cap Y\alpha$. Then $a \in A$ and $a \in Y\alpha$. We note that $(a, y) \in E$ since $y \in A$. Thus $a = b\alpha$ for some $b \in Y$ and so $(b\alpha, x\alpha) = (a, y) \in E$ which implies that $(b, x) \in E$. Hence $b \in B \cap Y$ and so $a = b\alpha \in (B \cap Y)\alpha$. Therefore, $A \cap Y\alpha \subseteq (B \cap Y)\alpha$. For the other containment, let $z\alpha \in (B \cap Y)\alpha$. Then $z \in B \cap Y$ from which it follows that $(x, z) \in E$ since $x \in B \cap Y$. Hence $(y, z\alpha) = (x\alpha, z\alpha) \in E$. Then $z\alpha \in Y\alpha$ and $z\alpha \in A$ since $y \in A \cap Y\alpha$. Thus $z\alpha \in A \cap Y\alpha$ and so $(B \cap Y)\alpha \subseteq A \cap Y\alpha$.

Therefore, $A \cap Y\alpha \in \{(A \cap Y)\alpha : A \in X/E, A \cap Y \neq \emptyset\}$ which implies that $Y\alpha/E_{Y\alpha} \subseteq \{(A \cap Y)\alpha : A \in X/E, A \cap Y \neq \emptyset\}$.

On the other hand, let $C \in X/E$ such that $C \cap Y \neq \emptyset$. There exists $p \in C \cap Y$ and then $p\alpha \in (C \cap Y)\alpha$ from which it follows that $p\alpha \in Y\alpha$ and $p\alpha \in C\alpha \subseteq D$ for some $D \in X/E$. Hence $p\alpha \in D \cap Y\alpha$.

We claim that $(C \cap Y)\alpha = D \cap Y\alpha$. Indeed, let $c\alpha \in (C \cap Y)\alpha$. Then $c \in C \cap Y$ and so $(c, p) \in E$ since $p \in C \cap Y$. Now, we have $(c\alpha, p\alpha) \in E$. Then $c\alpha \in Y\alpha$ and $c\alpha \in D$ since $p\alpha \in D \cap Y\alpha$. Thus $c\alpha \in D \cap Y\alpha$ and so $(C \cap Y)\alpha \subseteq D \cap Y\alpha$. For the other containment, let $d \in D \cap Y\alpha$. Then $d \in D$ and $d \in Y\alpha$. We note that $(d, p\alpha) \in E$ since $p\alpha \in D \cap Y\alpha$. We obtain $d = q\alpha$ for some $q \in Y$ and hence $(q\alpha, p\alpha) = (d, p\alpha) \in E$ which implies that $(q, p) \in E$. Then $q \in C \cap Y$. Hence $d = q\alpha \in (C \cap Y)\alpha$ and so $D \cap Y\alpha \subseteq (C \cap Y)\alpha$.

Therefore, $\{(A \cap Y)\alpha : A \in X/E, A \cap Y \neq \emptyset\} \subseteq Y\alpha/E_{Y\alpha}$. Consequently, $Y\alpha/E_{Y\alpha} = \{(A \cap Y)\alpha : A \in X/E, A \cap Y \neq \emptyset\}$. □

Theorem 2.5. *The following statements hold.*

(1) $E = X \times X$ *if and only if* $T_{E^*}(X, Y) = T(X, Y)$.

(2) $E = \Delta(X)$ *if and only if* $T_{E^*}(X, Y) = \{\alpha \in T(X, Y) : \alpha$ *is injective.*$\}$.

Proof. (1) ($\Rightarrow$) Suppose that $E = X \times X$. It remains to show that $T(X, Y) \subseteq T_{E^*}(X, Y)$. Let $\alpha \in T(X, Y)$. For each $x, y \in X$, we obtain that $(x, y) \in E$ if and only if $(x\alpha, y\alpha) \in E$ since $E = X \times X$. Therefore, $\alpha \in T_{E^*}(X, Y)$ which implies that $T(X, Y) \subseteq T_{E^*}(X, Y)$.

($\Leftarrow$) Assume that $T_{E^*}(X, Y) = T(X, Y)$. Obviously, $E \subseteq X \times X$, it remains to show that $X \times X \subseteq E$. Let a be a fixed element in Y, define a function $\alpha : X \to Y$ by $x\alpha = a$ for all $x \in X$. It is easy to see that $\alpha \in T(X, Y) = T_{E^*}(X, Y)$. Let $(x, y) \in X \times X$. We obtain $(x, y) \in E$ since $(x\alpha, y\alpha) = (a, a) \in E$. Hence $X \times X \subseteq E$.

(2) For convenience, we let $S = \{\alpha \in T(X, Y) : \alpha \text{ is injective.}\}$.

($\Rightarrow$) Suppose $E = \Delta(X)$. Let $\alpha \in T_{E^*}(X, Y)$. To show that $\alpha \in S$, let $x, y \in X$ with $x\alpha = y\alpha$. Then $(x\alpha, y\alpha) \in E = \Delta(X)$ which implies that $x = y$. Thus $\alpha \in S$ and so $T_{E^*}(X, Y) \subseteq S$. On the other hand, let $\alpha \in S$ and $(x, y) \in X \times X$. If $(x, y) \in E = \Delta(X)$, then $x = y$ and so $x\alpha = y\alpha$. Further $(x\alpha, y\alpha) \in E$. Conversely, if $(x\alpha, y\alpha) \in E = \Delta(X)$, then $x\alpha = y\alpha$ which implies that $x = y$ since $\alpha \in S$. Hence $(x, y) \in E$ and thus $\alpha \in T_{E^*}(X, Y)$. Therefore, $S \subseteq T_{E^*}(X, Y)$

($\Leftarrow$) Assume that $T_{E^*}(X, Y) = S$. By Theorem 2.3, we have $|Y/E_Y| = |X/E|$. Then we can write $X/E = \{A_i : i \in I\}$ and $Y/E_Y = \{B_i : i \in I\}$. Choose $b_i \in B_i$ for all $i \in I$. For each $A_i \in X/E$, define a function $\alpha : X \to Y$ by $z\alpha = b_i$ where $z \in A_i$. It is easy to verify that $\alpha \in T_{E^*}(X, Y) = S$. To show that $E = \Delta(X)$, it remains to show that $E \subseteq \Delta(X)$. Let $(x, y) \in E$. Then $x, y \in A_i$ for some $A_i \in X/E$. By the definition of α, we have $x\alpha = b_i = y\alpha$ which implies that $x = y$ since $\alpha \in S$. Therefore, $E = \Delta(X)$. □

For $\alpha \in T_{E^*}(X, Y)$, let

$$E(\alpha) = \{A\alpha^{-1} : A \in X\alpha/E_\alpha\}.$$

The following two theorems are consequences of Theorems 1.4 and 1.5 in [1], respectively.

Theorem 2.6. *Let* $\alpha \in T_{E^*}(X, Y)$ *and* $A \in X\alpha/E_\alpha$. *Then* $A\alpha^{-1} \in X/E$.

Theorem 2.7. *Let* $\alpha \in T_{E^*}(X, Y)$. *Then* $E(\alpha) = X/E$.

3 Regularity

In this section, we characterize regular elements in $T_{E^*}(X,Y)$ and then give a necessary and sufficient condition for $T_{E^*}(X,Y)$ to be regular.

In [5], the authors defined a subset F of $T(X,Y)$ by

$$F = \{\alpha \in T(X,Y) : X\alpha \subseteq Y\alpha\}$$

and proved that F is the largest regular subsemigroup of $T(X,Y)$. Now, we define the subset F_{E^*} of $T_{E^*}(X,Y)$ by $F_{E^*} = F \cap T_{E^*}(X,Y)$. Equivalently,

$$F_{E^*} = \{\alpha \in T_{E^*}(X,Y) : X\alpha \subseteq Y\alpha\} = \{\alpha \in T_{E^*}(X,Y) : X\alpha = Y\alpha\}.$$

It is clear that F_{E^*} is an appropriate extension of F in the sense that if $E = X \times X$, then $F_{E^*} = F$. In addition, if $X = Y$, then $F_{E^*} = T_{E^*}(X)$.

By Lemma 2.2 in [5], the authors proved that F is a right ideal of $T(X,Y)$. Hence we obtain the following lemma immediately.

Lemma 3.1. *F_{E^*} is a right ideal of $T_{E^*}(X,Y)$. Consequently, it is a subsemigroup of $T_{E^*}(X,Y)$.*

Lemma 3.2. *If $F_{E^*} \neq \emptyset$, then $Y \cap A \neq \emptyset$ for all $A \in X/E$*

Proof. Let $\alpha \in F_{E^*}$, $A \in X/E$ and $x \in A$. Then $x\alpha \in X\alpha \subseteq Y\alpha$ which implies that $x\alpha = y\alpha$ for some $y \in Y$. Hence $(x,y) \in E$ and so $y \in A$. Therefore, $y \in Y \cap A \neq \emptyset$. □

By Lemma 3.2, we have the following corollary.

Corollary 3.3. *If $Y \cap A = \emptyset$ for some $A \in X/E$, then $F_{E^*} = \emptyset$.*

Theorem 3.4. *Let $\alpha \in T_{E^*}(X,Y)$. Then α is regular if and only if $\alpha \in F_{E^*}$ and $A \cap Y\alpha \neq \emptyset$ for any $A \in X/E$.*

Proof. ($\Rightarrow$) Suppose that α is regular. Then $\alpha = \alpha\beta\alpha$ for some $\beta \in T_{E^*}(X,Y)$. Let $A \in X/E$ and $x \in A$. Then $x\beta = y$ for some $y \in Y$. Moreover, since $y\alpha = y\alpha\beta\alpha$, we obtain $(y\alpha, y\alpha\beta\alpha) \in E$ which implies that $(x\beta\alpha, y\alpha\beta\alpha) \in E$. Hence $(x, y\alpha) \in E$. Therefore, $y\alpha \in A \cap Y\alpha \neq \emptyset$. In addition, we have $X\alpha = X\alpha\beta\alpha \subseteq Y\beta\alpha \subseteq Y\alpha$ and so $\alpha \in F_{E^*}$.

($\Leftarrow$) For each $A \in X/E$, we have $A \cap Y\alpha \neq \emptyset$ which implies that $(A \cap Y\alpha)\alpha^{-1} \cap Y$ is nonempty. Let b_A be a fixed element in $(A \cap Y\alpha)\alpha^{-1} \cap Y$. Define a function β by

$$x\beta = \begin{cases} a & \text{if } x \in Y\alpha \quad \text{where } a \in x\alpha^{-1} \cap Y \\ b_A & \text{if } x \in A \setminus Y\alpha \end{cases}$$

The proof that $\beta \in T_{E^*}(X,Y)$ is routine. By the definition of β, we obtain $x\alpha\beta\alpha \in x\alpha\alpha^{-1}\alpha = \{x\alpha\}$ for all $x \in X$ since $X\alpha = Y\alpha$. Thus $\alpha\beta\alpha = \alpha$. Consequently, α is regular. □

Now, we give a necessary and sufficient condition for $T_{E^*}(X,Y)$ to be regular.

Theorem 3.5. *$T_{E^*}(X,Y)$ is regular if and only if the following statements hold.*

(1) $|Y/E_Y|$ is finite.

(2) Either $E\cap(Y\times Y)=\Delta(Y)$ or $X=Y$.

Proof. ($\Rightarrow$) Suppose that $T_{E^*}(X,Y)$ is regular. By Theorem 3.4, we conclude that F_{E^*} is nonempty. Hence $Y\cap A\neq\emptyset$ for all $A\in X/E$ by Lemma 3.2.

(1) Assume that $|Y/E_Y|$ is infinite. We note that $|X/E|=|Y/E_Y|$. Choose $B_1\in Y/E_Y$, then $|(Y/E_Y)\setminus\{B_1\}|=|Y/E_Y|=|X/E|$ since $|Y/E_Y|$ is infinite. Then we can write $X/E=\{A_i : i\in I\}$ and $Y/E_Y\setminus\{B_1\}=\{B_i : i\in I\}$. We assume that $B_1=Y\cap A_1$ for some $A_1\in X/E$ and define a function α by

$$x\alpha=b_i \text{ where } b_i\in B_i \text{ for any } x\in A_i,\ i\in I.$$

Clearly, $\alpha\in T_{E^*}(X,Y)$ and $A_1\cap Y\alpha=\emptyset$. By Theorem 3.4, α is not regular and so $T_{E^*}(X,Y)$ is not regular which is a contradiction. Hence $|Y/E_Y|$ is finite.

(2) Suppose to the contrary that $E\cap(Y\times Y)\neq\Delta(Y)$ and $X\neq Y$. Then there exists $A\in X/E$ such that $A\setminus Y$ and $Y\cap A$ are nonempty. By $E\cap(Y\times Y)\neq\Delta(Y)$, there is a class $B\in X/E$ such that $|B\cap Y|>1$. Let a,b be distinct elements in $B\cap Y$. Moreover, since $|X/E|=|Y/E_Y|$, we can write $(X/E)\setminus\{A\}=\{A_i : i\in I\}$ and $(Y/E_Y)\setminus\{B\cap Y\}=\{B_i : i\in I\}$. Choose $c_i\in B_i$ and define a function $\alpha : X\to Y$ by

$$x\alpha=\begin{cases} a & \text{if } x\in Y\cap A\\ b & \text{if } x\in A\setminus Y\\ c_i & \text{if } x\in A_i\end{cases}$$

It is obvious that $\alpha\in T_{E^*}(X,Y)$ and $b\in X\alpha\setminus Y\alpha$. Hence $\alpha\notin F_{E^*}$ which implies that α is not regular. It leads to a contradiction.

($\Leftarrow$) Assume that (1) and (2) hold. If $|Y/E_Y|$ is finite and $X=Y$, then $T_{E^*}(X,Y)$ is regular by Theorem 3.2 of [1]. Now, we suppose that $|Y/E_Y|$ is finite and $E\cap(Y\times Y)=\Delta(Y)$. We note that $|X/E|=|Y/E_Y|$ is also finite and $|Y\cap A|=1$ for all $A\in X/E$. Let $a\in T_{E^*}(X,Y)$ and $x\alpha\in X\alpha$. Then $x\in A$ for some $A\in X/E$ and there is $y\in Y\cap A$. Moreover, since $E\cap(Y\times Y)=\Delta(Y)$ we obtain that $A\alpha$ is a singleton. Thus $x\alpha=y\alpha\in Y\alpha$ and so $X\alpha\subseteq Y\alpha$. Hence $\alpha\in F_{E^*}$. Let $B\in X/E$. We obtain $B\alpha^{-1}=C$ for some $C\in X/E$ since $|X/E|=|Y/E_Y|$ is finite. There is $z\in Y\cap C$ and hence $z\alpha\in C\alpha\cap Y\alpha\subseteq B\cap Y\alpha\neq\emptyset$. Therefore, α is regular. □

4 Green's relations

To study Green's relations, we introduce some definitions for using throughout this paper. Actually, we extend the notions of E-admissibility and E^*-admissibility presented in [1].

Let $\alpha, \beta \in T_{E^*}(X,Y)$ and let φ be a mapping from $\pi(\alpha)$ into $\pi(\beta)$. We say that φ is $\tilde{E}$-*admissible* if and only if for each $A \in X/E$, there exists $B \in X/E$ such that

$$\pi_A(\alpha)\varphi \subseteq \tilde{\pi}_B(\beta).$$

Equivalently, $\varphi : \pi(\alpha) \to \pi(\beta)$ is $\tilde{E}$-admissible if and only if for each $A \in X/E$, there exists $B \in X/E$ such that for each $P \in \pi_A(\alpha)$, $B \cap P\varphi \cap Y \neq \emptyset$. If φ is a bijection such that φ and φ^{-1} are $\tilde{E}$-admissible, then φ is called $\tilde{E}^*$-*admissible*. We remark that if $X = Y$, then the notions of E-admissibility (resp. $\tilde{E}$-admissibility) and $\tilde{E}$-admissibility (resp. $\tilde{E}^*$-admissibility) are the same.

If $\eta \in T_{E^*}(X,Y)$, then denote by η_* the map from $\pi(\eta)$ onto $X\eta$ by $(x\eta^{-1})\eta_* = x$ for each $x \in X\eta$.

We note that, in general, if $X \neq Y$, then the semigroup $T_{E^*}(X,Y)$ does not contain the identity element. Hence $T_{E^*}(X,Y)^1 \neq T_{E^*}(X,Y)$. We are now in a position to prove some characterization of Green's $\mathscr{L}$-relation.

Theorem 4.1. *Let $\alpha, \beta \in F_{E^*}$. Then the following statements are equivalent.*

(1) $(\alpha,\beta) \in \mathscr{L}$ in $T_{E^*}(X,Y)$.

(2) $X\alpha = X\beta$.

(3) There exists an $\tilde{E}^*$-admissible bijection $\varphi : \pi(\alpha) \to \pi(\beta)$ such that $\alpha_* = \varphi\beta_*$.

Proof. The implication (1) $\Rightarrow$ (2) follows from [1, Theorem 2.1] .

(2) $\Rightarrow$ (3). Suppose that (2) holds. Then we have $X\alpha = Y\alpha = Y\beta = X\beta$ which implies that $E_\alpha = E_{Y\beta}$. For each $A \in X/E$, we have $A\alpha \in X\alpha/E_\alpha = Y\beta/E_{Y\beta}$ and so there exist $B \in X/E$ such that $A\alpha = (B \cap Y)\beta$ by Lemma 2.4. Similarly, $A\beta = (C \cap Y)\alpha$ for some $C \in X/E$.

Define a function $\varphi : \pi(\alpha) \to \pi(\beta)$ by $(x\alpha^{-1})\varphi = x\beta^{-1}$ for all $x \in X\alpha = X\beta$. It is easy to verify that φ is a bijection. Furthermore, for each $x\alpha^{-1} \in \pi(\alpha)$, we have $(x\alpha^{-1})\varphi\beta_* = (x\beta^{-1})\beta_* = x = (x\alpha^{-1})\alpha_*$. Thus $\alpha_* = \varphi\beta_*$.

Next, we show that φ is $\tilde{E}$-admissible. Let $A \in X/E$. Then there exists $B \in X/E$ such that $A\alpha = (B \cap Y)\beta$. Let $P \in \pi_A(\alpha)$. Then $P = x\alpha^{-1}$ for some $x \in X\alpha$ and $P \cap A \neq \emptyset$ which implies that there is $y \in P \cap A$. Hence $x = y\alpha \in A\alpha = (B \cap Y)\beta$ and so there exists $z \in B \cap Y$ such that $x = z\beta$. We see that $z \in x\beta^{-1} = (x\alpha^{-1})\varphi = P\varphi$. Thus $z \in P\varphi \cap B \cap Y \neq \emptyset$ and so φ is an $\tilde{E}$-admissible. Similarly, we can prove that φ^{-1} is also $\tilde{E}$-admissible. Therefore, φ is an $\tilde{E}^*$-admissible.

(3) $\Rightarrow$ (1). Suppose that (3) holds. Then for any $x \in A \in X/E$, $(x\alpha)\alpha^{-1}\varphi \cap B \cap Y \neq \emptyset$ for some $B \in X/E$. Define a function $\eta : X \to Y$ by $x\eta \in (x\alpha)\alpha^{-1}\varphi \cap B \cap Y$.

Next, we show that $\eta \in T_{E^*}(X,Y)$. Let $x, y \in X$ and $(x,y) \notin E$. Then $(x\alpha, y\alpha) \notin E$ since $\alpha \in T_{E^*}(X,Y)$. We see that $\Big(\{x\alpha\} \times \{y\alpha\}\Big) \cap E = \emptyset$. Moreover,

$$(x\alpha)\alpha^{-1}\varphi\beta_* = (x\alpha)\alpha^{-1}\alpha_* = x\alpha \text{ and } (y\alpha)\alpha^{-1}\varphi\beta_* = (y\alpha)\alpha^{-1}\alpha_* = y\alpha.$$

Thus $[(x\alpha)\alpha^{-1}\varphi]\beta = \{x\alpha\}$ and $[(y\alpha)\alpha^{-1}\varphi]\beta = \{y\alpha\}$ which implies that

$$\Big([(x\alpha)\alpha^{-1}\varphi]\beta \times [(y\alpha)\alpha^{-1}\varphi]\beta\Big) \cap E = \emptyset.$$

Hence

$$\Big([(x\alpha)\alpha^{-1}\varphi] \times [(y\alpha)\alpha^{-1}\varphi]\Big) \cap E = \emptyset.$$

Therefore, $(x\eta, y\eta) \notin E$. Conversely, let $x, y \in X$ be such that $(x, y) \in E$. Then there exists $A \in X/E$ such that $x, y \in A$. Hence $x\eta \in (x\alpha)\alpha^{-1}\varphi \cap B \cap Y$ and $y\eta \in (y\alpha)\alpha^{-1}\varphi \cap B \cap Y$ for some $B \in X/E$ and so $(x\eta, y\eta) \in E$. Therefore, $\eta \in T_{E^*}(X, Y)$.

Let $x \in X$. Then $x\eta\beta \in [(x\alpha)\alpha^{-1}\varphi \cap B \cap Y]\beta \subseteq [(x\alpha)\alpha^{-1}\varphi]\beta = \{x\alpha\}$. Thus $x\eta\beta = x\alpha$ for any $x \in X$. Hence $\alpha = \eta\beta$. Similarly, we can find a function $\theta \in T_{E^*}(X, Y)$ such that $\beta = \theta\alpha$. Consequently, $(\alpha, \beta) \in \mathscr{L}$. □

By the above theorem, if $X = Y$, then we obtain Theorem 2.1 of [1].

Recall that for each element a in a semigroup S, we denote $\mathscr{L}$-class, $\mathscr{R}$-class, $\mathscr{H}$-class, $\mathscr{D}$-class and $\mathscr{J}$-class containing a by L_a, R_a, H_a, D_a and J_a, respectively.

Theorem 4.2. *For $\alpha \in T_{E^*}(X, Y)$, the following statements hold.*

(1) If $\alpha \in T_{E^*}(X, Y) \setminus F_{E^*}$, then $L_\alpha = \{\alpha\}$.

(2) If $\alpha \in F_{E^*}$, then $L_\alpha = \{\beta \in F_{E^*} : X\alpha = X\beta\}$.

Proof. (1) Let $\alpha \in T_{E^*}(X, Y) \setminus F_{E^*}$ and let $\beta \in L_\alpha$. Then $\alpha\mathscr{L}\beta$ which implies that $\alpha = \eta\beta$ and $\beta = \theta\alpha$ for some $\eta, \theta \in T_{E^*}(X, Y)^1$. If $\eta, \theta \in T_{E^*}(X, Y)$, then $X\alpha = X\eta\beta = X\eta\theta\alpha \subseteq Y\alpha$. Thus $\alpha \in F_{E^*}$ which is a contradiction and so $\eta = 1$ or $\theta = 1$. Hence $\beta = \alpha$.

(2) Let $\alpha \in F_{E^*}$ and let $\beta \in L_\alpha$. Then $(\alpha, \beta) \in \mathscr{L}$ which implies that $\alpha = \eta\beta$ and $\beta = \theta\alpha$ for some $\eta, \theta \in T_{E^*}(X, Y)^1$. The case $\alpha = \beta$ is obvious. If $\alpha \neq \beta$, then $\eta, \theta \in T_{E^*}(X, Y)$. We obtain $X\beta = X\theta\alpha \subseteq Y\alpha = Y\eta\beta \subseteq Y\beta$ which implies that $\beta \in F_{E^*}$. In addition, $X\alpha = X\beta$ by Theorem 4.1. The other containment is clear. □

Now, we have already done for Green's $\mathscr{L}$-relation of $T_{E^*}(X, Y)$. To study the remaining Green's relations, we introduce some definitions for using throughout this paper. Let $\alpha \in T_{E^*}(X, Y)$, as in [1], the authors defined $Z(\alpha) = \{A \in X/E : A \cap X\alpha = \emptyset\}$. Moreover, we define

$$Z_Y(\alpha) = \{A \in X/E : A \cap X\alpha = \emptyset, A \cap Y \neq \emptyset\} = \{A \in Z(\alpha) : A \cap Y \neq \emptyset\}.$$

It is clear that $Z_Y(\alpha)$ is an appropriate extension of $Z(\alpha)$ in the sense that if $Y = X$, then $Z_Y(\alpha) = Z(\alpha)$. Furthermore, we have the following two lemmas.

Lemma 4.3. *Let $\alpha \in F_{E^*}$. Then $\pi(\alpha) = \pi_Y(\alpha)$.*

Proof. Let $\alpha \in F_{E^*}$. We have known that $\pi_Y(\alpha) \subseteq \pi(\alpha)$. It remains to show that $\pi(\alpha) \subseteq \pi_Y(\alpha)$. Let $M \in \pi(\alpha)$. Then $M = x\alpha^{-1}$ for some $x \in X\alpha = Y\alpha$ and so there exists $y \in Y$ such that $y\alpha = x$. Hence $y \in x\alpha^{-1} = M$ which implies that $M \cap Y \neq \emptyset$. Thus $M \in \pi_Y(\alpha)$. Therefore, $\pi(\alpha) = \pi_Y(\alpha)$. □

Lemma 4.4. *Let $\alpha \in F_{E^*}$. Then $Z(\alpha) = Z_Y(\alpha)$.*

Proof. Let $\alpha \in F_{E^*}$. Then $\alpha \in T_{E^*}(X,Y)$ and $X\alpha = Y\alpha$. Obviously, $Z_Y(\alpha) \subseteq Z(\alpha)$. It remains to show that $Z(\alpha) \subseteq Z_Y(\alpha)$. We note by Lemma 4.3 that $\pi(\alpha) = \pi_Y(\alpha)$. Let $A \in Z(\alpha)$. Then there is $x \in A$ and $(x\alpha)\alpha^{-1} \in \pi(\alpha) = \pi_Y(\alpha)$. Hence $(x\alpha)\alpha^{-1} \cap Y \neq \emptyset$ which implies that there exists $y \in (x\alpha)\alpha^{-1} \cap Y$. Thus $y\alpha = x\alpha$ and $(x\alpha, y\alpha) \in E$. Then $(x,y) \in E$ from which it follows that $y \in A \cap Y \neq \emptyset$ and so $A \in Z_Y(\alpha)$. □

Lemma 4.5. *Let $\alpha, \beta \in T_{E^*}(X,Y)$. If $\alpha = \beta\gamma$ for some $\gamma \in T_{E^*}(X,Y)$, then* $\ker(\beta) \subseteq \ker(\alpha)$.

Proof. Suppose that $\alpha = \beta\gamma$ for some $\gamma \in T_{E^*}(X,Y)$. Let $(a,b) \in \ker(\beta)$. Then $a\beta = b\beta$ and so $a\alpha = a\beta\gamma = b\beta\gamma = b\alpha$. Thus $(a,b) \in \ker(\alpha)$. □

Now, we prove the characterization of Green's $\mathscr{R}$-relation.

Theorem 4.6. *Let $\alpha, \beta \in T_{E^*}(X,Y)$ be such that $\alpha \neq \beta$. Then the following statements are equivalent.*

(1) $(\alpha, \beta) \in \mathscr{R}$.

(2) $\pi(\alpha) = \pi(\beta)$ *and* $|Z(\alpha)| = |Z_Y(\alpha)| = |Z_Y(\beta)| = |Z(\beta)|$.

(3) There exists $\delta \in T_{E^*}(X,Y)$ such that $\delta|_{X\alpha} : X\alpha \to X\beta$ is a bijection and $\beta = \alpha\delta$.

There exists $\sigma \in T_{E^*}(X,Y)$ such that $\sigma|_{X\beta} : X\beta \to X\alpha$ is a bijection and $\alpha = \beta\sigma$.

Proof. (1) ⇒ (2). Suppose that $(\alpha, \beta) \in \mathscr{R}$. Then $\alpha = \beta\theta$, $\beta = \alpha\eta$ for some $\eta, \theta \in T_{E^*}(X,Y)^1$. We see that θ and η belong to $T_{E^*}(X,Y)$ since $\alpha \neq \beta$. Hence $\ker(\alpha) = \ker(\beta)$ by Lemma 4.5 and thus $\pi(\alpha) = X/\ker(\alpha) = X/\ker(\beta) = \pi(\beta)$.

Let $A \in Z(\alpha)$. We claim that $A\eta \subseteq B$ for some $B \in Z(\beta)$. Now, we have $A \in X/E$, $A \cap X\alpha = \emptyset$ and $A\eta \subseteq B$ for some $B \in X/E$. Assume to the contrary that $B \notin Z(\beta)$. Then $B \cap X\alpha\eta = B \cap X\beta \neq \emptyset$ since $X\alpha\eta = X\beta$. Hence there exists $b \in B \cap X\alpha\eta$ and so $b = x\alpha\eta$ for some $x \in X$. Let $a \in A$. Then $a\eta \in A\eta \subseteq B$ from which it follows that $(a\eta, x\alpha\eta) \in E$ and so $(a, x\alpha) \in E$. Hence $x\alpha \in A \cap X\alpha \neq \emptyset$ which contradicts to $A \in Z(\alpha)$. Thus $A\eta \subseteq B$ for some $B \in Z(\beta)$. Moreover, $B \cap Y \neq \emptyset$ since $\eta \in T_{E^*}(X,Y)$. Hence $A\eta \subseteq B$ for some $B \in Z_Y(\beta)$.

Now, we show that $|Z(\alpha)| = |Z_Y(\beta)|$. For each $A \in Z(\alpha)$, $A\eta \subseteq B$ for some $B \in Z_Y(\beta)$. Define a function $\Psi : Z(\alpha) \to Z_Y(\beta)$ by $A\Psi = B$. It is easy to verify that Ψ is injective since $\eta \in T_{E^*}(X,Y)$. Thus $|Z(\alpha)| \leq |Z_Y(\beta)|$. By the same argument as above, we can show that $|Z(\beta)| \leq |Z_Y(\alpha)|$. Hence $|Z(\alpha)| \leq |Z_Y(\beta)| \leq |Z(\beta)| \leq |Z_Y(\alpha)| \leq |Z(\alpha)|$ and so $|Z(\alpha)| = |Z_Y(\alpha)| = |Z_Y(\beta)| = |Z(\beta)|$.

(2) ⇒ (3). Suppose that $\pi(\alpha) = \pi(\beta)$ and $|Z(\alpha)| = |Z_Y(\alpha)| = |Z_Y(\beta)| = |Z(\beta)|$. We can write $Z(\alpha) = \{A_i : i \in I\}$ and $Z_Y(\beta) = \{B_i : i \in I\}$. Choose $y_i \in Y \cap B_i$ for each $i \in I$ and define an E^*-preserving mapping

$$\rho : \bigcup_{A_i \in Z(\alpha)} A_i \to Y \text{ by } z\rho = y_i \text{ for each } z \in A_i.$$

Define a function $\delta : X \to Y$ by

$$x\delta = \begin{cases} x\alpha^{-1}\beta_* & \text{, if } x \in X\alpha \\ y\alpha^{-1}\beta_* & \text{, if } x \in A \setminus X\alpha \text{ where } A \in X/E \text{ with } A \cap X\alpha \neq \emptyset \text{ and } y \in A \cap X\alpha \\ x\rho & \text{, if } x \in A \quad\quad \text{ where } A \in X/E \text{ with } A \cap X\alpha = \emptyset. \end{cases}$$

The proof that $\delta \in T_{E^*}(X,Y)$ is routine. Now, we show that $\delta|_{X\alpha} : X\alpha \to X\beta$ is a bijection. Let $y \in X\beta$. Then $y\beta^{-1} \in \pi(\beta) = \pi(\alpha)$ which implies that $y\beta^{-1}\alpha_* \in X\alpha$. Hence $y\beta^{-1}\alpha_*\delta = y\beta^{-1}\alpha_*\alpha^{-1}\beta_* = y$ from which it follows that $\delta|_{X\alpha}$ is a surjection. Let $x, y \in X\alpha$ with $x\delta = y\delta$. Then $x\alpha^{-1}\beta_* = y\alpha^{-1}\beta_*$ and so $x = y$. Hence $\delta|_{X\alpha}$ is an injection. Consequently, $\delta|_{X\alpha} : X\alpha \to X\beta$ is a bijection and $x\alpha\delta = x\alpha\alpha^{-1}\beta_* = x\beta$ for any $x \in X$. Therefore $\beta = \alpha\delta$. Similarly, we can find a function $\sigma \in T_{E^*}(X,Y)$ such that $\sigma|_{X\beta} : X\beta \to X\alpha$ is a bijection and $\alpha = \beta\sigma$.

The implication (3) ⇒ (1) is clear. □

By Lemma 4.4 and Theorem 4.6, we obtain the following corollary which covers Theorem 2.2 of [1].

Corollary 4.7. *Let $\alpha, \beta \in F_{E^*}$. Then the following statements are equivalent.*

(1) $(\alpha, \beta) \in \mathscr{R}$ *in* $T_{E^*}(X,Y)$.

(2) $\pi(\alpha) = \pi(\beta)$ *and* $|Z(\alpha)| = |Z(\beta)|$.

(3) There exists $\delta \in T_{E^*}(X,Y)$ *such that* $\delta|_{X\alpha} : X\alpha \to X\beta$ *is a bijection and* $\beta = \alpha\delta$.

There exists $\sigma \in T_{E^*}(X,Y)$ *such that* $\sigma|_{X\beta} : X\beta \to X\alpha$ *is a bijection and* $\alpha = \beta\sigma$.

Lemma 4.8. *Let $\alpha, \beta \in T_{E^*}(X,Y)$. If $\pi(\alpha) = \pi(\beta)$, then either both α and β are in F_{E^*}, or neither is in F_{E^*}.*

Proof. Assume that $\pi(\alpha) = \pi(\beta)$ and $\alpha \in F_{E^*}$. It suffices to show $\beta \in F_{E^*}$. Let $x\beta \in X\beta$. Then $(x\beta)\beta^{-1} \in \pi(\beta) = \pi(\alpha)$ which implies that $(x\beta)\beta^{-1} = (z\alpha)\alpha^{-1}$ for some $z \in X$. We have $z\alpha \in X\alpha \subseteq Y\alpha$ implies $z\alpha = y\alpha$ for some $y \in Y$. Thus $y \in (z\alpha)\alpha^{-1} = (x\beta)\beta^{-1}$ and so $x\beta = y\beta \in Y\beta$. Therefore, $\beta \in F_{E^*}$. □

By Theorem 4.6, Corollary 4.7 and Lemma 4.8, we have the following result.

Corollary 4.9. *For $\alpha \in T_{E^*}(X,Y)$, the following statements hold.*

(1) If $\alpha \in F_{E^*}$, *then* $R_\alpha = \{\beta \in F_{E^*} : \pi(\alpha) = \pi(\beta) \text{ and } |Z(\alpha)| = |Z(\beta)|\}$.

(2) If $\alpha \in T_{E^*}(X,Y) \setminus F_{E^*}$, *then*

$$\begin{aligned} R_\alpha \;=\; & \{\alpha\} \cup \{\beta \in T_{E^*}(X,Y) \setminus F_{E^*} : \pi(\alpha) = \pi(\beta) \\ & \text{and } |Z(\alpha)| = |Z_Y(\alpha)| = |Z_Y(\beta)| = |Z(\beta)|\}. \end{aligned}$$

Theorem 4.10. *Let $\alpha, \beta \in F_{E^*}$. Then the following statements are equivalent.*

(1) $(\alpha, \beta) \in \mathscr{H}$ *in* $T_{E^*}(X, Y)$.

(2) $\pi(\alpha) = \pi(\beta)$ *and* $X\alpha = X\beta$.

(3) There exists an $\tilde{E}^*$*-admissible bijection* $\varphi : \pi(\alpha) \to \pi(\beta)$ *such that* $\alpha_* = \varphi\beta_*$.
There exist $\delta, \sigma \in T_{E^*}(X, Y)$ *such that* $\delta|_{X\alpha} : X\alpha \to X\beta, \sigma|_{X\beta} : X\beta \to X\alpha$ *are bijections and* $\beta = \alpha\delta, \alpha = \beta\sigma$.

Proof. (1) $\Rightarrow$ (2). Suppose that $(\alpha, \beta) \in \mathscr{H}$ in $T_{E^*}(X, Y)$. Then $(\alpha, \beta) \in \mathscr{L}$ and $(\alpha, \beta) \in \mathscr{R}$. By Theorems 4.1 and 4.6, we obtain that $\pi(\alpha) = \pi(\beta)$ and $X\alpha = X\beta$.

(2) $\Rightarrow$ (3). Suppose that $\pi(\alpha) = \pi(\beta)$ and $X\alpha = X\beta$. Then

$$Z(\alpha) = \{A \in X/E : A \cap X\alpha = \emptyset\} = \{A \in X/E : A \cap X\beta = \emptyset\} = Z(\beta)$$

which implies that $|Z_Y(\alpha)| = |Z_Y(\beta)|$. By Lemma 4.4, we obtain $|Z(\alpha)| = |Z_Y(\alpha)| = |Z_Y(\beta)| = |Z(\beta)|$. Hence (3) is true by Theorems 4.1 and 4.6.

The implication (3) $\Rightarrow$ (1) follows by Theorems 4.1 and 4.6. □

As an immediate consequence of the previous theorems, we get the following corollary.

Corollary 4.11. *For* $\alpha \in T_{E^*}(X, Y)$, *the following statements hold.*

(1) If $\alpha \in T_{E^*}(X, Y) \setminus F_{E^*}$, *then* $H_\alpha = \{\alpha\}$.

(2) If $\alpha \in F_{E^*}$, *then* $H_\alpha = \{\beta \in F_{E^*} : \pi(\alpha) = \pi(\beta)$ *and* $X\alpha = X\beta\}$.

Next, we consider Green's relation $\mathscr{D}$.

Theorem 4.12. *Let $\alpha, \beta \in F_{E^*}$. Then the following statements are equivalent.*

(1) $(\alpha, \beta) \in \mathscr{D}$ *in* $T_{E^*}(X, Y)$.

(2) $|Z(\alpha)| = |Z(\beta)|$ *and there exists* $\delta \in T_{E^*}(X, Y)$ *such that* $\delta|_{X\alpha} : X\alpha \to X\beta$ *is a bijection.*

Proof. (1) $\Rightarrow$ (2). Suppose that $(\alpha, \beta) \in \mathscr{D}$ in $T_{E^*}(X, Y)$. Then $(\alpha, \gamma) \in \mathscr{L}$ and $(\gamma, \beta) \in \mathscr{R}$ for some $\gamma \in T_{E^*}(X, Y)$. By Theorem 4.2 and Corollary 4.9, we have $\pi(\gamma) = \pi(\beta)$, $X\alpha = Y\alpha = Y\gamma = X\gamma$ and $|Z(\gamma)| = |Z(\beta)|$. Moreover, since $X\alpha = X\gamma$, we obtain $Z(\alpha) = Z(\gamma)$ which implies that $|Z(\alpha)| = |Z(\gamma)|$. Hence $|Z(\alpha)| = |Z(\gamma)| = |Z(\beta)|$. We note that $|Z_Y(\beta)| = |Z(\beta)| = |Z(\gamma)|$ since $\beta \in F_{E^*}$.

Next, let $Z(\gamma) = \{A_i : i \in I\}$ and $Z_Y(\beta) = \{B_i : i \in I\}$. Then we choose $y_i \in Y \cap B_i$ for each $i \in I$. Define a function

$$\rho : \bigcup_{A_i \in Z(\gamma)} A_i \to Y \text{ by } z\rho = y_i \text{ where } z \in A_i.$$

We see that ρ is an E^*-preserving mapping. For each $A \in X/E$, define

$$x\delta = \begin{cases} x\gamma^{-1}\beta_* & \text{if } x \in A \cap X\gamma \quad \text{where } A \cap X\gamma \neq \emptyset \\ b & \text{if } x \in A \setminus X\gamma \quad \text{where } A \cap X\gamma \neq \emptyset \text{ and } b \in (A \cap X\gamma)\gamma^{-1}\beta \\ x\rho & \text{if } x \in A \quad \text{where } A \cap X\gamma = \emptyset. \end{cases}$$

The proof that $\delta \in T_{E^*}(X,Y)$ is routine. Next, we will show that $\delta|_{X\alpha} : X\alpha \to X\beta$ is a bijection. Let $z \in X\beta$. Then $z\beta^{-1}\gamma_* \in X\gamma = X\alpha$. Hence $z\beta^{-1}\gamma_*\delta = z\beta^{-1}\gamma_*\gamma^{-1}\beta_* = z$ which implies that $\delta|_{X\alpha}$ is surjective. Let $x, y \in X\alpha$ with $x\delta = y\delta$. Then $x\gamma^{-1}\beta_* = y\gamma^{-1}\beta_*$. Thus $x = y$ implies $\delta|_{X\alpha}$ is injective.

(2) $\Rightarrow$ (1). Suppose that (2) holds. Define $\gamma : X \to Y$ by $x\gamma = x\beta\delta^{-1} \in X\alpha$. We see that $\gamma \in T_{E^*}(X,Y)$ since β and δ are E^*-preserving. Moreover, since $\alpha, \beta \in F_{E^*}$, we obtain $X\gamma = X\beta\delta^{-1} = X\alpha = Y\alpha$ and $Y\gamma = Y\beta\delta^{-1} = X\beta\delta^{-1} = X\alpha$. Then $X\alpha = Y\alpha = Y\gamma = X\gamma$ and so $(\alpha,\gamma) \in \mathscr{L}$ by Theorem 4.1. We see that $Z(\alpha) = Z(\gamma)$ since $X\alpha = X\gamma$ from which it follows that $|Z(\alpha)| = |Z(\gamma)|$. In addition, we have $\gamma \in F_{E^*}$ since $X\gamma = Y\gamma$. Hence $|Z(\gamma)| = |Z_Y(\gamma)|$ by Lemma 4.4. Similarly, we obtain $|Z(\beta)| = |Z_Y(\beta)|$ and so $|Z_Y(\gamma)| = |Z(\gamma)| = |Z(\alpha)| = |Z(\beta)| = |Z_Y(\beta)|$. Furthermore,

$$\begin{aligned} \ker(\gamma) &= \{(x,y) : x\gamma = y\gamma\} \\ &= \{(x,y) : x\beta\delta^{-1} = y\beta\delta^{-1}\} \\ &= \{(x,y) : x\beta = y\beta\} \\ &= \ker(\beta). \end{aligned}$$

Hence $\pi(\gamma) = X/\ker(\gamma) = X/\ker(\beta) = \pi(\beta)$. Therefore, $(\gamma,\beta) \in \mathscr{R}$ by Theorem 4.6. Consequently, $(\alpha,\beta) \in \mathscr{D}$ since $(\alpha,\gamma) \in \mathscr{L}$ and $(\gamma,\beta) \in \mathscr{R}$. □

We remark that the above theorem extends Theorem 2.4 of [1].

Theorem 4.13. *For $\alpha \in T_{E^*}(X,Y)$, the following statements hold.*

(1) If $\alpha \in T_{E^*}(X,Y) \setminus F_{E^*}$, then $D_\alpha = R_\alpha$.

(2) If $\alpha \in F_{E^*}$, then

$$D_\alpha = \{\beta \in F_{E^*} : \beta \text{ satisfies the condition (2) of Theorem 4.12}\}.$$

Proof. (1) Let $\alpha \in T_{E^*}(X,Y) \setminus F_{E^*}$ and let $\beta \in D_\alpha$. Then $\alpha\mathscr{L}\gamma$ and $\gamma\mathscr{R}\beta$ for some $\gamma \in T_{E^*}(X,Y)$. By Theorem 4.2, we obtain that $\gamma = \alpha$ and and thus $\alpha\mathscr{R}\beta$. Hence $\beta \in R_\alpha$. The other containment is clear since $\mathscr{R} \subseteq \mathscr{D}$.

(2) Let $\alpha \in F_{E^*}$ and let $\beta \in D_\alpha$. Then $\alpha\mathscr{L}\gamma$ and $\gamma\mathscr{R}\beta$ for some $\gamma \in T_{E^*}(X,Y)$. It is clear that $\beta \in F_{E^*}$ by Theorem 4.2 and Corollary 4.9. The remaining part of (2) has a straightforward proof. □

Finally, we consider Green's $\mathscr{J}$-relation.

Lemma 4.14. *Let $\alpha, \beta \in T_{E^*}(X,Y)$ and $A \in X/E$. If $\alpha = \lambda\beta\rho$ for some $\lambda, \rho \in T_{E^*}(X,Y)$, then $|X\alpha| \leq |X\beta|$ and $A\alpha \subseteq (B \cap Y)\beta\rho$ for some class B.*

Proof. It is clear that $|X\alpha| = |X\lambda\beta\rho| \leq |X\lambda\beta| \leq |X\beta|$ and $A\alpha = A\lambda\beta\rho \subseteq (B \cap Y)\beta\rho$ for some class B. □

Lemma 4.15. *Let $\alpha \in F_{E^*}$. Then $A\alpha \subseteq (A \cap Y)\alpha$ for all $A \in X/E$.*

Proof. Let $A \in X/E$ and $a\alpha \in A\alpha$. Then $a\alpha \in X\alpha \subseteq Y\alpha$ which implies that $a\alpha = y\alpha$ for some $y \in Y$. Hence $(a\alpha, y\alpha) \in E$ implies $(a, y) \in E$. Thus $y \in A$. Therefore, $a\alpha = y\alpha \in (A \cap Y)\alpha$. □

Lemma 4.16. *Let $\alpha, \beta \in F_{E^*}$. If $(\alpha, \beta) \in \mathscr{D}$ in $T_{E^*}(X, Y)$, then $|X\alpha| = |X\beta|$ and there exist $\rho, \tau \in T_{E^*}(X, Y)$ such that for any $A \in X/E$, $A\alpha \subseteq (B \cap Y)\beta\rho$ and $A\beta \subseteq (C \cap Y)\alpha\tau$ for some $B, C \in X/E$.*

Proof. Assume that $(\alpha, \beta) \in \mathscr{D}$ in $T_{E^*}(X, Y)$. Then, by Theorem 4.12, it is clear that $|X\alpha| = |X\beta|$. In addition, since $(\beta, \alpha) \in \mathscr{D}$, there exists $\rho \in T_{E^*}(X, Y)$ such that $\rho|_{X\beta} : X\beta \to X\alpha$ is a bijection by Theorem 4.12. Let $A \in X/E$. Then $A\alpha \subseteq G \cap X\alpha = (G' \cap X\beta)\rho$ for some E-classes G and G' since $\rho \in T_{E^*}(X, Y)$ and $\rho|_{X\beta} : X\beta \to X\alpha$ is a bijection. Moreover, we obtain $G' \cap X\beta = B\beta$ for some $B \in X/E$ since $\beta \in T_{E^*}(X, Y)$. Thus $A\alpha \subseteq B\beta\rho \subseteq (B \cap Y)\beta\rho$ by Lemma 4.15. Similarly, there exists $\tau \in T_{E^*}(X, Y)$ such that $A\beta \subseteq (C \cap Y)\alpha\tau$ for some $C \in X/E$. □

Theorem 4.17. *Let $\alpha, \beta \in T_{E^*}(X, Y)$ be such that $\alpha \neq \beta$. Then $(\alpha, \beta) \in \mathscr{J}$ if and only if either*

(1) $\pi(\alpha) = \pi(\beta)$ and $|Z(\alpha)| = |Z_Y(\alpha)| = |Z_Y(\beta)| = |Z(\beta)|$; or

(2) $|X\alpha| = |X\beta|$ and there exist $\rho, \tau \in T_{E^*}(X, Y)$ such that for any $A \in X/E$, $A\alpha \subseteq (B \cap Y)\beta\rho$ and $A\beta \subseteq (C \cap Y)\alpha\tau$ for some $B, C \in X/E$.

Proof. (⇒). Suppose that $(\alpha, \beta) \in \mathscr{J}$. Then $\alpha = \theta\beta\eta$, $\beta = \mu\alpha\nu$ for some $\theta, \eta, \mu, \nu \in T_{E^*}(X, Y)^1$. If $\theta = 1 = \mu$, then $\alpha = \beta\eta$ and $\beta = \alpha\nu$ which implies that $(\alpha, \beta) \in \mathscr{R}$. Hence (1) holds by Theorem 4.6. If $\theta \in T_{E^*}(X, Y)$ or $\mu \in T_{E^*}(X, Y)$, then we have $\alpha = \lambda\beta\xi$ and $\beta = \rho\alpha\delta$ for some $\lambda, \rho \in T_{E^*}(X, Y)$ and $\xi, \delta \in T_{E^*}(X, Y)^1$. For example, if $\theta = 1$ and $\mu \in T_{E^*}(X, Y)$, then $\alpha = \beta\eta = \mu\alpha\nu\eta = \mu\beta(\eta\nu\eta)$. We split the proof into four cases.
Case 1. $\xi = 1 = \delta$. In this case, $\alpha = \lambda\beta$ and $\beta = \rho\alpha$. Thus, $(\alpha, \beta) \in \mathscr{L} \subseteq \mathscr{D}$. Then, by Theorem 4.13 and Lemma 4.16, we obtain (1) or (2).
Case 2. $\xi, \delta \in T_{E^*}(X, Y)$. We have (2) holds by Lemma 4.14.
Case 3. $\xi = 1$ and $\delta \in T_{E^*}(X, Y)$. In this case, $\alpha = \lambda\beta$ and $\beta = \rho\alpha\delta$. Then $\alpha = \lambda\beta = \lambda\rho\alpha\delta = (\lambda\rho\lambda)\beta\delta$, which reduces to Case 2.
Case 4. $\xi \in T_{E^*}(X, Y)$ and $\delta = 1$. In this case, $\alpha = \lambda\beta\xi$ and $\beta = \rho\alpha$. Then $\beta = \rho\alpha = \rho\lambda\beta\xi = (\rho\lambda\rho)\alpha\xi$, which also reduces to Case 2.

(⇐). If (1) holds, then $(\alpha, \beta) \in \mathscr{R} \subseteq \mathscr{J}$ by Theorem 4.6. Now, we suppose (2) holds. Define a function $\theta : X \to Y$ by $x\theta \in x\alpha\rho^{-1}\beta^{-1} \cap Y$. Next, we will prove that $\theta \in T_{E^*}(X, Y)$. Let $(m, n) \in E$. Then $(m\alpha, n\alpha) \in E$. Moreover, since $\beta, \rho \in T_{E^*}(X, Y)$, we get $(m\theta, n\theta) \in m\alpha\rho^{-1}\beta^{-1} \times n\alpha\rho^{-1}\beta^{-1} \subseteq E$. Conversely, let

$(a,b)\notin E$. Then $(a\alpha,b\alpha)\notin E$. We obtain that $\left(a\alpha\rho^{-1}\beta^{-1}\times b\alpha\rho^{-1}\beta^{-1}\right)\cap E=\emptyset$. Thus $(a\theta,b\theta)\notin E$. Consequently, $\theta\in T_{E^*}(X,Y)$.

To show that $\theta\beta\rho=\alpha$, let $x\in X$. Then $x\theta\in x\alpha\rho^{-1}\beta^{-1}\cap Y$ which implies that $x\theta\beta\rho\in\left(x\alpha\rho^{-1}\beta^{-1}\cap Y\right)\beta\rho$. Hence there exists $y\in x\alpha\rho^{-1}\beta^{-1}\cap Y$ such that $y\beta\rho=x\theta\beta\rho$ and so $x\theta\beta\rho=y\beta\rho=x\alpha$. Similarly, we can construct a function $\mu\in T_{E^*}(X,Y)$ such that $\mu\alpha\tau=\beta$. Therefore, $(\alpha,\beta)\in\mathscr{J}$. □

Finally, we characterize Green's $\mathscr{D}$ and $\mathscr{J}$ relations on F_{E^*} when the set of all equivalent classes is finite.

Theorem 4.18. *Let $\alpha,\beta\in F_{E^*}$, $|X/E|=n$ (n is a positive integer). Then the following statements are equivalent.*

(1) $(\alpha,\beta)\in\mathscr{D}$.

(2) $(\alpha,\beta)\in\mathscr{J}$.

Proof. The implication (1) ⇒ (2) is clear.

(2) ⇒ (1). Suppose that $(\alpha,\beta)\in\mathscr{J}$. Clearly, if $\alpha=\beta$, then $(\alpha,\beta)\in\mathscr{D}$. Now, we assume that $\alpha\neq\beta$. Then the items (1) or (2) of Theorem 4.17 holds. If the item (1) of Theorem 4.17 is true, then $(\alpha,\beta)\in\mathscr{R}\subseteq\mathscr{D}$ by Theorem 4.6. Suppose that the item (2) of Theorem 4.17 holds, that is, $|X\alpha|=|X\beta|$ and there exist $\rho,\tau\in T_{E^*}(X,Y)$ such that for any $A\in X/E$, $A\alpha\subseteq(B\cap Y)\beta\rho$ and $A\beta\subseteq(C\cap Y)\alpha\tau$ for some $B,C\in X/E$.

Let

$$X/E=\{A_i : A_i\cap A_j=\emptyset, i\neq j\}.$$

For any $A\in X/E$, $A\alpha\subseteq(B\cap Y)\beta\rho$ and $A\beta\subseteq(C\cap Y)\alpha\tau$ for some $B,C\in X/E$. Let $A_i,A_j\in X/E$ with $A_i\neq A_j$. Then there exist $B_i,B_j\in X/E$ with $B_i\neq B_j$ such that

$$A_i\alpha\subseteq(B_i\cap Y)\beta\rho \text{ and } A_j\alpha\subseteq(B_j\cap Y)\beta\rho.$$

Similarly, there exist $C_i,C_j\in X/E$ with $C_i\neq C_j$ such that

$$B_i\beta\subseteq(C_i\cap Y)\alpha\tau \text{ and } B_j\beta\subseteq(C_j\cap Y)\alpha\tau.$$

Let $A_1\in X/E$. Then there exists $B_1\in X/E$ such that $A_1\alpha\subseteq(B_1\cap Y)\beta\rho$. Thus

$$|A_1\alpha|\leq|(B_1\cap Y)\beta\rho|\leq|B_1\beta\rho|\leq|B_1\beta|.$$

Similarly, there exists $A_2\in X/E$ such that $B_1\beta\subseteq(A_2\cap Y)\alpha\tau$. Thus

$$|B_1\beta|\leq|(A_2\cap Y)\alpha\tau|\leq|A_2\alpha\tau|\leq|A_2\alpha|.$$

Moreover, there exists $B_2\in X/E$ such that $A_2\alpha\subseteq(B_2\cap Y)\beta\rho$. Thus

$$|A_2\alpha|\leq|(B_2\cap Y)\beta\rho|\leq|B_2\beta\rho|\leq|B_2\beta|.$$

Repeat above processes, we get

$$|A_1\alpha|\leq|B_1\beta|\leq|A_2\alpha|\leq|B_2\beta|\leq\cdots\leq|A_i\alpha|\leq|B_i\beta|\leq\cdots$$

Then there exists $A_k \in X/E$ such that $A_k = A_1$ since $|X/E|$ is finite. Thus

$$|A_1\alpha| = |B_1\beta| = \cdots = |A_{k-1}\alpha| = |B_{k-1}\beta|$$

and $A_i\alpha \subseteq (B_i \cap Y)\beta\rho, B_i\beta \subseteq (A_{i+1} \cap Y)\alpha\tau$ where $i = 1, 2, \ldots, k-1$. For any $A \in (X/E)\setminus\{A_i : i = 1, 2, \ldots, k-1\}$, $A\alpha \subseteq (B\cap Y)\beta\rho$ for some $B \in (X/E)\setminus\{B_i : i = 1, 2, \ldots, k-1\}$. In addition, for any $B \in (X/E) \setminus \{B_i : i = 1, 2, \ldots, k-1\}$, $B\beta \subseteq (A \cap Y)\alpha\tau$ for some $A \in (X/E) \setminus \{A_i : i = 1, 2, \ldots, k-1\}$.

Repeat above processes between $(X/E)\setminus\{A_i : i = 1, 2, \ldots, k-1\}$ and $(X/E)\setminus\{B_i : i = 1, 2, \ldots, k-1\}$. Finally, we get $|A_i\alpha| = |B_i\beta|$ where $i = 1, 2, \ldots, n$ and so

$$X/E = \{A_i : i = 1, 2, \ldots, n\} = \{B_i : i = 1, 2, \ldots, n\}.$$

Then there exist bijections $\delta_i : A_i\alpha \to B_i\beta$ where $i = 1, 2, \ldots, n$. Let

$$x\delta = \begin{cases} x\delta_i & \text{, if } x \in A_i\alpha \\ y\delta_i & \text{, if } x \in A \setminus X\alpha \qquad \text{where } y \in A_i\alpha, \quad A_i\alpha \subseteq A \in X/E. \end{cases}$$

The prove that $\delta \in T_{E^*}(X, Y)$ is routine. We can see that $X\alpha = \bigcup_{i=1}^{n} A_i\alpha$ and $X\beta = \bigcup_{i=1}^{n} B_i\beta$ since $X/E = \{A_i : i = 1, 2, \ldots, n\} = \{B_i : i = 1, 2, \ldots, n\}$. Thus $\delta|_{X\alpha} : X\alpha \to X\beta$ is a bijection since $\delta_i : A_i\alpha \to B_i\beta$ for all $i = 1, 2, \ldots, n$ are bijections. Moreover, since X/E is finite, we have $Z(\alpha) = Z(\beta) = \emptyset$. Thus $|Z(\alpha)| = |Z(\beta)| = 0$. By Theorem 4.12, $(\alpha, \beta) \in \mathscr{D}$. □

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