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Convergence in Hausdorff Content of Padé-Faber Approximants and Its Applications

Waraporn Chonlapap[†1] **and Nattapong Bosuwan**[†‡2]

[†]Department of Mathematics, Faculty of Science, Mahidol University,
Rama VI Road, Ratchathewi District, Bangkok 10400, Thailand,
e-mail : jantinku@gmail.com (W. Chonlapap)
e-mail : nattapong.bos@mahidol.ac.th (N. Bosuwan)
[‡]Centre of Excellence in Mathematics, CHE,
Si Ayutthaya Road, Bangkok 10400, Thailand

Abstract : A convergence in Hausdorff content of Padé-Faber approximants (recently introduced in [1]) on some certain sequences is proved. As applications of this result, we give an alternate proof of a Montessus de Ballore type theorem for these Padé-Faber approximants and a proof of a convergence of Padé-Faber approximants in the maximal canonical domain in which the approximated function can be continued to a meromorphic function.

Keywords : Padé approximation; Faber polynomials; Montessus de Ballore's theorem; Hausdorff content

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[2]Corresponding author email: nattapong.bos@mahidol.ac.th (Nattapong Bosuwan)

1 Introduction

Let E be a compact subset of the complex plane $\mathbb{C}$ such that $\overline{\mathbb{C}} \setminus E$ is simply connected and E contains more than one point. It is convenient to assume that $0 \in E$ and this can be done, if necessary, without loss of generality making a change of variables. By the Riemann mapping theorem, there exists a unique exterior conformal mapping Φ from $\overline{\mathbb{C}} \setminus E$ onto $\overline{\mathbb{C}} \setminus \{w \in \mathbb{C} : |w| \leq 1\}$ satisfying $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. For any $\rho > 1$, we define

$$\Gamma_\rho := \{z \in \mathbb{C} : |\Phi(z)| = \rho\} \qquad \text{and} \qquad D_\rho := E \cup \{z \in \mathbb{C} : |\Phi(z)| < \rho\},$$

as the *level curve of index* ρ and the *canonical domain of index* ρ, respectively. We denote by $\rho_0(F)$ the index $\rho > 1$ of the largest canonical domain D_ρ to which F can be extended as a holomorphic function, and by $\rho_m(F)$ the index $\rho > 1$ of the largest canonical domain D_ρ to which F can be extended as a meromorphic function with at most m poles (counting multiplicities). We denote by

$$D_{\rho_\infty(F)} := \bigcup_{m=0}^{\infty} D_{\rho_m(F)}$$

the maximum canonical domain in which F can be continued to a meromorphic function.

The *Faber polynomial of* E *of degree* n is defined by the formula

$$\Phi_n(z) := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\Phi^n(t)}{t-z} dt, \qquad z \in D_\rho, \qquad n = 0, 1, 2, \ldots.$$

Denote by $\mathcal{H}(E)$ the space of all functions holomorphic in some neighborhood of E. The *n-th Faber coefficient of* $F \in \mathcal{H}(E)$ with respect to Φ_n is given by

$$[F]_n := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{F(t)\Phi'(t)}{\Phi^{n+1}(t)} dt,$$

where $1 < \rho < \rho_0(F)$. Denote by $\mathbb{N}$ the set of all positive integers. Set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The definition of Padé-Faber approximants (first introduced in [1]) is stated below.

Definition 1.1. Let $F \in \mathcal{H}(E)$ and $(n, m) \in \mathbb{N} \times \mathbb{N}$ be fixed. Then, there exist polynomials $q^E_{n,m}$, $p^E_{n,m,k}$, $k = 0, 1, \ldots, m-1$ such that

$$\deg(p^E_{n,m,k}) \leq n-1, \qquad \deg(q^E_{n,m}) \leq m, \qquad q^E_{n,m} \not\equiv 0, \tag{1.1}$$

$$[z^k q^E_{n,m} F - p^E_{n,m,k}]_j = 0, \qquad j = 0, 1, 2, \ldots, n. \tag{1.2}$$

For each $k = 0, 1, \ldots, m-1$, the rational function

$$R^E_{n,m,k} := \frac{p^E_{n,m,k}}{q^E_{n,m}}$$

is called an (n, m, k) *Padé-Faber approximant of* F.

To solve for ordered pairs $(p^E_{n,m,k}, q^E_{n,m})$, we need to find $nm+m+1$ unknown coefficients in (1.1) from $nm+m$ linear equations in (1.2). Then, $R^E_{n,m,k}$ always exist but they may not be unique. Moreover, since $q^E_{n,m} \not\equiv 0$, we normalize it to have leading coefficient equal to 1. Note that the definition of Padé-Faber approximants in Definition 1.1 is totally different from the definition of "classical" Padé-Faber approximants (see, e.g. [2]). Since this new definition of Padé-Faber approximants was recently introduced, there are only two publications [1, 3] studying this approximation. In [1], Bosuwan and López gave necessary and sufficient conditions for the convergence with geometric rate of $\{q^E_{n,m}\}_{n\in\mathbb{N}}$ (when m is fixed), namely, proving the analogue of the Montessus de Ballore-Gonchar theorem for Padé-Faber approximants on row sequences (see [1, Corollary 1.6]). Later, Bosuwan [3] further studied the convergence of zeros of $\{q^E_{n,m}\}_{n\in\mathbb{N}}$ (when m is fixed). These two results show that the zeros of $\{q^E_{n,m}\}_{n\in\mathbb{N}}$ can be used to detect the location of the poles of the approximated function $F \in \mathcal{H}(E)$.

Next, let us introduce a concept of convergence in Hausdorff content. Let B be a subset of the complex plane $\mathbb{C}$. By $\mathcal{U}(B)$, we denote the class of all coverings of B by at most a numerable set of disks. Let $\beta > 0$ and set

$$h_\beta(B) := \inf\left\{\sum_{j=1}^{\infty} |U_j|^\beta : \{U_j\} \in \mathcal{U}(B)\right\},$$

where $|U_j|$ stands for the radius of the disk U_j. The quantity $h_\beta(B)$ is called the *β-dimensional Hausdorff content of the set B*. This set function is not a measure but it is subadditive and monotonic. Clearly, if B is a disk, then $h_\beta(B) = |B|^\beta$.

Definition 1.2. Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of complex valued functions defined on a domain $D \subset \mathbb{C}$ and g be another complex function defined on D. We say that $\{g_n\}_{n\in\mathbb{N}}$ *converges in β-dimensional Hausdorff content to the function g inside D* if for every compact subset K of D and for each $\varepsilon > 0$, we have

$$\lim_{n\to\infty} h_\beta\{z \in K : |g_n(z) - g(z)| > \varepsilon\} = 0.$$

Such a convergence will be denoted by $h_\beta\text{-}\lim_{n\to\infty} g_n = g$ in D.

The objective of this paper is to investigate a convergence in Hausdorff content of the sequences of Padé-Faber approximants $R^E_{n,m_n,k}$ as $n \to \infty$ when the sequences $\{m_n\}_{n\in\mathbb{N}}$ satisfy

$$\lim_{n\to\infty} \frac{m_n \ln n}{n} = 0. \tag{1.3}$$

This type of sequences of indices $\{(n, m_n)\}_{n\in\mathbb{N}}$ when $\{m_n\}_{n\in\mathbb{N}}$ satisfy the limit (1.3) was first considered by Gonchar [4] for Padé (α, β)-approximants. In the current paper, we prove many results analogous to those in the paper by Gonchar (see Theorem 2, Corollary 1, and Corollary 2 in [4]). As a consequence of our main theorem in this paper, we give an alternative proof of a Montessus de Ballore type theorem for row sequences of Padé-Faber approximants which was originally proved in [1]. Note that the normalization of $q^E_{n,m}$ introduced in the next section is different from the one in [1].

An outline of the paper is as follows. In section 2, we state the main theorem and its corollaries. All auxiliary lemmas are in section 3. Section 4 is devoted to the proofs of all results in section 2.

2 Main Results

An analogue of Theorem 2 in [4] is the following theorem. This theorem constitutes our main result.

Theorem 2.1. Let $\rho > 1$, $F \in \mathcal{H}(E)$ be meromorphic in D_ρ. Assume that

$$m^* := \liminf_{n\to\infty} m_n \geq d_k \tag{2.1}$$

and

$$\lim_{n\to\infty} \frac{m_n \ln n}{n} = 0, \tag{2.2}$$

where k is a fixed number in $\{0, 1, \ldots, m^*-1\}$ and d_k denotes the number of poles of z^kF in D_ρ. Then, for any $\beta > 0$, each sequence $\{R^E_{n,m_n,k}\}_{n\in\mathbb{N}}$ converges in β-dimensional Hausdorff content to z^kF inside D_ρ as $n \to \infty$.

One of the consequences of Theorem 2.1 is a Montessus de Ballore type theorem for Padé-Faber approximants stated below.

Corollary 2.2. *Let $k \in \{0, 1, \ldots, m-1\}$ be fixed. Suppose that $z^kF \in \mathcal{H}(E)$ has poles of total multiplicity exactly m in $D_{\rho_m(z^kF)}$ at the (not necessarily distinct) points $\lambda_1, \lambda_2, \ldots, \lambda_m$. Then, $R^E_{n,m,k}$ is uniquely determined for all sufficiently large n and the sequence $\{R^E_{n,m,k}\}_{n\in\mathbb{N}}$ converges uniformly to z^kF inside $D_{\rho_m(z^kF)} \setminus \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ as $n \to \infty$. Moreover, for any compact subset K of $D_{\rho_m(z^kF)} \setminus \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$,*

$$\limsup_{n\to\infty} \|z^kF - R^E_{n,m,k}\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_m(z^kF)},$$

where $\|\cdot\|_K$ denotes the sup-norm on K and if $K \subset E$, then $\|\Phi\|_K$ is replaced by 1.

Here and in what follows, the phrase "uniformly inside a domain" means "uniformly on each compact subset of the domain".

The following corollary is an analogue of Corollary 2 in [4].

Corollary 2.3. *Let $k \in \mathbb{N}_0$ be fixed and $F \in \mathcal{H}(E)$. Denote by $D_{\rho_\infty(z^kF)}$ the maximal canonical domain in which z^kF can be continued to a meromorphic function. Assume that*

$$\lim_{n\to\infty} m_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{m_n \ln n}{n} = 0.$$

Then, for any $\beta > 0$, each sequence $\{R^E_{n,m_n,k}\}_{n\in\mathbb{N}}$ converges in β-dimensional Hausdorff content to z^kF inside $D_{\rho_\infty(z^kF)}$ as $n \to \infty$.

3 Notation and Auxiliary Results

For each $n \in \mathbb{N}$, let Q^E_{n,m_n} be the polynomial q^E_{n,m_n} normalized in terms of its zeros $\lambda_{n,j}$ so that

$$Q^E_{n,m_n}(z) := \prod_{|\lambda_{n,j}|\le 1}(z-\lambda_{n,j})\prod_{|\lambda_{n,j}|>1}\left(1-\frac{z}{\lambda_{n,j}}\right) \tag{3.1}$$

and for all $k = 0, 1, \ldots, m_n - 1$,

$$R^E_{n,m_n,k} = \frac{p^E_{n,m_n,k}}{q^E_{n,m_n}} = \frac{P^E_{n,m_n,k}}{Q^E_{n,m_n}}.$$

Now, we discuss some upper and lower estimates on the normalized Q^E_{n,m_n} in (3.1). Let $\varepsilon > 0$, $d \in \mathbb{N}$, $k \in \mathbb{N}_0$, and $F \in \mathcal{H}(E)$ be fixed. Suppose that the poles of z^kF in $D_{\rho_d(z^kF)}$ are $\lambda_1, \lambda_2, \ldots, \lambda_{d'}$ (they are not necessarily distinct and $d' \le d$) and the zeros of Q^E_{n,m_n} for F are $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,l_{m_n}}$ (they are not necessarily distinct and $l_{m_n} \le m_n$). We would like to emphasize that since $0 \in E$, for any $k \in \mathbb{N}_0$, $D_{\rho_d(z^kF)} = D_{\rho_d(F)}$ and $\lambda_1, \lambda_2, \ldots, \lambda_{d'}$ are exactly all the poles of F in $D_{\rho_d(F)}$. We cover each pole of z^kF in $D_{\rho_d(z^kF)}$ with an open disk of radius $(\varepsilon/(6d))^{1/\beta}$ and denote by $J^\beta_{0,\varepsilon}(F,d)$ the union of these disks. For each $n \in \mathbb{N}$, we cover each zero of Q^E_{n,m_n} with an open disk of radius $(\varepsilon/(6m_nn^2))^{1/\beta}$ and denote by $J^\beta_{n,\varepsilon}(F)$ the union of these disks. Set for each $\ell \in \mathbb{N}$,

$$J^\beta_\varepsilon(F,d;\ell) := J^\beta_{0,\varepsilon}(F,d)\bigcup\left(\bigcup_{n=\ell}^{\infty} J^\beta_{n,\varepsilon}(F)\right) \tag{3.2}$$

and

$$J^\beta_\varepsilon(F,d) := J^\beta_\varepsilon(F,d;1).$$

Using the monotonicity and subadditivity of h_β, we have

$$h_\beta(J^\beta_\varepsilon(F,d)) \le h_\beta(J^\beta_{0,\varepsilon}(F,d)) + \sum_{n=1}^{\infty} h_\beta(J^\beta_{n,\varepsilon}(F))$$

$$\le \frac{\varepsilon}{6} + \sum_{n=1}^{\infty}\frac{\varepsilon}{6n^2} = \varepsilon\left(\frac{1}{6}+\frac{\pi^2}{6^2}\right) < \varepsilon.$$

Note that $J^\beta_{\varepsilon_1}(F,d) \subset J^\beta_{\varepsilon_2}(F,d)$ for $\varepsilon_1 < \varepsilon_2$. For any set $B \subset D_{\rho_d(z^kF)}$, we put $B(\varepsilon) := B \setminus J^\beta_\varepsilon(F,d)$. Clearly, if $\{g_n\}_{n\in\mathbb{N}}$ converges uniformly to g on $K(\varepsilon)$ for any compact $K \subset D_{\rho_d(F)}$ and $\varepsilon > 0$, then h_β-$\lim_{n\to\infty} g_n = g$ in $D_{\rho_d(z^kF)}$.

The normalization of Q^E_{n,m_n} provides the following useful upper and lower bounds on the estimation of Q^E_{n,m_n}.

Lemma 3.1. *Fix $k \in \mathbb{N}_0$ and $d \in \mathbb{N}$. Let $F \in \mathcal{H}(E)$, $K \subset D_{\rho_d(z^k F)}$ be a compact set, $\varepsilon > 0$ be fixed, and $\ell \in \mathbb{N}$ be fixed. Suppose that*

$$\liminf_{n\to\infty} m_n \geq d',$$

where d' is the total multiplicity of poles of $z^k F$ in $D_{\rho_d(z^k F)}$, and

$$\lim_{n\to\infty} \frac{m_n \ln n}{n} = 0.$$

Then, there exist constants $C_1 > 0$ and $C_2 > 0$ independent of n such that for all sufficiently large n,

$$\|Q^E_{n,m_n}\|_K \leq C_1^{m_n}, \tag{3.3}$$

where $\|\cdot\|_K$ is the sup-norm on K and

$$\min_{z\in K\setminus J^\beta_\varepsilon(F,d;\ell)} |Q^E_{n,m_n}(z)| \geq (C_2 m_n n^2)^{-2m_n/\beta}, \tag{3.4}$$

where the above inequality is meaningful when $K \setminus J^\beta_\varepsilon(F,d;\ell)$ is a nonempty set.

Proof of Lemma 3.1. Without loss of generality, we assume that K is a nonempty compact subset of $D_{\rho_d(z^k F)}$. Moreover, it is easy to check that if $K = \{0\}$, the inequalities (3.3) and (3.4) hold. Then, we can assume further that $K \neq \{0\}$ and set $M := \|z\|_K > 0$. Therefore, there exists $S \in \mathbb{N}$ such that $SM > 1$. From the normalization of Q^E_{n,m_n},

$$\|Q^E_{n,m_n}\|_K = \max_{z\in K}\left|\prod_{|\lambda_{n,j}|\leq 1}(z-\lambda_{n,j})\prod_{|\lambda_{n,j}|>1}\left(1-\frac{z}{\lambda_{n,j}}\right)\right| \leq (M+1)^{m_n}$$

and for $z \in K \setminus J^\beta_\varepsilon(F,d;\ell)$ and $n \geq \ell$,

$$\begin{aligned}
|Q^E_{n,m_n}(z)| &= \left|\prod_{|\lambda_{n,j}|\leq 1}(z-\lambda_{n,j})\prod_{|\lambda_{n,j}|>1}\left(1-\frac{z}{\lambda_{n,j}}\right)\right| \\
&= \left|\prod_{|\lambda_{n,j}|\leq 1}(z-\lambda_{n,j})\prod_{1<|\lambda_{n,j}|\leq SM}\left(1-\frac{z}{\lambda_{n,j}}\right)\prod_{|\lambda_{n,j}|>SM}\left(1-\frac{z}{\lambda_{n,j}}\right)\right| \\
&= \left|\prod_{|\lambda_{n,j}|\leq 1}(z-\lambda_{n,j})\prod_{1<|\lambda_{n,j}|\leq SM}\left(\frac{\lambda_{n,j}-z}{\lambda_{n,j}}\right)\prod_{|\lambda_{n,j}|>SM}\left(1-\frac{z}{\lambda_{n,j}}\right)\right| \\
&\geq \prod_{|\lambda_{n,j}|\leq 1}\left(\frac{\varepsilon}{6m_n n^2}\right)^{1/\beta}\prod_{1<|\lambda_{n,j}|\leq SM}\left[\left(\frac{\varepsilon}{6m_n n^2}\right)^{1/\beta}\frac{1}{SM}\right]\prod_{|\lambda_{n,j}|>SM}\left(1-\frac{1}{S}\right).
\end{aligned} \tag{3.5}$$

Since $(\varepsilon/(6m_n n^2))^{1/\beta} \to 0$ as $n \to \infty$, it is easy to see that for n sufficiently large,

$$\left(1 - \frac{1}{S}\right) \geq \left(\frac{\varepsilon}{6m_n n^2}\right)^{1/\beta} \qquad \text{and} \qquad \frac{1}{SM} \geq \left(\frac{\varepsilon}{6m_n n^2}\right)^{1/\beta}.$$

Therefore, there exists a constant $C_2 > 0$ such that the expression in (3.5) is greater than $(C_2 m_n n^2)^{-(2m_n/\beta)}$. This completes the proof. □

Next, the following lemma (see, e.g., [5]) concerns the formula for computing $\rho_0(F)$ and the domain of convergence of Faber polynomial expansions of holomorphic functions.

Lemma 3.2. *Let $F \in \mathcal{H}(E)$. Then,*

$$\rho_0(F) = \left(\limsup_{n\to\infty} |[F]_n|^{1/n}\right)^{-1}.$$

Moreover, the series $\sum_{n=0}^{\infty}[F]_n \Phi_n$ converges to F uniformly inside $D_{\rho_0(F)}$.

As a consequence of Lemma 3.2 and Definition 1.1, if $F \in \mathcal{H}(E)$, then for any $k = 0, 1, \ldots, m_n$,

$$z^k Q^E_{n,m_n}(z)F(z) - P^E_{n,m_n,k}(z) = \sum_{\ell=n+1}^{\infty} [z^k Q^E_{n,m_n} F]_\ell \, \Phi_\ell(z), \qquad z \in D_{\rho_0(z^k F)}, \tag{3.6}$$

and $P^E_{n,m_n,k} = \sum_{\ell=0}^{n-1}[z^k Q^E_{n,m_n} F]_\ell \, \Phi_\ell$ are uniquely determined by Q^E_{n,m_n}.

The next lemma (see [6, p. 43] or [7, p. 583] for its proof) gives an estimate of Faber polynomials Φ_n on a level curve.

Lemma 3.3. *Let $\rho > 1$ be fixed. Then, there exists $c > 0$ such that*

$$\|\Phi_n\|_{\Gamma_\rho} \leq c\rho^n, \qquad n \geq 0. \tag{3.7}$$

Indeed, by the maximum modulus principle, the inequalities in (3.7) can be replaced by the inequalities

$$\|\Phi_n\|_{\overline{D}_\rho} \leq c\rho^n, \qquad n \geq 0, \tag{3.8}$$

which are used frequently in this paper.

The following lemma is about the uniqueness of $Q^E_{n,m}$ (and $q^E_{n,m}$).

Lemma 3.4. *Let $(n, m) \in \mathbb{N} \times \mathbb{N}$ be fixed. Assume that for all $q^E_{n,m}$ in Definition 1.1, $\deg(q^E_{n,m}) = m$. Then, $q^E_{n,m}$ is unique.*

Proof of Lemma 3.4. Let $(n, m) \in \mathbb{N} \times \mathbb{N}$ be fixed. From (1.1) and (1.2) in Definition 1.1, it is easy to check that a polynomial $c_m z^m + c_{m-1}z^{m-1} + \ldots + c_0$ is $q^E_{n,m}$ if and only if $c_m z^m + c_{m-1} z^{m-1} + \ldots + c_0$ is monic and the constants $c_m, c_{m-1}, \ldots, c_0$ must satisfy the following equation

$$\begin{bmatrix} [z^m F]_n & [z^{m-1}F]_n & \ldots & [F]_n \\ [z^{m+1}F]_n & [z^m F]_n & \ldots & [zF]_n \\ \vdots & \vdots & \ldots & \vdots \\ [z^{2m-1}F]_n & [z^{2m-2}F]_n & \ldots & [z^{m-1}F]_n \end{bmatrix} \begin{bmatrix} c_m \\ c_{m-1} \\ \vdots \\ c_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3.9}$$

For contradiction, let us suppose that there are distinct polynomials $\hat{q} = z^m + \hat{c}_{m-1} z^{m-1} + \hat{c}_{m-2} z^{m-2} + \ldots + \hat{c}_0$ and $\tilde{q} = z^m + \tilde{c}_{m-1} z^{m-1} + \tilde{c}_{m-2} z^{m-2} + \ldots + \tilde{c}_0$ satisfying (3.9). Let $\check{q}$ be the polynomial $\hat{q} - \tilde{q}$ normalized to be monic. Clearly, $\deg(\check{q}) < m$ and $\check{q} \not\equiv 0$ is a monic polynomial where all coefficients satisfying (3.9). Therefore, $\check{q}$ is $q^E_{n,m}$. This contradicts with the assumption that for all $q^E_{n,m}$, $\deg(q^E_{n,m}) = m$. □

The final lemma proved by Gonchar (see [4, Lemma 1]) allows us to derive uniform convergence on compact subsets of the region under consideration from convergence in h_1-content under appropriate assumptions.

Lemma 3.5. *Suppose that h_1-$\lim_{n\to\infty} g_n = g$ in D. Then the following assertions hold true:*

(i) If the functions $g_n, n \in \mathbb{N}$, are holomorphic in D, then the sequence $\{g_n\}_{n\in\mathbb{N}}$ converges uniformly inside D and g is holomorphic in D.

(ii) If each of the functions g_n is meromorphic in D and has no more than $k < +\infty$ poles in this domain, then the limit function g is also meromorphic and has no more than k poles in D.

(iii) If each function g_n is meromorphic and has no more than $k < +\infty$ poles in D and the function g is meromorphic and has exactly k poles in D, then all $g_n, n \geq N$, also have k poles in D; the poles of g_n tend to the poles $\lambda_1, \lambda_2, \ldots, \lambda_k$ of g (taking account of their orders) and the sequence $\{g_n\}_{n\in\mathbb{N}}$ tends to g uniformly inside the domain $D' = D \setminus \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$.

4 Proofs of main results

Proof of Theorem 2.1. Let $k \in \{0, 1, \ldots, m^* - 1\}$ be fixed and d be the number of poles of $z^k F$ (counting multiplicities) in D_ρ (particularly, in $D_{\rho_d(z^k F)}$). For $j = 1, 2, \ldots, \gamma$, let α_j be a distinct pole of $z^k F$ in $D_{\rho_d(z^k F)}$, and τ_j be the order of α_j. Note that since $0 \in E$, $D_{\rho_d(z^k F)} = D_{\rho_d(F)}$ and $\alpha_1, \alpha_2, \ldots, \alpha_\gamma$ are all the poles of F in $D_{\rho_d(F)}$ with orders $\tau_1, \tau_2, \ldots, \tau_\gamma$, respectively.

In the first step, we want to show that for each $j = 1, 2, \ldots, \gamma$,

$$\limsup_{n\to\infty} |(Q^E_{n,m_n})^{(u)}(\alpha_j)|^{1/n} \leq \frac{|\Phi(\alpha_j)|}{\rho_d(F)}, \tag{4.1}$$

where $u = 0, 1, \ldots, \tau_j - 1$. This can be done by induction. Let $j \in \{1, 2, \ldots, \gamma\}$ be fixed. Define

$$\omega_d(z) := \prod_{j=1}^{\gamma} (z - \alpha_j)^{\tau_j},$$

where $d = \sum_{j=1}^{\gamma} \tau_j$,

$$G_\ell(z) := \frac{\omega_d(z) F(z)}{(z - \alpha_j)^\ell}, \qquad \text{and} \qquad H_\ell(z) := (z - \alpha_j)^\ell G_\ell(z),$$

where $\ell = 1, 2, \ldots, \tau_j$. Note that $H_\ell(\alpha_j) \neq 0$ for all $\ell = 1, 2, \ldots, \tau_j$. By Definition 1.1, since $\deg(\omega_d/(z - \alpha_j)^\ell) = d - \ell \leq m_n - 1$, it is not difficult to check that

$$a_{n,n}^{(\ell)} := [G_\ell Q_{n,m_n}^E]_n = \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} \frac{G_\ell(z) Q_{n,m_n}^E(z) \Phi'(z)}{\Phi^{n+1}(z)} dz = 0, \tag{4.2}$$

where $1 < \rho_1 < |\Phi(\alpha_j)|$. Define

$$\tau_{n,n}^{(\ell)} := \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \frac{G_\ell(z) Q_{n,m_n}^E(z) \Phi'(z)}{\Phi^{n+1}(z)} dz,$$

where $|\Phi(\alpha_j)| < \rho_2 < \rho_d(F)$.

Because $G_1 Q_{n,m_n}^E \Phi'/\Phi^{n+1}$ is meromorphic on $\{z \in \mathbb{C} : \rho_1 \leq |z| \leq \rho_2\}$ and has a pole at α_j of order at most 1, it follows from Cauchy's Residue theorem to $G_1 Q_{n,m_n}^E \Phi'/\Phi^{n+1}$ at α_j that

$$\begin{aligned}
&\frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \frac{G_1(z) Q_{n,m_n}^E(z) \Phi'(z)}{\Phi^{n+1}(z)} dz - \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} \frac{G_1(z) Q_{n,m_n}^E(z) \Phi'(z)}{\Phi^{n+1}(z)} dz \\
&= \operatorname{res}\left(G_1 Q_{n,m_n}^E \Phi'/\Phi^{n+1}, \alpha_j\right) \\
&= \lim_{z \to \alpha_j} \frac{(z - \alpha_j) G_1(z) Q_{n,m_n}^E(z) \Phi'(z)}{\Phi^{n+1}(z)} \\
&= \frac{H_1(\alpha_j) Q_{n,m_n}^E(\alpha_j) \Phi'(\alpha_j)}{\Phi^{n+1}(\alpha_j)}.
\end{aligned} \tag{4.3}$$

From (4.2) and (4.3), we have

$$\frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \frac{G_1(z) Q_{n,m_n}^E(z) \Phi'(z)}{\Phi^{n+1}(z)} dz = \frac{H_1(\alpha_j) Q_{n,m_n}^E(\alpha_j) \Phi'(\alpha_j)}{\Phi^{n+1}(\alpha_j)}, \tag{4.4}$$

and by Lemma 3.1, we know that for all $\ell = 1, 2, \ldots, \tau_j$,

$$\left| \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \frac{G_\ell(z) Q_{n,m_n}^E(z) \Phi'(z)}{\Phi^{n+1}(z)} dz \right| \leq \frac{c_1 c^{m_n}}{\rho_2^n}, \tag{4.5}$$

where the numbers c and c_1 do not depend on n (from now on, we will denote some constants that do not depend on n by $c_2, c_3, c_4, \ldots$). By (4.4) and (4.5), we obtain

$$|Q^E_{n,m_n}(\alpha_j)| \le \frac{c_2 c^{m_n} |\Phi(\alpha_j)|^n}{\rho_2^n}.$$

Letting $\rho_2 \to \rho_d(F)$, it is easy to check that

$$\limsup_{n\to\infty} |Q^E_{n,m_n}(\alpha_j)|^{1/n} \le \frac{|\Phi(\alpha_j)|}{\rho_d(F)}.$$

Next, we suppose that the inequality (4.1) is true for $u = 0, 1, \ldots, \ell-2$, where $\ell = 2, 3, \ldots, \tau_j$, and we will show that the inequality (4.1) holds for $\ell - 1$. Since $G_\ell Q^E_{n,m_n} \Phi'/\Phi^{n+1}$ is meromorphic on $\{z \in \mathbb{C} : \rho_1 \le |z| \le \rho_2\}$ and has poles at α_j of order at most ℓ, it follows from Cauchy's Residue theorem to $G_\ell Q^E_{n,m_n} \Phi'/\Phi^{n+1}$ at α_j that

$$\begin{aligned}
\tau^{(\ell)}_{n,n} - a^{(\ell)}_{n,n} &= \frac{1}{2\pi i}\int_{\Gamma_{\rho_2}} \frac{G_\ell(z) Q^E_{n,m_n}(z)\Phi'(z)}{\Phi^{n+1}(z)}dz - \frac{1}{2\pi i}\int_{\Gamma_{\rho_1}} \frac{G_\ell(z) Q^E_{n,m_n}(z)\Phi'(z)}{\Phi^{n+1}(z)}dz \\
&= \operatorname{res}\left(G_\ell Q^E_{n,m_n}\Phi'/\Phi^{n+1}, \alpha_j\right) \\
&= \frac{1}{(\ell-1)!}\lim_{z\to\alpha_j}\left(\frac{(z-\alpha_j)^\ell G_\ell(z) Q^E_{n,m_n}(z)\Phi'(z)}{\Phi^{n+1}(z)}\right)^{(\ell-1)}.
\end{aligned}$$

Using (4.2) and the Leibniz formula, we have

$$\tau^{(\ell)}_{n,n} = \frac{1}{(\ell-1)!}\sum_{t=0}^{\ell-1}\binom{\ell-1}{t}\left(\frac{H_\ell \Phi'}{\Phi^{n+1}}\right)^{(\ell-1-t)}(\alpha_j)(Q^E_{n,m_n})^{(t)}(\alpha_j).$$

Consequently,

$$\begin{aligned}
(Q^E_{n,m_n})^{(\ell-1)}(\alpha_j) =& (\ell-1)!\tau^{(\ell)}_{n,n}\left(\frac{\Phi^{n+1}}{H_\ell \Phi'}\right)(\alpha_j) \\
&- \sum_{t=0}^{\ell-2}\binom{\ell-1}{t}\left(\frac{H_\ell \Phi'}{\Phi^{n+1}}\right)^{(\ell-1-t)}(\alpha_j)(Q^E_{n,m_n})^{(t)}(\alpha_j)\left(\frac{\Phi^{n+1}}{H_\ell \Phi'}\right)(\alpha_j).
\end{aligned} \tag{4.6}$$

Let $\delta > 0$ such that $\rho_2 := \rho_d(F) - \delta > |\Phi(\alpha_j)|$. Moreover, by (4.5),

$$|\tau^{(\ell)}_{n,n}| = \left|\frac{1}{2\pi i}\int_{\Gamma_{\rho_2}} \frac{G_\ell(z) Q^E_{n,m_n}(z)\Phi'(z)}{\Phi^{n+1}(z)}dz\right| \le \frac{c_1 c^{m_n}}{\rho_2^n}, \tag{4.7}$$

and by Cauchy's integral formula, for all $t = 0, 1, \ldots, \ell-2$,

$$\begin{aligned}
\left|\left(\frac{H_\ell \Phi'}{\Phi^{n+1}}\right)^{(\ell-1-t)}(\alpha_j)\right| &= \left|\frac{(\ell-1-t)!}{2\pi i}\int_{|z-\alpha_j|=\varepsilon}\frac{H_\ell(z)\Phi'(z)}{(z-\alpha_j)^{\ell-t}\Phi^{n+1}(z)}dz\right| \\
&\le \frac{c_2}{(|\Phi(\alpha_j)|-\delta)^n},
\end{aligned} \tag{4.8}$$

where $\{z \in \mathbb{C} : |z-\alpha_j| = \varepsilon\} \subset \{z \in \mathbb{C} : |\Phi(z)| > |\Phi(\alpha_j)| - \delta\}$. From (4.7) and (4.8), the equality (4.6) implies that

$$\begin{aligned}
&\limsup_{n\to\infty}|(Q^E_{n,m_n})^{(\ell-1)}(\alpha_j)|^{1/n}\\
&\qquad= \limsup_{n\to\infty}\Bigg|(\ell-1)!\tau^{(\ell)}_{n,n}\left(\frac{\Phi^{n+1}}{H_\ell\Phi'}\right)(\alpha_j)\\
&\qquad-\sum_{t=0}^{\ell-2}\binom{\ell-1}{t}\left(\frac{H_\ell\Phi'}{\Phi^{n+1}}\right)^{(\ell-1-t)}(\alpha_j)(Q^E_{n,m_n})^{(t)}(\alpha_j)\left(\frac{\Phi^{n+1}}{H_\ell\Phi'}\right)(\alpha_j)\Bigg|^{1/n}\\
&\qquad\le \max\left\{\frac{|\Phi(\alpha_j)|}{\rho_2}, \left(\frac{|\Phi(\alpha_j)|}{\rho_d(F)}\right)\left(\frac{|\Phi(\alpha_j)|}{|\Phi(\alpha_j)|-\delta}\right)\right\}.
\end{aligned}$$

Letting $\delta \to 0$, we obtain the inequality

$$\limsup_{n\to\infty}|(Q^E_{n,m_n})^{(\ell-1)}(\alpha_j)|^{1/n} \le \frac{|\Phi(\alpha_j)|}{\rho_d(F)}.$$

Therefore, we have the inequality (4.1) for all $u = 0, 1, \ldots, \tau_j - 1$.

From (3.6), we obtain

$$z^k Q^E_{n,m_n}F - P^E_{n,m_n,k} = \sum_{\ell=n+1}^{\infty} a^{(k)}_{\ell,n}\Phi_\ell, \tag{4.9}$$

where

$$a^{(k)}_{\ell,n} := [z^k Q^E_{n,m_n}F]_\ell.$$

Multiplying the equation (4.9) by ω_d and expanding the result in terms of Faber polynomial expansion, we have

$$\begin{aligned}
z^k\omega_d Q^E_{n,m_n}F - \omega_d P^E_{n,m_n,k} &= \sum_{\ell=n+1}^{\infty} a^{(k)}_{\ell,n}\omega_d\Phi_\ell = \sum_{\nu=0}^{\infty} b^{(k)}_{\nu,n}\Phi_\nu\\
&= \sum_{\nu=0}^{n+d} b^{(k)}_{\nu,n}\Phi_\nu + \sum_{\nu=n+d+1}^{\infty} b^{(k)}_{\nu,n}\Phi_\nu, \qquad (4.10)
\end{aligned}$$

where $b^{(k)}_{\nu,n} := \sum_{\ell=n+1}^{\infty} a^{(k)}_{\ell,n}[\omega_d\Phi_\ell]_\nu$ or $b^{(k)}_{\nu,n} := [z^k\omega_d Q^E_{n,m_n}F - \omega_d P^E_{n,m_n,k}]_\nu$.

Let K be a compact subset of $D_{\rho_d(z^kF)}$ and set

$$\sigma := \max\{||\Phi||_K, 1\}$$

($\sigma = 1$ when $K \subset E$). Next, we will estimate $\sum_{\nu=n+d+1}^{\infty}|b^{(k)}_{\nu,n}||\Phi_\nu(z)|$ on $\overline{D}_\sigma$. Since $\deg(\omega_d P^E_{n,m_n,k}) < d+n$, for all $\nu \ge n+d+1$,

$$b^{(k)}_{\nu,n} := [z^k\omega_d Q^E_{n,m_n}F - \omega_d P^E_{n,m_n,k}]_\nu = [z^k\omega_d Q^E_{n,m_n}F]_\nu$$

$$= \frac{1}{2\pi i}\int_{\Gamma_{\rho_2}} \frac{z^k \omega_d(z) Q^E_{n,m_n}(z)F(z)\Phi'(z)}{\Phi^{\nu+1}(z)}dz,$$

where $\sigma < \rho_2 < \rho_d(z^k F)$. From Lemma 3.1, for sufficiently large n, it is easy to see that

$$|b^{(k)}_{\nu,n}| \le \frac{c_3 c^{m_n}}{\rho_2^\nu}. \tag{4.11}$$

By (3.8) and (4.11), we get

$$\left\| \sum_{\nu=n+d+1}^{\infty} |b^{(k)}_{\nu,n}||\Phi_\nu| \right\|_{\overline{D}_\sigma} \le \sum_{\nu=n+d+1}^{\infty} \left(\frac{c_3 c^{m_n}}{\rho_2^\nu}\right)(c_4\sigma^\nu) = c_5 c^{m_n}\left(\frac{\sigma}{\rho_2}\right)^n. \tag{4.12}$$

Consequently, as $\rho_2 \to \rho_d(z^k F)$, we have

$$\limsup_{n\to\infty} \left\| \sum_{\nu=n+d+1}^{\infty} |b^{(k)}_{\nu,n}||\Phi_\nu| \right\|_{\overline{D}_\sigma}^{1/n} \le \frac{\sigma}{\rho_d(z^k F)}. \tag{4.13}$$

Now, we find the estimate of $\sum_{\nu=0}^{n+d} |b^{(k)}_{\nu,n}||\Phi_\nu(z)|$ on $\overline{D}_\sigma$. By Definition 1.1, we know

$$a^{(k)}_{\ell,n} := \frac{1}{2\pi i}\int_{\Gamma_{\rho_1}} \frac{z^k Q^E_{n,m_n}(z)F(z)\Phi'(z)}{\Phi^{\ell+1}(z)}dz,$$

where $1 < \rho_1 < \rho_0(z^k F)$, and we define

$$\tau^{(k)}_{\ell,n} := \frac{1}{2\pi i}\int_{\Gamma_{\rho_2}} \frac{z^k Q^E_{n,m_n}(z)F(z)\Phi'(z)}{\Phi^{\ell+1}(z)}dz, \tag{4.14}$$

where $\rho_{d-1}(z^k F) < \rho_2 < \rho_d(z^k F)$. Because $z^k Q^E_{n,m_n} F\Phi'/\Phi^{\ell+1}$ is meromorphic on $\{z \in \mathbb{C} : \rho_1 \le |z| \le \rho_2\}$ and has poles at $\alpha_1, \alpha_2, \ldots, \alpha_d$ of orders at most $\tau_1, \tau_2, \ldots, \tau_d$, respectively, it follows from Cauchy's Residue theorem that

$$\tau^{(k)}_{\ell,n} - a^{(k)}_{\ell,n} = \sum_{j=1}^{\gamma} \operatorname{res}\left(\frac{z^k Q^E_{n,m_n}(z)F(z)\Phi'(z)}{\Phi^{\ell+1}(z)}, \alpha_j\right)$$

$$= \sum_{j=1}^{\gamma} \frac{1}{(\tau_j-1)!}\lim_{z\to\alpha_j}\left(\frac{(z-\alpha_j)^{\tau_j} z^k Q^E_{n,m_n}(z)F(z)\Phi'(z)}{\Phi^{\ell+1}(z)}\right)^{(\tau_j-1)}$$

$$= \sum_{j=1}^{\gamma} \frac{1}{(\tau_j-1)!}\sum_{u=0}^{\tau_j-1}\binom{\tau_j-1}{u}\left(\frac{(z-\alpha_j)^{\tau_j} z^k F\Phi'}{\Phi^{\ell+1}}\right)^{(\tau_j-1-u)}(\alpha_j)(Q^E_{n,m_n})^{(u)}(\alpha_j). \tag{4.15}$$

Let $\delta > 0$. By computations similar to (4.7) and (4.8), we have

$$|\tau^{(k)}_{\ell,n}| \le \frac{c_6 c^{m_n}}{\rho_2^\ell} \tag{4.16}$$

and

$$\left|\left(\frac{(z-\alpha_j)^{\tau_j} z^k F\Phi'}{\Phi^{\ell+1}}\right)^{(\tau_j-1-u)}(\alpha_j)\right| \le \frac{c_7}{(|\Phi(\alpha_j)|-\delta)^{\ell}}. \tag{4.17}$$

Moreover, the inequalities (4.1) imply that for all $u = 0, 1, \ldots, \tau_j - 1$,

$$|(Q^E_{n,m_n})^{(u)}(\alpha_j)| \le c_8 \left(\frac{|\Phi(\alpha_j)|+\delta}{\rho_d(z^k F)+\delta}\right)^n \tag{4.18}$$

(recall that $D_{\rho_d(z^kF)} = D_{\rho_d(F)}$). From (4.15), (4.16), (4.17), and (4.18), we obtain

$$\begin{aligned}|a^{(k)}_{\ell,n}| \le & |\tau^{(k)}_{\ell,n}| \\ & + \left|\sum_{j=1}^{\gamma} \frac{1}{(\tau_j-1)!} \sum_{u=0}^{\tau_j-1} \binom{\tau_j-1}{u} \left(\frac{(z-\alpha_j)^{\tau_j} z^k F\Phi'}{\Phi^{\ell+1}}\right)^{(\tau_j-1-u)}(\alpha_j)(Q^E_{n,m_n})^{(u)}(\alpha_j)\right| \\ \le & \frac{c_6 c^{m_n}}{\rho_2^{\ell}} + \frac{c_9}{(\rho_d(z^kF)+\delta)^n} \sum_{j=1}^{\gamma} \frac{(|\Phi(\alpha_j)|+\delta)^n}{(|\Phi(\alpha_j)|-\delta)^{\ell}}.\end{aligned}$$

Next, we estimate $|[\omega_d \Phi_\ell]_\nu|$. Suppose that $\delta > 0$ is sufficiently small so that $\rho_1 - \delta > 1$. Then, by (3.7),

$$|[\omega_d \Phi_\ell]_\nu| = \left|\frac{1}{2\pi i}\int_{\Gamma_{\rho_1-\delta}} \frac{\omega_d(z)\Phi_\ell(z)\Phi'(z)}{\Phi^{\nu+1}(z)} dz\right| \le \frac{c_{10}(\rho_1-\delta)^{\ell}}{(\rho_1-\delta)^{\nu}}.$$

Consequently, we get

$$\begin{aligned}|b^{(k)}_{\nu,n}| &\le \sum_{\ell=n+1}^{\infty} |a^{(k)}_{\ell,n}||[\omega_d\Phi_\ell]_\nu| \\ &\le \sum_{\ell=n+1}^{\infty} \left(\frac{c_6 c^{m_n}}{\rho_2^{\ell}} + \frac{c_9}{(\rho_d(z^kF)+\delta)^n} \sum_{j=1}^{\gamma} \frac{(|\Phi(\alpha_j)|+\delta)^n}{(|\Phi(\alpha_j)|-\delta)^{\ell}}\right)\left(\frac{c_{10}(\rho_1-\delta)^{\ell}}{(\rho_1-\delta)^{\nu}}\right) \\ &= \frac{c_{11}c^{m_n}}{(\rho_1-\delta)^{\nu}}\left(\frac{\rho_1-\delta}{\rho_2}\right)^n + \frac{c_{12}(\rho_1-\delta)^n}{(\rho_d(z^kF)+\delta)^n(\rho_1-\delta)^{\nu}} \sum_{j=1}^{\gamma}\left(\frac{|\Phi(\alpha_j)|+\delta}{|\Phi(\alpha_j)|-\delta}\right)^n.\end{aligned} \tag{4.19}$$

Applying (3.8) and (4.19), we have

$$\sum_{\nu=0}^{n+d} |b^{(k)}_{\nu,n}|\|\Phi_\nu\|_{\overline{D}_\sigma}$$

$$\le \left(c_{13}c^{m_n}\left(\frac{\rho_1-\delta}{\rho_2}\right)^n + \frac{c_{14}(\rho_1-\delta)^n}{(\rho_d(z^kF)+\delta)^n}\sum_{j=1}^{\gamma}\left(\frac{|\Phi(\alpha_j)|+\delta}{|\Phi(\alpha_j)|-\delta}\right)^n\right)\sum_{\nu=0}^{n+d}\left(\frac{\sigma}{(\rho_1-\delta)}\right)^{\nu}$$

$$\leq \left(c_{13} c^{m_n} \left(\frac{\rho_1 - \delta}{\rho_2} \right)^n + \frac{c_{14}(\rho_1 - \delta)^n}{(\rho_d(z^k F) + \delta)^n} \sum_{j=1}^{\gamma} \left(\frac{|\Phi(\alpha_j)| + \delta}{|\Phi(\alpha_j)| - \delta} \right)^n \right) \sum_{\nu=0}^{n+d} \sigma^\nu$$

$$\leq \left(c_{13} c^{m_n} \left(\frac{\rho_1 - \delta}{\rho_2} \right)^n + \frac{c_{14}(\rho_1 - \delta)^n}{(\rho_d(z^k F) + \delta)^n} \sum_{j=1}^{\gamma} \left(\frac{|\Phi(\alpha_j)| + \delta}{|\Phi(\alpha_j)| - \delta} \right)^n \right) (n + d + 1)\sigma^{n+d}. \tag{4.20}$$

This implies that

$$\limsup_{n\to\infty} \left\| \sum_{\nu=0}^{n+d} |b_{\nu,n}^{(k)}||\Phi_\nu| \right\|_{\overline{D}_\sigma}^{1/n} \leq \max \left\{ \frac{\sigma(\rho_1 - \delta)}{\rho_2}, \frac{\sigma(\rho_1 - \delta)}{\rho_d(z^k F) + \delta} \max_{j=1,\ldots,\gamma} \left(\frac{|\Phi(\alpha_j)| + \delta}{|\Phi(\alpha_j)| - \delta} \right) \right\}.$$

Letting $\delta \to 0$, $\rho_1 \to 1^+$, and $\rho_2 \to \rho_d(z^k F)$, we have

$$\limsup_{n\to\infty} \left\| \sum_{\nu=0}^{n+d} |b_{\nu,n}^{(k)}||\Phi_\nu| \right\|_{\overline{D}_\sigma}^{1/n} \leq \frac{\sigma}{\rho_d(z^k F)}. \tag{4.21}$$

Finally, by (3.4), (4.10), (4.13) and (4.21), we obtain for sufficiently large ℓ,

$$\limsup_{n\to\infty} \left\| z^k F - R_{n,m_n,k}^E \right\|_{\overline{D}_\sigma \setminus J_\varepsilon^\beta(F,d;\ell)}^{1/n}$$

$$\leq \limsup_{n\to\infty} \left\| \sum_{\nu=0}^{n+d} \frac{b_{\nu,n}^{(k)} \Phi_\nu}{w_d Q_{n,m_n}^E} + \sum_{\nu=n+d+1}^{\infty} \frac{b_{\nu,n}^{(k)} \Phi_\nu}{w_d Q_{n,m_n}^E} \right\|_{\overline{D}_\sigma \setminus J_\varepsilon^\beta(F,d;\ell)}^{1/n},$$

$$\leq \frac{\sigma}{\rho_d(z^k F)} \cdot \limsup_{n\to\infty} \left(\frac{1}{\min_{z \in K \setminus J_\varepsilon^\beta(F,d;\ell)} |Q_{n,m_n}^E(z)|} \right)^{1/n}$$

$$\leq \frac{\sigma}{\rho_d(z^k F)} \cdot \limsup_{n\to\infty} (c_{15} m_n n^2)^{\frac{2m_n}{n\beta}} = \frac{\sigma}{\rho_d(z^k F)}, \tag{4.22}$$

where $c_{15} > 0$ and the last equality follows from the limit condition (2.2). Therefore, for any $\beta > 0$, $h_\beta\text{-}\lim_{n\to\infty} R_{n,m_n,k}^E = z^k F$ in $D_{\rho_d(z^k F)}$. Since $D_\rho \subset D_{\rho_d(z^k F)}$, $h_\beta\text{-}\lim_{n\to\infty} R_{n,m_n,k}^E = z^k F$ in D_ρ.

□

Proof of Corollary 2.2. Let $k \in \{0, 1, \ldots, m-1\}$ be fixed. By the assumption of Corollary 2.2, we have $m_n = m$. Then, the conditions (2.1) and (2.2) in Theorem 2.1 are obtained. By Theorem 2.1, we get $h_1\text{-}\lim_{n\to\infty} R_{n,m_n,k}^E = z^k F$ in $D_{\rho_d(z^k F)}$. Applying (*iii*) in Lemma 3.5, we get that each pole of $z^k F$ in $D_{\rho_m(z^k F)}$ attracts as many zeros of $Q_{n,m}^E$ as its order. Therefore, since $z^k F$ has m poles in $D_{\rho_m(z^k F)}$, $\deg Q_{n,m}^E = m$ for all sufficiently large n. Applying Lemma 3.4, $Q_{n,m}^E$ is unique for all such n. From the discussion below (3.6), since $P^E_{n,m,k}$ is uniquely determined by $Q^E_{n,m}$, $R^E_{n,m,k}$ is also unique for all such n.

Let $K \subset D_{\rho_d(z^kF)} \backslash \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ be a compact set. Choose $\sigma := \max\{\|\Phi\|_K, 1\}$. Since all points $\lambda_1, \lambda_2, \ldots, \lambda_m$ attract all zeros of $Q^E_{n,m}$, for sufficiently small $\epsilon > 0$ and large ℓ,

$$K \subset \overline{D}_\sigma \setminus J^\beta_\epsilon(F, d : \ell).$$

By the inequality (4.22), we have

$$\begin{aligned}\limsup_{n\to\infty} \left\| z^kF - R^E_{n,m,k} \right\|_K^{1/n} &\le \limsup_{n\to\infty} \left\| z^kF - R^E_{n,m,k} \right\|^{1/n}_{\overline{D}_\sigma \setminus J^\beta_\epsilon(F,d;\ell)} \\ &\le \frac{\sigma}{\rho_d(z^kF)}.\end{aligned}$$

This implies that the sequence $\{R^E_{n,m,k}\}_{n\in\mathbb{N}}$ converges uniformly to z^kF inside $D_{\rho_m(z^kF)} \setminus \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ as $n \to \infty$. The proof is completed. □

Proof of Corollary 2.3. Let K be a compact subset of $D_{\rho_\infty(z^kF)}$, and let $\varepsilon > 0$, $\beta > 0$, and $k \in \mathbb{N}_0$ be fixed. Then, since K is compact, $K \subset D_{\rho_d(z^kF)}$ for some $d \in \mathbb{N}$. Clearly, $\lim_{n\to\infty} m_n \ge d$. Applying Theorem 2.1, because h_β-$\lim_{n\to\infty} R^E_{n,m_n,k} = z^kF$ in $D_{\rho_d(z^kF)}$,

$$\lim_{n\to\infty} h_\beta\{z \in K : |R^E_{n,m_n,k}(z) - z^kF(z)| > \varepsilon\} = 0.$$

This completes the proof. □

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