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Common Fixed Point Theorem for Weakly Compatible Non-Continuous Mappings

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Abstract : A common fixed point theorem involving two pairs of weakly compatible mappings is proved under a Lipschitz type contractive condition, which is independent of the known contractive definitions.

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1. Introduction

The study of common fixed point of mappings satisfying contractive type conditions has been a very active field of research activity during the last three decades. The most general of the common fixed point theorems pertain to four mappings, say A, B, S and T of a metric space (X, d), and use either a Banach type contractive condition of the form,

$$d(Ax, By) \leq h\ m(x, y), 0 \leq h < 1, \text{where} \tag{0.1}$$

$$m(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), [d(Sx, By) + d(Ax, Ty)]/2\},$$

or, a Meir-Keeler type (ϵ, δ)-contractive condition of the form,
given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\epsilon \leq m(x, y) < \epsilon + \delta \Rightarrow d(Ax, By) < \epsilon, \tag{0.2}$$

or, a ϕ-contractive condition of the form

$$d(Ax, By) \leq \phi(m(x, y)), \tag{0.3}$$

involving a contractive gauge function $\phi : R_+ \to R_+$ is such that $\phi(t) < t$ for each $t > 0$.
The weak form of contractive condition (2) is of the form

$$\epsilon < m(x, y) < \epsilon + \delta \Rightarrow d(Ax, By) \leq \epsilon. \tag{0.4}$$

Clearly, condition (1) is a special case of both conditions (2) and (3). A ϕ-contractive condition (3) does not guarantee the existence of a fixed point unless some additional condition is assumed. Therefore, to ensure the existence of common fixed point under the contractive condition (3), the following conditions on the function ϕ have been introduced and used by various authors.

(I) $\phi(t)$ is non decreasing and $t/(t - f(t))$ is non increasing (Carbone *et.al.*[2]),

(II) $\phi(t)$ is non decreasing and $\lim_n \phi^n(t) = 0$ for each $t > 0$ (Jachymski [3]),

(III) ϕ is upper semi continuous (Boyd and Wong [1], Jachymski [3], Maiti and Pal [11]), Pant [14]), or equivalently,

(IV) ϕ is non decreasing and continuous from right (Park and Rhoades [23]).

It is now known (e.g. Jachymski [3], Pant *et.al.*[15])that if any of the conditions (I), (II), (III) or (IV) is assumed on ϕ, then a ϕ-contractive condition (3) implies an analogous (ϵ, δ)-contractive condition (2) and both the contractive conditions hold simultaneously. Similarly, a Meir-Keeler type (ϵ, δ)-contractive condition does not ensure the existence of a fixed point. The following example illustrates that an (ϵ, δ)-contractive condition of type (2) neither ensures the existence of a fixed point nor implies an analogous ϕ-contractive condition (3).

Example 1 *(Pant et.al. [16]): Let $X = [0, 2]$ and d be the Euclidean metric on X. Define $f : X \to X$ by $fx = (1 + x)/2$ if, $x < 1$ and $fx = 0$ if, $x \geq 1$. Then, it satisfies the contractive condition*

$$\epsilon \leq max\{d(x, y), d(x, fx), d(y, fy), [d(x, fy) + d(fx, y)]/2\} < \epsilon + \delta$$

$\Rightarrow d(fx, fy) < \epsilon$, with $\delta(\epsilon) = 1$ for $\epsilon \geq 1$ and $\delta(\epsilon) = 1 - \epsilon$ for $\epsilon < 1$ but f does not have a fixed point . Also, f does not satisfy the contractive condition

$$d(fx, fy) \leq \phi(\max\{d(x, y), d(x, fx), d(y, fy), [d(x, fy) + d(fx, y)]/2\}),$$

since the desired function $\phi(t)$ cannot be defined at $t = 1$.

Hence, the two types of contractive conditions (2) and (3) are independent of each other. Thus, to ensure the existence of common fixed point under the contractive condition (2), the following conditions on the function δ have been introduced and used by various authors.

(V) δ is non decreasing (Pant [13, 14]),

(VI) δ is lower semi-continuous (Jungck [8], Jungck *et.al.*[9]).

Jachymski [3] has shown that the (ϵ, δ)-contractive condition (2) with a nondecreasing δ implies a ϕ-contractive condition (3). Also, Pant *et.al.* [16] have shown that the (ϵ, δ)-contractive condition (2) with a lower semi continuous δ, implies a ϕ-contractive condition (3). Thus, we see that if additional conditions are assumed on δ then the $(\epsilon, \delta)-$ contractive condition (2) implies an analogous ϕ-contractive condition (3) and both the contractive conditions hold simultaneously.

It is thus clear that contractive conditions (2) and (3) hold simultaneously whenever (2) or (3) is assumed with additional condition on δ or ϕ respectively. It follows, therefore, that the known common fixed point theorems can be extended and generalized if instead of assuming one of the contractive condition (2) or (3) with additional conditions on δ and ϕ, we assume contractive condition (2) together with the following condition of the form.

$$d(Ax, By) < \alpha[d(Sx, Ty) + d(Ax, Sx) + d(By, Ty) + d(Sx, By) + d(Ax, Ty)],$$

for $\quad 0 \leq \alpha \leq 1/3$.

We prove a common fixed point theorem for the sequence of mappings adopting this approach in this paper. This gives a new approach of ensuring the existence of fixed points under an (ϵ, δ)-contractive condition consists of assuming additional conditions which are independent of the ϕ-contractive condition implied by (V) and (VI). As the fixed point theorem is established removing the assumption of continuity, relaxing the compatibility to weak compatibility property and also replacing the completeness of the space, this result generalizes and improves various other similar results of fixed points.

Two self-mappings A and S of a metric space (X, d) are said to be *compatible* (see Jungck [8]) if, $lim_n d(ASx_n, SAx_n) = 0$, whenever x_n is a sequence in X such that $lim_n Ax_n = lim_n Sx_n = t$ for some t in X. It is easy to see that compatible mappings commute at their coincidence points.

Two self-mappings A and S of a metric space (X, d) are called *weakly compatible* (see Jungck and Rhoades [10]) if they commute at coincidence points. That is, if $Ax = Sx$ implies that $ASx = SAx$ for x in X. It is noted that a compatible maps are weakly compatible but weakly compatible maps need not be compatible [25].

To prove our theorem, we shall use the following lemma of Jachymski [3]:

LEMMA(2.2 **of** [3]**):** Let A, B, S and T be self mappings of a metric space (X, d) such that $AX \subset TX$, $BX \subset SX$. Assume further that given $\epsilon > 0$ there exists $\delta > 0$ such that for all x, y in X

$$\epsilon < M(x, y) < \epsilon + \delta \Rightarrow d(Ax, By) \leq \epsilon \tag{0.5}$$

and

$$d(Ax, By) < M(x, y), \text{whenever } M(x, y) > 0, \tag{0.6}$$

where $M(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), [d(Sx, By)+d(Ax, Ty)]/2\}$. Then for each x_0 in X, the sequence y_n in X defined by the rule

$$y_{2n} = Ax_{2n} = Tx_{2n+1}; \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$$

is a Cauchy sequence.

Jachymski [3] has shown that contractive condition (2) implies (4) but contractive condition (4) does not imply the contractive condition (2).

2. The Main Result

Let $\{A_i\}, i = 1, 2, 3, ...,$ S and T be self mappings of a metric space (X, d). In the sequel, we shall denote,
$M_{1i}(x, y) = \max\{d(Sx, Ty), d(A_1x, Sx), d(A_iy, Ty), [d(Sx, A_iy) + d(A_1x, Ty)]/2\}$.

Theorem 1. Let $\{A_i\}, i = 1, 2, 3, ...,$ S and T be self mappings of a metric space (X, d) such that

(i) $A_1X \subset TX$, $A_iX \subset SX$ for $i > 1$,

(ii) given $\epsilon > 0$ there exists a $\delta > 0$ such that for all x, y in X
$\epsilon < M_{12}(x, y) < \epsilon + \delta \Rightarrow d(A_1x, A_2y) \le \epsilon$, and

(iii) $d(A_1x, A_iy) < \alpha[d(Sx, Ty)+d(A_1x, Sx)+d(A_iy, Ty)+d(Sx, A_iy)+d(A_1x, Ty)]$, for $0 \le \alpha \le 1/3$.

If one of A_iX, SX or TX is complete subspace of X and if the pairs (A_1, S) and (A_k, T), for some $k > 1$, are weakly compatible, then all the A_i, S and T have unique common fixed point.

PROOF. Let x_0 be any point in X. Define sequences x_n and y_n in X given by the rule

$$y_{2n} = A_1x_{2n} = Tx_{2n+1}, \quad y_{2n+1} = A_2x_{2n+1} = Sx_{2n+2}, n = 0, 1, 2, ... \tag{0.7}$$

This can be done by virtue of (i). Then, using the same proof as that in paper of Pant *et.al.*[16], we conclude that y_n is a Cauchy sequence in X. Suppose that TX is a complete subspace of X, then the subsequence $y_{2n} = Tx_{2n+1}$ is a Cauchy sequence in TX and hence has a limit u. Let v be in $T^{-1}u$, then $Tv = u$. Since y_{2n} is convergent, so y_n is convergent to u and hence y_{2n+1} also converges to u. Now, setting $x = x_{2n}$ and $y = v$ in (iii), we have for some i,

$$d(A_1x_{2n}, A_iv) < \alpha[d(Sx_{2n}, Tv)+d(A_1x_{2n}, Sx_{2n})+d(A_iv, Tv)+d(Sx_{2n}, A_iv)+d(A_1x_{2n}, Tv)]. \tag{0.8}$$

Letting $n \to \infty$, we have $d(u, A_iv) \le 2\alpha\ d(u, A_iv)$, which implies that $A_iv = u$. Also, since $A_iX \subset SX$, so $u = A_iv$ implies that $u \in SX$. Let $w \in S^{-1}u$, then $Sw = u$. Sothat, setting $x = w$ and $y = x_{2n+1}$ in (iii), we get

$$d(A_1w, A_ix_{2n+1}) < \alpha[d(Sw, Tx_{2n+1}) + d(A_1w, Sw) + d(A_ix_{2n+1}, Tx_{2n+1}) + d(Sw, A_ix_{2n+1}) + d(A_1w, Tx_{2n+1})],$$

and letting n tend to infinity, we get $d(A_1w, u) \le 2\alpha\ d(u, A_1w)$ which implies that $u = A_1w$. This means that

$$u = Tv = A_iv = A_1w = Sw. \tag{0.9}$$

Now, since $u = Tv = A_iv$, so by the weak compatibility of (A_i, T), it follows that $TA_iv = A_iTv$ and so we get $A_iu = A_iTv = TA_iv = Tu$. Also, since $u = A_1w = Sw$, so by the weak compatibility of (A_1, S), it follows that $SA_1w = A_1Sw$ and so we get $A_1u = A_1Sw = SA_1w = Su$. Thus. from (iii), we have, for some i,
$d(A_1w, A_iu) < \alpha[d(Sw, Tu)+d(A_1w, Sw)+d(A_iu, Tu)+d(Sw, A_iu)+d(A_1w, Tu)]$;
that is, $d(u, A_iu) < 3\alpha\ d(u, A_iu)]$, which is a contradiction for $0 \leq \alpha \leq 1/3$. This implies that $u = A_iu$. Similarly, using (iii), one can show that $A_1u = u$. Therefore, we have $u = A_iu = Tu = A_1u = Su$. Hence, the point u is a common fixed point of all $\{A_i\}, S$ and T.

If we assume SX is complete, then the argument analogue to the previous completeness argument proves the theorem. If A_1X is complete, then $u \in A_1X \subset TX$. Similarly, if A_iX is complete, then $u \in A_iX \subset SX$. So, the theorem is established. The uniqueness of the common fixed point follows easily from the condition (iii). This completely proves the theorem.

We now give an example to illustrate the above theorem.

Example 2. Let $X = [2, 20]$ and d be the Euclidean metric on X. Define $\{A_i\}, S$ and $T : X \to X$ as follows:
$A_1x = 2$ for each x;
$Sx = x$ if, $x \leq 8$, $Sx = 8$ if, $8 < x < 14$, $Sx = (x + 10)/3$ if, $14 \leq x \leq 17$ and $Sx = (x + 7)/3$ if, $x > 17$;
$Tx = 2$ if, $x = 2$ or $x > 6$, $Tx = x + 12$ if, $2 < x < 4$, $Tx = (x + 9)/3$ if, $4 \leq x < 5$ and $Tx = 8$ if $5 \leq x \leq 6$;
$A_2x = 2$ if, $x < 4$ or $x > 6$, $A_2x = x + 3$ if, $4 \leq x < 5$, $A_2x = x + 2$ if, $5 \leq x \leq 6$;
and for each $i > 2$,
$A_ix = 2$ if, $x = 2$ or $x \geq 4$, $A_ix = (x + 30)/4$ if, $2 < x < 4$.

Then $\{A_i\}, S$ and T satisfy all the conditions of the above theorem and have a unique common fixed point $x = 2$. Being compatible mappings, all $\{A_i\}, S$ and T are weakly compatible mappings. It can be seen in this example that A_1, A_2, S and T satisfy the condition (4) when $\delta(\epsilon) = 14 - \epsilon$ if, $\epsilon \geq 6$ and $\delta(\epsilon) = 6 - \epsilon$ if, $\epsilon < 6$. It may also be noted that the mappings A_1, A_2, S and T do not satisfy the contractive condition (2). To see this, we can take $x > 17$and $5 \leq y \leq 6$, then we have $5 \leq d(A_1x, A_2y) \leq 6$ whereas $6 < M_{12}(x, y) < 8$. Thus, the contractive condition (4) is satisfied but not (2) when $x > 17$ and $5 \leq y \leq 6$. Also, we see that $\delta(\epsilon)$ is neither non-decreasing nor lower semi-continuous. However, A_1, A_2, S and T do not satisfy the contractive condition $d(A_1x, A_2y) \leq \phi(M_{12}(x, y))$ since the required condition ϕ does not satisfy $\phi(t) < t$ at $t = 6$.

Hence, we see that the present example does not satisfy the condition of any previously known common fixed point theorem for continuous mappings since neither the mappings satisfy a ϕ-contractive condition nor δ is lower semi continuous or is non-decreasing.

Remarks: Pant [18] has shown that condition (iii) of the above Theorem 1 is independent of ϕ-contractive conditions. Our result extends the result of Jha *et.al*. [5, 6], Pant *et.al*. [16], Jha and Pant [7], Pant and Jha [17] and Pant [18] and gives a new generalization of Meir-Keeler type common fixed point theorem. Also, as various assumptions either on ϕ or on δ have been considered to ensure the existence of common fixed points under contractive conditions, so this Theorem 1 improves various results of Popa [21], Singh and Tomar [24], Vats [24] and also all other similar results for fixed points.

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