



# Coincidence Point Theorems for Geraghty's Type Contraction in Generalized Metric Spaces Endowed with a Directed Graph

Narawadee Phudolsitthiphat and Anchalee Khemphet <sup>1</sup>

Center of Excellence in Mathematics and Applied Mathematics,  
Department of Mathematics, Faculty of Science,  
Chiang Mai University, Chiang Mai 50200, Thailand  
e-mails : [narawadee\\_n@hotmail.co.th](mailto:narawadee_n@hotmail.co.th) (N. Phudolsitthiphat)  
[anchalee.k@cmu.ac.th](mailto:anchalee.k@cmu.ac.th) (A. Khemphet)

**Abstract :** In this work, we present some existence results for coincidence point theorems for Geraghty's type contraction mappings in a space with two generalized metrics endowed with a directed graph. Further, some examples are provided to support our main theorems. Finally, these results can be applied to the integral equation.

**Keywords :** fixed point; coincidence point; common fixed point; generalized metric

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## 1 Introduction

There was a link between fixed point theory and optimization. Many ideas in fixed point theory were applied to solve optimization problems. Specifically, solutions to structural optimizations and inverse problems can be found using some tools in fixed point theory. Consequently, finding the sufficient condition for the existence of fixed points of some related functions have been an interesting topic in the literature. Endowing a space with a function that has a nice property is another approach to generate known results in this area.

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<sup>1</sup>Corresponding author email: [anchalee.k@cmu.ac.th](mailto:anchalee.k@cmu.ac.th) (Anchalee Khemphet)

In 1973, Geraghty [2] defined a class  $\mathcal{B}$  of all functions  $\beta : [0, \infty) \rightarrow [0, 1)$  such that

$$\beta(t_n) \rightarrow 1 \implies t_n \rightarrow 0,$$

which can be used to generalize the Banach's contraction principle.

Recently, Juan Martínez-Moreno, Wutiphol Sintunavarat and Yeol Je Cho [5] were interested in finding a fixed point of Geraghty's type contraction mappings on a space with two metrics. They proved new common fixed point theorems for Geraghty's type contraction mappings using the monotone property and  $g$ -uniform continuity with two metrics and provided some examples to illustrate their results.

**Definition 1.1.** [5] Let  $(X, d)$  and  $(Y, d')$  be two metric spaces, and  $g$  a self-mapping on  $X$ . A mapping  $f : X \rightarrow Y$  is said to be  $g$ -uniformly continuous on  $X$  if for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $d(gx, gy) < \delta$  for  $x, y \in X$ , then  $d'(fx, fy) < \epsilon$ . If  $g$  is the identity mapping, then  $f$  is uniformly continuous on  $X$ .

In addition, Mohamed Jleli and Bessem Samet [4] introduced a generalized metric space that covered many topological spaces, for example, standard metric spaces,  $b$ -metric spaces, dislocated metric spaces, and modular spaces.

For every  $x \in X$  and a function  $D : X \times X \rightarrow [0, +\infty]$ , denote

$$C(D, X, x) = \{\{x_n\} \subseteq X : \lim_{n \rightarrow \infty} D(x_n, x) = 0\}.$$

**Definition 1.2.** [4] A mapping  $D : X \times X \rightarrow [0, +\infty]$  is said to be a generalized metric on a set  $X$  if it satisfies the following conditions. For any  $x, y \in X$ ,

(D<sub>1</sub>) if  $D(x, y) = 0$  then  $x = y$ ;

(D<sub>2</sub>)  $D(x, y) = D(y, x)$ ;

(D<sub>3</sub>) there exists  $K > 0$  such that if  $\{x_n\} \in C(D, X, x)$ , then

$$D(x, y) \leq K \limsup_{n \rightarrow \infty} D(x_n, y).$$

We call  $(X, D)$  a generalized metric space, also known as a JS-metric space.

**Definition 1.3.** [4] Let  $(X, D)$  be a JS-metric space. Let  $\{x_n\}$  be a sequence in  $X$  and  $x \in X$ .

(i)  $\{x_n\}$  is said to  $D$ -converge to  $x$  if  $\{x_n\} \in C(D, X, x)$ .

(ii)  $\{x_n\}$  is said to be  $D$ -Cauchy if  $\lim_{n, m \rightarrow \infty} D(x_n, x_m) = 0$ .

(iii)  $X$  is said to be  $D$ -complete if every  $D$ -Cauchy sequence in  $X$   $D$ -converges to some element in  $X$ .

**Proposition 1.4.** [4] Let  $(X, D)$  be a JS-metric space. Let  $\{x_n\}$  be a sequence in  $X$  and  $x, y \in X$ . If  $\{x_n\}$   $D$ -converges to  $x$  and  $y$ , then  $x = y$ .

**Definition 1.5.** [4] Let  $(X, D)$  be a JS-metric space. A mapping  $f : X \rightarrow X$  is said to be continuous at a point  $x_0 \in X$  if, for any  $\{x_n\} \in C(D, X, x_0)$ ,  $\lim_{n \rightarrow \infty} D(fx_n, fx_0) = 0$ . A mapping  $f : X \rightarrow X$  is said to be continuous if it is continuous at each  $x$  in  $X$ .

Inspired by the above work, we study some existence results for coincidence point theorems for Geraghty's type contraction mappings in a space with two generalized metrics endowed with a directed graph.

## 2 Main Results

First, we recall some standard graph notations and terminology. Let  $X \neq \emptyset$  and  $\Delta$  a diagonal of  $X \times X$ . Suppose that  $G$  is a directed graph such that the set  $V(G)$  of its vertices coincides with  $X$  and  $\Delta \subseteq E(G)$ , where  $E(G)$  is the set of the edges of  $G$ . Assume that  $G$  has no parallel edges. A directed graph  $G$  is said to be transitive if, for any  $x, y, z \in V(G)$ , if  $(x, y)$  and  $(y, z)$  are in  $E(G)$ , then  $(x, z)$  is in  $E(G)$ .

**Definition 2.1.** Let  $(X, D)$  be a JS-metric space, and let  $f$  and  $g$  be two self-mappings on  $X$ . Suppose that  $G$  is a directed graph. Then,  $f$  is said to be  $(g, D)$ -edge preserving w.r.t  $G$  if

$$(gx, gy) \in E(G) \Rightarrow (fx, fy) \in E(G) \text{ and } D(gx, gy) < \infty.$$

**Definition 2.2.** Let  $(X, D)$  be a JS-metric space and  $G$  a directed graph. A mapping  $f : (X, D) \rightarrow (X, D)$  is said to be  $G$ -continuous if for any  $x \in X$ , if there exists a sequence  $\{x_n\}$  in  $C(D, X, x)$  such that  $(x_n, x_{n+1}) \in E(G)$  for each  $n \in \mathbb{N}$ , then  $\{fx_n\}$  is in  $C(D, X, fx)$ .

**Definition 2.3.** Let  $(X, D)$  be a JS-metric space and  $x \in X$ . Suppose that  $G$  is a directed graph. We say that the triple  $(X, D, G)$  has the property A if for any sequence  $\{x_n\}$  in  $C(D, X, x)$ , if  $(x_n, x_{n+1}) \in E(G)$  for each  $n \in \mathbb{N}$ , then  $(x_n, x) \in E(G)$ .

Based on the original idea of Geraghty, we define the class  $\Theta$  of all functions  $\theta : [0, \infty] \rightarrow [0, 1)$  such that

$$\theta(t_n) \rightarrow 1 \implies t_n \rightarrow 0.$$

Then, a new class of the Geraghty's type contractions is defined as follows.

**Definition 2.4.** Let  $(X, D)$  be a JS-metric space endowed with a directed graph  $G$ , and let  $f$  and  $g$  be two self-mappings on  $X$ . Then, the pair  $(f, g)$  is said to be a  $\theta$ -M-contraction w.r.t  $D$  if

- (1)  $f$  is  $(g, D)$ -edge preserving w.r.t  $G$ ;

- (2) there exists a function  $\theta \in \Theta$  such that for all  $x, y \in X$ , if  $(gx, gy) \in E(G)$ , then

$$D(fx, fy) \leq \theta(M(gx, gy))M(gx, gy), \quad (2.1)$$

where  $M(gx, gy) = \max\{D(gx, gy), D(gx, fx), D(gy, fy)\}$ .

Let  $X$  be a space with a directed graph  $G$ . Suppose that  $f$  and  $g$  are two self-mappings on  $X$ . Denote the set of all coincidence points of  $f$  and  $g$  by

$$C(f, g) = \{u \in X : fu = gu\},$$

the set of all common fixed points of  $f$  and  $g$  by

$$Cm(f, g) = \{u \in X : fu = u = gu\},$$

and

$$X(f, g) = \{u \in X : (gu, fu) \in E(G)\}.$$

**Definition 2.5.** Let  $(X, D)$  and  $(Y, D')$  be two JS-metric spaces, and  $g$  a self-mapping on  $X$ . The mapping  $f : X \rightarrow Y$  is said to be  $g$ -Cauchy on  $X$  if for any sequence  $\{x_n\}$  in  $X$ , we have that  $\{fx_n\}$  is a  $D'$ -Cauchy sequence in  $(Y, D')$  whenever  $\{gx_n\}$  is a  $D$ -Cauchy sequence in  $(X, D)$ .

For any two generalized metrics  $D$  and  $D'$  on  $X$ , the notation  $D \geq D'$  represents  $D(u, v) \geq D'(u, v)$  for every  $u, v \in X$ . If there exists  $u_0, v_0 \in X$  such that  $D(u_0, v_0) < D'(u_0, v_0)$ , then we use the notation  $D \not\geq D'$ . We can define other inequalities similarly as above.

**Lemma 2.6.** Let  $(X, D)$  be a JS-metric space endowed with a directed graph  $G$ . Suppose that  $f$  and  $g$  are two self-mappings on  $X$  such that  $(f, g)$  is a  $\theta$ - $M$ -contraction w.r.t  $D$ . Then, for any  $x, y \in C(f, g)$ ,

- (1)  $D(gx, gx) = 0$ ;  
 (2) if  $(gx, gy) \in E(G)$ , then  $gx = gy$ .

*Proof.* (1) Let  $x \in C(f, g)$ . That is,  $fx = gx$ . Since  $(gx, gx) \in \Delta \subseteq E(G)$ ,  $M(gx, gx) = \max\{D(gx, gx), D(gx, fx), D(gx, gx)\} = D(gx, gx) < \infty$ . Consider

$$D(gx, gx) = D(fx, fx) \leq \theta(D(gx, gx))D(gx, gx).$$

Since  $0 \leq \theta(t) < 1$ ,  $D(gx, gx) = 0$ .

- (2) Let  $x, y \in C(f, g)$  such that  $(gx, gy) \in E(G)$ . That is,  $fx = gx$  and  $fy = gy$ . From (1),  $D(gx, fx) = D(gy, fy) = 0$ . Since  $(f, g)$  is a  $\theta$ - $M$ -contraction w.r.t.  $D$ ,  $M(gx, gy) = \max\{D(gx, gy), D(gx, fx), D(gy, fy)\} = D(gx, gy) < \infty$ . Consider

$$\begin{aligned} D(gx, gy) &= D(fx, fy) \\ &\leq \theta(M(gx, gy))M(gx, gy) \\ &= \theta(D(gx, gy))D(gx, gy). \end{aligned}$$

Since  $0 \leq \theta(t) < 1$ ,  $D(gx, gy) = 0$ . Therefore,  $gx = gy$ . □

**Theorem 2.7.** *Let  $(X, D')$  be a  $D'$ -complete JS-metric space endowed with a directed graph  $G$ , and let  $D$  be another generalized metric on  $X$ . Suppose that*

- (1)  $f$  and  $g$  are self-mappings on  $X$  such that  $(f, g)$  is a  $\theta$ - $M$ -contraction w.r.t  $D$ ;
- (2)  $g : (X, D') \rightarrow (X, D')$  is continuous and  $f(X) \subseteq g(X)$ ;
- (3)  $G$  is transitive;
- (4) if  $D \not\preceq D'$ , then  $f : (X, D) \rightarrow (X, D')$  is  $g$ -Cauchy on  $X$ ;
- (5)  $f : (X, D') \rightarrow (X, D')$  is  $G$ -continuous;
- (6)  $f$  and  $g$  commute;
- (7)  $X(f, g) \neq \emptyset$ .

Then,  $C(f, g) \neq \emptyset$ . Moreover, if  $(gx, gy) \in E(G)$  for any  $x, y \in C(f, g)$ , then  $Cm(f, g) \neq \emptyset$ .

*Proof.* From assumption (7), there exists an  $x_0 \in X$  such that  $(gx_0, fx_0) \in E(G)$ . Since  $f(X) \subseteq g(X)$  and  $f(x_0) \in X$ , one can construct a sequence  $\{x_n\}$  in  $X$  so that

$$gx_{n+1} = fx_n$$

for all  $n \in \mathbb{N}$ . If  $gx_{n_0} = gx_{n_0+1}$  for some  $n_0 \in \mathbb{N}$ , then  $x_{n_0}$  is a coincidence point of  $f$  and  $g$ . Assume that  $gx_n \neq gx_{n+1}$  for each  $n \in \mathbb{N}$ . Then,  $D(gx_n, gx_{n+1}) \neq 0$  for all  $n \in \mathbb{N}$ .

Since  $(gx_0, gx_1) = (gx_0, fx_0) \in E(G)$  and  $f$  is  $(g, D)$ -edge preserving w.r.t  $G$ ,  $(gx_1, gx_2) = (fx_0, fx_1) \in E(G)$  and  $D(gx_1, gx_2) < \infty$ . Repeat this recursively, we can conclude that

$$(gx_n, gx_{n+1}) \in E(G) \text{ and } D(gx_n, gx_{n+1}) < \infty \text{ for each } n \in \mathbb{N}. \quad (2.2)$$

Next, we will show that  $\lim_{n \rightarrow \infty} D(gx_n, gx_{n+1}) = 0$ . Consider

$$\begin{aligned} D(gx_{n+1}, gx_{n+2}) &= D(fx_n, fx_{n+1}) \\ &\leq \theta(M(gx_n, gx_{n+1}))M(gx_n, gx_{n+1}) \\ &< M(gx_n, gx_{n+1}) \end{aligned} \quad (2.3)$$

and

$$\begin{aligned} M(gx_n, gx_{n+1}) &= \max\{D(gx_n, gx_{n+1}), D(gx_n, fx_n), D(gx_{n+1}, fx_{n+1})\} \\ &= \max\{D(gx_n, gx_{n+1}), D(gx_{n+1}, gx_{n+2})\}. \end{aligned}$$

If  $M(gx_n, gx_{n+1}) = D(gx_{n+1}, gx_{n+2})$ , then, by (2.3),

$$D(gx_{n+1}, gx_{n+2}) < D(gx_{n+1}, gx_{n+2}).$$

This is a contradiction. Thus, for all  $n \geq 1$ , we obtain that

$$M(gx_n, gx_{n+1}) = D(gx_n, gx_{n+1}). \quad (2.4)$$

Therefore,

$$D(gx_{n+1}, gx_{n+2}) < D(gx_n, gx_{n+1}) \text{ for all } n \in \mathbb{N}.$$

Then,  $\{D(gx_n, gx_{n+1})\}$  is nonnegative and decreasing. Consequently, there exists  $r \geq 0$  such that  $\lim_{n \rightarrow \infty} D(gx_n, gx_{n+1}) = r$ . Suppose that  $r > 0$ . From (2.3) and (2.4), we have that

$$\frac{D(gx_{n+1}, gx_{n+2})}{D(gx_n, gx_{n+1})} = \frac{D(gx_{n+1}, gx_{n+2})}{M(gx_n, gx_{n+1})} \leq \theta(M(gx_n, gx_{n+1})) < 1.$$

It follows that  $\lim_{n \rightarrow \infty} \theta(M(gx_n, gx_{n+1})) = 1$ . Since  $\theta \in \Theta$ ,  $\lim_{n \rightarrow \infty} D(gx_n, gx_{n+1}) = \lim_{n \rightarrow \infty} M(gx_n, gx_{n+1}) = 0$ , a contradiction. Thus,

$$\lim_{n \rightarrow \infty} D(gx_n, gx_{n+1}) = 0. \quad (2.5)$$

Now, we claim that  $\{gx_n\}$  is a  $D$ -Cauchy sequence. Suppose that this is not true. That is, there exists an  $\epsilon > 0$  such that for any  $k \in \mathbb{N}$ , there are subsequences  $\{x_{n_k}\}$  and  $\{x_{m_k}\}$  of  $\{x_n\}$  so that  $D(gx_{n_k}, gx_{m_k}) \geq \epsilon$  for  $m_k \geq n_k \geq k$ .

Note that, by (2.2),  $(gx_{n_k}, gx_{m_k}) \in E(G)$  for all  $k \in \mathbb{N}$  since  $G$  is transitive. Then,

$$\begin{aligned} D(gx_{n_k}, gx_{m_k}) &= D(fx_{n_k-1}, fx_{m_k-1}) \\ &\leq \theta(M(gx_{n_k-1}, gx_{m_k-1}))M(gx_{n_k-1}, gx_{m_k-1}), \end{aligned}$$

where

$$M(gx_{n_k-1}, gx_{m_k-1}) = \max\{D(gx_{n_k-1}, gx_{m_k-1}), D(gx_{n_k-1}, fx_{n_k-1}), D(gx_{m_k-1}, fx_{m_k-1})\}.$$

If  $M(gx_{n_k-1}, gx_{m_k-1})$  coincides with either  $D(gx_{n_k-1}, fx_{n_k-1})$  or  $D(gx_{m_k-1}, fx_{m_k-1})$ , then, by (2.5),  $\lim_{k \rightarrow \infty} D(gx_{n_k}, gx_{m_k}) = 0$ . This contradicts to the fact that  $\{gx_n\}$  is not a  $D$ -Cauchy sequence. Thus,  $M(gx_{n_k-1}, gx_{m_k-1}) = D(gx_{n_k-1}, gx_{m_k-1})$ . Since  $n_k$  and  $m_k$  are arbitrary, this is valid for each  $n_k$  and  $m_k$ . Therefore,

$$D(gx_{n_k}, gx_{m_k}) \leq \prod_{i=1}^{n_k} \theta(M(gx_{n_k-i}, gx_{m_k-i}))M(gx_0, gx_{m_k-n_k}).$$

Let  $i_k \in \{1, 2, \dots, n_k\}$  so that

$$\theta(M(gx_{n_k-i_k}, gx_{m_k-i_k})) = \max\{\theta(M(gx_{n_k-i}, gx_{m_k-i})) : 1 \leq i \leq n_k\}.$$

Define  $\eta = \limsup_{k \rightarrow \infty} \{\theta(M(gx_{n_k - i_k}, gx_{m_k - i_k}))\}$ . If  $\eta < 1$ , then  $\lim_{k \rightarrow \infty} D(gx_{n_k}, gx_{m_k}) = 0$ , a contradiction. If  $\eta = 1$ , without loss of generality, we may assume that

$$\lim_{k \rightarrow \infty} \theta(M(gx_{n_k - i_k}, gx_{n_k + m_k - i_k})) = 1.$$

Then, we have that

$$\lim_{k \rightarrow \infty} M(gx_{n_k - i_k}, gx_{n_k + m_k - i_k}) = 0$$

because  $\theta \in \Theta$ . This implies that there exists  $k_0 \in \mathbb{N}$  such that

$$M(gx_{n_{k_0} - i_{k_0}}, gx_{n_{k_0} + m_{k_0} - i_{k_0}}) < \frac{\epsilon}{2}.$$

Consider

$$\begin{aligned} \epsilon &\leq D(gx_{n_{k_0}}, gx_{n_{k_0} + m_{k_0}}) \\ &\leq \prod_{j=1}^{i_{k_0}} \theta(M(gx_{n_{k_0} - j}, gx_{n_{k_0} + m_{k_0} - j})) M(gx_{n_{k_0} - i_{k_0}}, gx_{n_{k_0} + m_{k_0} - i_{k_0}}) \\ &< \frac{\epsilon}{2}. \end{aligned}$$

This is a contradiction. Therefore,  $\{gx_n\}$  is a  $D$ -Cauchy sequence.

Finally, we show that  $\{gx_n\}$  is also a  $D'$ -Cauchy sequence. Notice that if  $D \geq D'$ , we are done. Assume that  $D \not\geq D'$ . By assumption (4),  $f : (X, D) \rightarrow (X, D')$  is  $g$ -Cauchy on  $X$ . Since  $\{gx_n\}$  is a  $D$ -Cauchy sequence,  $\{fx_n\}$  is a  $D'$ -Cauchy sequence. Thus,

$$\lim_{n, m \rightarrow \infty} D'(gx_{n+1}, gx_{m+1}) = \lim_{n, m \rightarrow \infty} D'(fx_n, fx_m) = 0$$

and, consequently,  $\{gx_n\}$  is a  $D'$ -Cauchy sequence.

Since  $(X, D')$  is a  $D'$ -complete JS-metric space, there exists a  $u \in X$  such that

$$\lim_{n \rightarrow \infty} D'(gx_n, u) = \lim_{n \rightarrow \infty} D'(fx_n, u) = 0.$$

Equivalently,

$$\{gx_n\}, \{fx_n\} \in C(D', X, u).$$

It follows that

$$\{fgx_n\} \in C(D', X, fu) \quad \text{and} \quad \{gfx_n\} \in C(D', X, gu)$$

since  $f$  is  $G$ -continuous and  $g$  is continuous on  $(X, D')$ .

Since  $f$  and  $g$  commute,  $\{fgx_n\} \in C(D', X, gu)$ . By Proposition 1.4, we have that  $fu = gu$ . Hence,  $u$  is a coincidence point of  $f$  and  $g$ .

In addition, assume that for any  $x, y \in C(f, g)$ ,  $(gx, gy) \in E(G)$ . Let  $c = gu = fu$ . Observe that  $gc = gfxu = fguc = fc$ . Thus,  $c \in C(f, g)$ . By assumption, we obtain that  $(gu, gc) \in E(G)$ . From Lemma (2.6), we can conclude that  $fc = gc = gu = c$ . Hence,  $c$  is a common fixed point of  $f$  and  $g$ .  $\square$

In the case where  $D = D'$ , we have the following corollary.

**Corollary 2.1.** *Let  $(X, D)$  be a  $D$ -complete JS-metric space endowed with a directed graph  $G$ . Suppose that*

- (1)  $f$  and  $g$  are self-mappings on  $X$  such that  $(f, g)$  is a  $\theta$ - $M$ -contraction w.r.t  $D$ ;
- (2)  $g$  is continuous and  $f(X) \subseteq g(X)$ ;
- (3)  $G$  is transitive;
- (4)  $f$  is  $G$ -continuous;
- (5)  $f$  and  $g$  commute;
- (6)  $X(f, g) \neq \emptyset$ .

Then,  $C(f, g) \neq \emptyset$ . Moreover, if  $(gx, gy) \in E(G)$  for any  $x, y \in C(f, g)$ , then  $Cm(f, g) \neq \emptyset$ .

In the following theorem, we replace  $G$ -continuity of  $f$ , continuity of  $g$  and commutative property between them with other conditions as stated below.

**Theorem 2.8.** *Let  $(X, D)$  be a  $D$ -complete JS-metric space. Assume that*

- (1)  $f$  and  $g$  are self-mappings on  $X$  such that  $(f, g)$  is a  $\theta$ - $M$ -contraction w.r.t  $D$ ;
- (2)  $f(X) \subseteq g(X)$ ;
- (3)  $G$  is transitive;
- (4)  $(X, D, G)$  has the property  $A$ ;
- (5)  $(g(X), D)$  is  $D$ -complete;
- (6)  $X(f, g) \neq \emptyset$ .

Then, there exists  $\{x_n\}$  in  $X$  such that  $(gx_n, gx_{n+1}) \in E(G)$  and  $\{gx_n\} \in C(D, X, gu)$  for some  $u \in X$ . Moreover, if there exists  $0 < C \leq 1$  such that  $D(gu, fu) < \infty$  and

$$D(gu, fu) \leq C \limsup_{n \rightarrow \infty} D(fx_n, fu),$$

then  $u \in C(f, g)$ .

*Proof.* Let  $\{x_n\}$  be a sequence in  $X$  such that  $gx_{n+1} = fx_n$  for each  $n \in \mathbb{N}$ . By the same arguments as in the proof of the previous theorem,  $\{gx_n\}$  is a  $D$ -Cauchy sequence such that  $(gx_n, gx_{n+1}) \in E(G)$  for all  $n \in \mathbb{N}$ .

Since  $(g(X), D)$  is a  $D$ -complete JS-metric space, there exists a  $u \in X$  such that

$$\lim_{n \rightarrow \infty} D(gx_n, gu) = \lim_{n \rightarrow \infty} D(fx_n, gu) = 0.$$



That is,

$$\{gx_n\}, \{fx_n\} \in C(D, X, gu).$$

In addition, assume that there exists  $0 < C \leq 1$  such that  $D(gu, fu) < \infty$  and

$$D(gu, fu) \leq C \limsup_{n \rightarrow \infty} D(fx_n, fu).$$

From assumption (3),  $(gx_n, gu) \in E(G)$  for each  $n \in \mathbb{N}$ . Since  $(f, g)$  is a  $\theta$ - $M$ -contraction w.r.t  $D$ ,

$$D(gx_{n+1}, fu) = D(fx_n, fu) \leq \theta(M(gx_n, gu))M(gx_n, gu), \quad (2.6)$$

where

$$M(gx_n, gu) = \max\{D(gx_n, gu), D(gx_n, fx_n), D(gu, fu)\}.$$

In the cases where either  $M(gx_n, gu) = D(gx_n, gu)$  or  $M(gx_n, gu) = D(gx_n, fx_n) = D(gx_n, gx_{n+1})$ , it is easy to see that

$$\lim_{n \rightarrow \infty} D(fx_n, fu) = 0,$$

and so, by Proposition 1.4, we have that  $fu = gu$ .

Assume that

$$M(gx_n, gu) = D(gu, fu).$$

From (2.6),

$$D(gu, fu) \leq C \limsup_{n \rightarrow \infty} D(fx_n, fu) \leq CD(gu, fu).$$

Then,

$$D(gu, fu) \leq \frac{1}{C}D(gu, fu) \leq \limsup_{n \rightarrow \infty} D(fx_n, fu) \leq D(gu, fu).$$

It follows that

$$\limsup_{n \rightarrow \infty} D(fx_n, fu) = D(gu, fu).$$

Since  $D(gu, fu) < \infty$ , there exists a subsequence  $\{D(fx_{n(k)}, fu)\} \subseteq \{D(fx_n, fu)\}$  such that

$$\lim_{k \rightarrow \infty} D(fx_{n(k)}, fu) = D(gu, fu).$$

Now, consider the following two cases.

Case 1. Let  $D(gu, fu) = 0$ . Since  $D$  is a generalized metric,  $fu = gu$ .

Case 2. Let  $0 < D(gu, fu) < \infty$ . From (2.6), we have that

$$\frac{D(fx_{n(k)}, fu)}{D(gu, fu)} = \frac{D(fx_{n(k)}, fu)}{M(gx_{n(k)}, gu)} \leq \theta(M(gx_{n(k)}, gu)) < 1.$$

Let  $k \rightarrow \infty$ . Thus,

$$\lim_{k \rightarrow \infty} \theta(M(gx_{n(k)}, gu)) = 1.$$

By property of  $\theta$ ,

$$\lim_{k \rightarrow \infty} M(gx_{n(k)}, gu) = D(gu, fu) = 0.$$

Hence,  $gu = fu$ . □

We give examples to illustrate Theorems 2.7 and 2.8, respectively.

**Example 2.9.** Let  $X = [0, 1]$ , and let  $D$  and  $D'$  be generalized metrics on  $X$  defined by

$$D(x, y) = \begin{cases} x + y, & x \neq 0 \text{ and } y \neq 0, \\ \frac{x}{2}, & y = 0, \\ \frac{y}{2}, & x = 0, \end{cases}$$

and

$$D'(x, y) = \begin{cases} L(x + y), & x \neq 0 \text{ and } y \neq 0, \\ \frac{Lx}{2}, & y = 0, \\ \frac{Ly}{2}, & x = 0, \end{cases}$$

where  $x, y \in X$  and  $L$  is a real number such that  $L > 1$ .

We have that  $(X, D')$  is  $D'$ -complete. Now, suppose that

$$E(G) = \{(x, y) : x, y \in [0, \frac{1}{4}] \text{ with } x \neq 0 \text{ or } y = 0\}.$$

Let  $f$  and  $g$  be two self-mappings on  $X$  defined by

$$f(x) = x^4 \quad \text{and} \quad g(x) = x^2.$$

It can be observed that the assumptions (2), (5) and (6) in Theorem 2.7 are satisfied. Moreover, since  $\frac{1}{2} \in X$  such that  $(g\frac{1}{2}, f\frac{1}{2}) = (\frac{1}{4}, \frac{1}{16}) \in E(G)$ ,  $\frac{1}{2} \in X(f, g)$ .

Next, assume that  $(x, y) \in E(G)$  and  $(y, z) \in E(G)$ . If  $y = 0$ , then  $z = 0$ , and if  $y \neq 0$ , then  $x \neq 0$ . Thus,  $x \neq 0$  or  $z = 0$ . Therefore,  $(x, z) \in E(G)$ . This implies that  $G$  is transitive.

*Claim 1.*  $f$  is  $(g, D)$ -edge preserving w.r.t  $G$ .

Let  $x, y \in X$  such that  $(gx, gy) \in E(G)$ . Then,  $x^2, y^2 \in [0, \frac{1}{4}]$ , and  $gx = x^2 \neq 0$  or  $gy = y^2 = 0$ . That is,  $x^4, y^4 \in [0, \frac{1}{16}]$ , and  $fx = x^4 \neq 0$  or  $fy = x^4 = 0$ . It follows that  $(fx, fv) \in E(G)$ . Note that  $D(gx, gy) < \infty$ . Therefore, we have *Claim 1*.

*Claim 2.*  $(f, g)$  is an  $\theta - M$ -contraction w.r.t  $D$ , where  $\theta \in \Theta$  is defined by

$$\theta(t) = \begin{cases} \frac{1}{4}, & 0 \leq t < 1, \\ t^2 + 2, & t \geq 1 \end{cases}$$

for  $t \in [0, \infty]$ .

Let  $x, y \in X$  such that  $(gx, gy) \in E(G)$ . Then,  $x^2, y^2 \in [0, \frac{1}{4}]$ , and  $gx = x^2 \neq 0$  or  $gy = y^2 = 0$ . Consider the following two cases.

Case 1. Let  $gy = 0$ . Notice that

$$\begin{aligned} D(fx, fy) &= D(x^4, 0) \\ &= \frac{x^4}{2} \\ &\leq \frac{1}{4} \left( \frac{x^2}{2} \right) \\ &\leq \theta(M(gx, gy))M(gx, gy). \end{aligned}$$

Case 2. Let  $gy \neq 0$ . Then,  $gx \neq 0$ . We have that

$$\begin{aligned} D(fx, fy) &= D(x^4, y^4) \\ &= x^4 + y^4 \\ &\leq \frac{1}{4} (x^2 + y^2) \\ &\leq \theta(M(gx, gy))M(gx, gy). \end{aligned}$$

Thus, we obtain Claim 2.

Further, it is obvious that  $D \leq D'$ .

Claim 3.  $f : (X, D) \rightarrow (X, D')$  is  $g$ -Cauchy on  $X$ .

Let  $\{x_n\} \subseteq X$  such that  $\{gx_n\}$  is  $D$ -Cauchy. Given  $\epsilon > 0$ . There exists a  $k \in \mathbb{N}$  such that for all  $m, n \geq k$ ,

$$D(x_n^2, x_m^2) = D(gx_n, gx_m) < \frac{\epsilon}{L}.$$

Consider

$$D'(fx_n, fx_m) = LD(fx_n, fx_m) = LD(x_n^4, x_m^4) \leq LD(x_n^2, x_m^2) < L\left(\frac{\epsilon}{L}\right) = \epsilon$$

for any  $m, n \geq k$ . Then, Claim 3 holds. Hence, by Theorem 2.7,  $f$  and  $g$  have a coincidence point, precisely, 0.

**Example 2.10.** Let  $X = [0, 1]$  and  $D$  the generalized metric on  $X$  defined in Example 2.9. Thus,  $(X, D)$  is  $D$ -complete. Suppose that

$$E(G) = \{(x, y) : x \neq 0 \text{ or } y = 0\}.$$

Let  $f$  and  $g$  be two self-mappings on  $X$  defined by

$$f(x) = \frac{x}{x+6} \quad \text{and} \quad g(x) = \frac{x}{3}.$$

Then,  $f(X) \subseteq g(X)$  and  $g(X)$  is  $D$ -complete. Note that  $f$  and  $g$  does not commute. Moreover, we have that  $1 \in X$  and  $(g1, f1) \in E(G)$ . Further, by the previous

example, if  $(x, y) \in E(G)$  and  $(y, z) \in E(G)$ , then  $(x, z) \in E(G)$ . The transitivity of  $G$  is obtained.

*Claim 1.*  $f$  is  $(g, D)$ -edge preserving w.r.t  $G$ .

Let  $x, y, z \in X$ . Assume that  $(gx, gy) \in E(G)$ . That is,  $gx \neq 0$  or  $gy = 0$ . Thus,  $x \neq 0$  or  $y = 0$ . Therefore,  $fx \neq 0$  or  $fy = 0$ . This implies that  $(fx, fy) \in E(G)$ . Then, we are done.

*Claim 2.*  $(f, g)$  is an  $\theta$ - $M$ -contraction w.r.t  $D$ , where  $\theta \in \Theta$  defined by  $\theta(t) = 1/2$  for  $t \in [0, \infty]$ .

Let  $x, y \in X$  such that  $(gx, gy) \in E(G)$ . Consider the following two cases.

*Case 1.* Let  $gy = 0$ . We obtain that

$$\begin{aligned} D(fx, fy) &= D\left(\frac{x}{x+6}, 0\right) \\ &= \frac{1}{2} \left(\frac{x}{x+6}\right) \\ &\leq \frac{1}{2} \left(\frac{x}{6}\right) \\ &\leq \theta(M(gx, gy))M(gx, gy). \end{aligned}$$

*Case 2.* Let  $gy \neq 0$ . Then,  $gx \neq 0$ . Observe that

$$\begin{aligned} D(fx, fy) &= D\left(\frac{x}{x+6}, \frac{y}{y+6}\right) \\ &= \frac{x}{x+6} + \frac{y}{y+6} \\ &\leq \frac{1}{2} \left(\frac{x}{3} + \frac{y}{3}\right) \\ &\leq \theta(M(gx, gy))M(gx, gy). \end{aligned}$$

Therefore, *Claim 2* is obtained.

*Claim 3.*  $(X, D, G)$  has property  $A$ .

Let  $\{y_n\}$  be a sequence in  $C(D, X, c)$  such that  $(y_n, y_{n+1}) \in E(G)$  for some  $c \in X$  and any  $n \in \mathbb{N}$ . Then,

$$y_n \neq 0 \text{ or } y_{n+1} = 0 \text{ for each } n \in \mathbb{N}. \quad (2.7)$$

If  $y_n \neq 0$  for all  $n \in \mathbb{N}$ , then  $(y_n, c) \in E(G)$  for all  $n \in \mathbb{N}$ . Assume that there exists an  $n_0 \in \mathbb{N}$  such that  $y_{n_0} = 0$ . By (2.7),  $y_k = 0$  for all  $k \geq n_0$ . Suppose that  $c \neq 0$ . Then,

$$D(y_k, c) = D(0, c) = \frac{c}{2} \neq 0 \text{ for all } k \geq n_0.$$

This contradicts the fact that  $\{y_n\} \in C(D, X, c)$ . Thus,  $c = 0$  and, consequently,  $(y_n, c) \in E(G)$ . Then, *Claim 3* is followed.

Hence, by Theorem 2.8, there exists a sequence  $\{x_n\}$  in  $X$  such that  $(gx_n, gx_{n+1}) \in E(G)$  and  $\{gx_n\} \in C(D, X, gu)$  for some  $u \in X$ . In addition, by the proof of this

theorem,  $\{fx_n\} \in C(D, X, gu)$ . Note that

$$D(gu, fu) = \begin{cases} gu + fu, & u \neq 0, \\ 0, & u = 0. \end{cases}$$

Then,  $D(gu, fu) < \infty$ . Moreover, it can be concluded that  $x_n \neq 0$  for all  $n \in \mathbb{N}$  or there exists an  $n_0 \in \mathbb{N}$  such that  $x_n = 0$  for all  $n \geq n_0$  since  $(gx_n, gx_{n+1}) \in E(G)$ . Clearly, if  $u = 0$ , we are done. Assume that  $u \neq 0$ . If  $x_n = 0$  for all  $n \geq n_0$ , then  $u = 0$  because  $\{gx_n\} \in C(D, X, gu)$ , a contradiction. Therefore,  $x_n \neq 0$  for all  $n \in \mathbb{N}$ . Consider

$$\begin{aligned} \limsup_{n \rightarrow \infty} D(fx_n, fu) &= \limsup_{n \rightarrow \infty} (fx_n + fu) \\ &= gu + fu \\ &= D(gu, fu). \end{aligned}$$

Thus, there exists a  $C \in (0, 1]$  such that

$$D(gu, fu) \leq C \limsup_{n \rightarrow \infty} D(gx_n, fu).$$

Hence, a coincidence point of  $f$  and  $g$  exists.

### 3 Application

In this section, we solve the integral equation by applying our result to the problem. There were many publications on using outcomes from fixed point theory to find a solution of integral equations (see [1, 3, 6, 7, 8]). Define an integral equation

$$x(t) = \int_0^T p(t, s, x(s)) ds + b(t) \quad (3.1)$$

for  $t \in [0, T]$ , where  $T$  is a real number such that  $T > 0$ . Let  $X = C([0, T], \mathbb{R})$  and

$$D(x, y) = \max_{t \in [0, T]} |x(t)| + \max_{t \in [0, T]} |y(t)|,$$

where  $x, y \in C([0, T], \mathbb{R})$ . Note that  $(X, D)$  is a  $D$ -complete JS-metric space. Now, we provide the following theorem for the existence of a solution to the homogeneous part of the integral equation (3.1), i.e.,  $b(t) = 0$ .

**Theorem 3.1.** *From the equation (3.1), assume that*

- (1)  $p : [0, T] \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous;

(2) for any  $x, y \in \mathbb{R}$ , if  $x \leq y$ , then  $p(t, s, x) \leq p(t, s, y)$  and

$$|p(t, s, x)| + |p(t, s, y)| \leq \frac{k}{T} (|x| + |y|)$$

for some  $k \in [0, 1)$ , where  $s, t \in [0, T]$ ;

(3) there is an  $x_0 \in X$  such that  $x_0(t) \leq \int_0^T p(t, s, x_0(s)) ds$ , where  $t \in [0, T]$ .

Then, the homogeneous part of the integral equation (3.1) has a solution.

*Proof.* Let  $f$  and  $g$  be two self-mappings on  $X$  defined by

$$fx(t) = \int_0^T p(t, s, x(s)) ds,$$

and  $gx(t) = x(t)$  for  $x \in X$  and  $t \in [0, T]$ . Clearly,  $f(X) \subseteq g(X)$ , and  $f$  and  $g$  are continuous mappings that commute. Suppose that

$$E(G) = \{(x, y) : x(t) \leq y(t) \text{ for any } t \in [0, T]\}.$$

It is easy to see that  $G$  is transitive. Furthermore, by the assumption (3),  $X(f, g) \neq \emptyset$ . To show that all assumptions in Corollary 2.1 are satisfied, it remains only to show that  $(f, g)$  is an  $\theta$ - $M$ -contraction w.r.t  $D$  for some  $\theta \in \Theta$ .

First, we prove that  $f$  is  $(g, D)$ -edge preserving w.r.t  $G$ . Let  $x, y \in X$ . Observe that  $D(x(t), y(t)) < \infty$ . Next, assume that  $(gx, gy) \in E(G)$ . That is,  $gx(t) \leq gy(t)$ , and so,  $x(t) \leq y(t)$  for any  $t \in [0, T]$ . From the assumption (2), we have that  $p(t, s, x) \leq p(t, s, y)$ . Consider

$$\begin{aligned} fx(t) &= \int_0^T p(t, s, x(s)) ds \\ &\leq \int_0^T p(t, s, y(s)) ds \\ &= fy(t). \end{aligned}$$

Thus,  $(fx, fy) \in E(G)$ . Then, we can conclude that  $f$  is  $(g, D)$ -edge preserving w.r.t  $G$ .

Now, we present the proof of the inequality (2.1). Assume that  $x(t) \leq y(t)$  for all  $t \in [0, T]$ . By the assumption (2), we obtain the following.

$$\begin{aligned} &|fx(t)| + |fy(t)| \\ &\leq \int_0^T (|p(t, s, x(s))| + |p(t, s, y(s))|) ds \\ &\leq \frac{k}{T} \int_0^T (|x(s)| + |y(s)|) ds \\ &\leq k \left( \max_{t \in [0, T]} |gx(t)| + \max_{t \in [0, T]} |gy(t)| \right). \end{aligned}$$

It is to show that the inequality (2.1) for Let  $\theta(t) = k$ , where  $k \in [0, 1)$  and  $t \in [0, \infty)$ . Then, the inequality (2.1) holds. Therefore,  $(f, g)$  is an  $\theta$ - $M$ -contraction w.r.t  $D$ .

Hence, there exists a coincidence point of  $f$  and  $g$  which is a solution to the homogeneous part of the integral equation (3.1).  $\square$

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## References

- [1] J. Ahmad, N. Hussain, A. Azam and M. Arshad, Common fixed point results in complex valued metric space with applications to system of integral equations, *J. Nonlinear Convex Anal.*, 16(5) (2015), 855-871.
- [2] M. Geraghty, On contractive mappings, *Proc. Amer. Math. Soc.*, 40 (1973), 604-608.
- [3] N. Hussain, A. Azam, J. Ahmad and M. Arshad, Common fixed point results in complex valued metric spaces with application to integral equations, *Filomat*, 28:7 (2014), 1363-1380.
- [4] M. Jleli, B. Samet, A generalized metric space and related fixed point theorems, *Fixed Point Theory Appl.*, Vol. 2015:61 (2015).
- [5] J. Martínez-Moreno, W. Sintunavarat and Y. J. Cho, Common fixed point theorems for Geraghty's type contraction mappings using the monotone property with two metrics, *Fixed Point Theory Appl.*, Vol. 2015:174 (2015).
- [6] J.R. Roshan, V. Parvaneh and I. Altun, Some coincidence point results in ordered  $b$ -metric spaces and applications in a system of integral equations, *Appl. Math. Comput.*, 226 (2014), 725-737.
- [7] C. Thangthong and A. Khemphet, Coincidence point theorems for  $(\alpha, \beta, \gamma)$ -contraction mappings in generalized metric spaces, *Int. J. Math. Math. Sci.*, Vol. 2018 (2018), 1-7.
- [8] X. Wu and L. Zhao, Fixed point theorems for generalized  $\alpha$ - $\psi$  type contractive mappings in  $b$ -metric spaces and applications, *J. Math. Computer Sci.*, 18 (2018), 49-62.

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