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Common Fixed Point Theorems for Some Admissible Contraction Mapping in JS-Metric Spaces

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Abstract : In this work, we will to investigate some existence results for coincidence point and common fixed point theorems for some admissible contraction mappings, a generalization of Kannan and Chatterjea contraction mappings, in JS-Metric Spaces. Some examples supported our main results are also presented and the results generalize those presented in [2].

Keywords : Fixed point; Coincidence point, Admissible, Common fixed point.
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1 Introduction and preliminaries

In 2012, Samet et al. [4] studied the existing results for α-ψ-contractions. His concept was given in the following definition. Suppose that $X \neq \emptyset$ and α :

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$X \times X \to [0, \infty)$.

Definition 1.1. *[4] Let f be a self-mapping on X and $u, v \in X$. If $\alpha(fu, fv) \geq 1$ whenever $\alpha(u, v) \geq 1$, then we say that f is α-admissible.*

Later, Karapinar [5] added more condition to Definition 1.1.

Definition 1.2. *[5] Let f be an α-admissible self-mapping on X and $u, v, w \in X$. If $\alpha(u, w) \geq 1$ and $\alpha(w, v) \geq 1$ imply $\alpha(u, v) \geq 1$, then we say that f is triangular α-admissible.*

Furthermore, another essential part in this topic is a metric space. There were a large number of literatures that worked not only on a metric space, but also on other topological spaces. Appeared in 2015, a generalization of metric spaces which includes many classes of topological spaces such as metric spaces, b-metric spaces, dislocated metric spaces, and modular spaces are introduced by Jleli and Samet [3]. These new spaces are studied among many researchers. For instance, ElKouch and Marhrani [2] extended some fixed point theorems for Kannan and Chatterjea contraction mappings to this more general setting.

To begin with, let X be a nonempty set, and let $D : X \times X \to [0, +\infty]$ be a function. For each $x \in X$, we set

$$C(D, X, x) = \{\{x_n\} \subseteq X : \lim_{n\to\infty} D(x_n, x) = 0\}$$

Definition 1.3. *[3] Let X be a nonempty set. A function $D : X \times X \to [0, +\infty]$ is called a **generalized metric** on a set X if it satisfies the following conditions:*

(D_1) For any $x, y \in X$, $D(x, y) = 0$ implies $x = y$;

(D_2) For any $x, y \in X$, $D(x, y) = D(y, x)$; and

(D_3) There is a constant $C > 0$ such that

$$D(x, y) \leq C \limsup_{n\to\infty} D(x_n, y)$$

whenever $x, y \in X$ and $\{x_n\} \in C(D, X, x)$.

*In this case, (X, D) will be called a **JS-metric space**.*

Definition 1.4. *[3] Suppose that (X, D) is a JS-metric space, and $\{x_n\}$ is a sequence in X. We say that the sequence $\{x_n\}$ **D-converges** to $x \in X$ whenever $\{x_n\} \in C(D, X, x)$. Moreover, $\{x_n\}$ is called a **D-Cauchy sequence** if and only if $\lim_{m,n\to\infty} D(x_n, x_m) = 0$. Finally, (X, D) is said to be **D-complete** if each D-Cauchy sequence in X is D-converging to some element in X.*

Proposition 1.5. *[3] Given a JS-metric space (X, D), a sequence $\{x_n\}$ in X, and $x, y \in X$. Then $x = y$ whenever $\{x_n\} \in C(D, X, x) \cap C(D, X, y)$.*

The purpose of this work is to present some existence results for coincidence point theorems for admissible BKS-contraction mappings in JS-Metric Spaces. Some examples supported our main results are also presented.

2 Main Results

We begin this section by introducing terms and concepts employed later in this work.

Definition 2.1. *Let (X, D) be a JS-metric space. A function $f : X \to X$ is called* ***continuous at a point*** *$x_0 \in X$, if $\{x_n\} \in C(D, X, x_0)$ implies $\{fx_n\} \in C(D, X, fx_0)$.*

In addition, f is said to be ***continuous*** *if it is continuous at each x in X.*

Definition 2.2. *Let (X, D) be a JS-metric space and $f, g : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say that f is triangular-(α, D)-admissible with respect to g if, for all $x, y, z \in X$, we have*

(1) $\alpha(gx, gy) \geq 1$ implies $\alpha(fx, fy) \geq 1$ and $D(gx, gy) < \infty$;

(2) $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ imply $\alpha(x, y) \geq 1$.

Now, we introduce a generalization of contraction mappings.

Definition 2.3. *Let (X, D) be a JS-metric space endowed with a directed graph G, and let $f, g : X \to X$ be given functions and let $\alpha : X \times X \to [0, \infty)$. The pair (f, g) is called a* ***admissible*** *BKC**-contraction** if:*

(i) f is triangular-(α, D)-admissible with respect to g; and

(ii) There exists $\lambda \in [0, 1/2)$ such that for all $x, y \in X$ with $\alpha(gx, gy) \geq 1$, we have

$$\alpha(gx, gy)D(fx, fy) \leq \lambda \max\{2D(gx, gy), D(gx, fx) + D(gy, fy), D(gx, fy) + D(gy, fx)\}. \tag{2.1}$$

Let (X, D) be a JS-metric space, $f, g : X \to X$ be functions and let $\alpha : X \times X \to [0, \infty)$. We denote the **set of all coincidence points of mappings** f **and** g **of** X by

$$C(f, g) = \{u \in X : fu = gu\}.$$

We also define the **set of all common fixed points of mappings** f **and** g by

$$Cm(f, g) = \{u \in X : fu = gu = u\}.$$

For any sequence $\{x_n\} \subseteq X$ and $n \in \mathbb{N} \cup \{0\}$, we denote

$$\beta(D, f, x_n) = \sup\{D(fx_{n+i}, fx_{n+j}) : i, j \in \mathbb{N})\}.$$

Finally, we set

$$A(f,g) = \{x_0 \in X : \alpha(gx_0, fx_0) \geq 1 \text{ and } \beta(D,f,x_0) < \infty\}.$$

Next, we give a lemma for proving our main results.

Lemma 2.4. *Let (X,D) be a JS-metric space, $\alpha : X \times X \to [0,\infty)$ and let $f,g : X \to X$ be functions such that (f,g) is a admissible BKC-contraction. Then any $x,y \in C(f,g)$ satisfy the following properties.*

(i) If $\alpha(gx,gx) \geq 1$, then $D(gx,gx) = 0$.

(ii) If $\alpha(gx,gy) \geq 1$, then $gx = gy$.

Moreover, suppose that $f(X) \subseteq g(X)$ and there exists $x_0 \in A(f,g)$. Then we obtain a sequence $\{gx_n\}$ (defined in the following proof) which is a D-Cauchy sequence in (X,D).

Proof. (i) Let $x \in C(f,g)$. Since $\alpha(gx,gx) \geq 1$ and f is triangular-(α,D)-admissible with respect to g, we have $D(gx,gx) < \infty$

$$\begin{aligned} D(gx,gx) &= D(fx,fx) \\ &\leq \alpha(gx,gx)D(fx,fx) \\ &\leq \lambda \max\{2D(gx,gx), D(gx,fx)+D(gy,fy), D(gx,fy)+D(gy,fx)\} \\ &\leq 2\lambda D(gx,gx). \end{aligned}$$

Since $2\lambda < 1$, we have $D(gx,gx) = 0$.

(ii) Let $x,y \in C(f,g)$ and $\alpha(gx,gy) \geq 1$. By f is triangular-(α,D)-admissible with respect to g, $\alpha(gx,gy) \geq 1$, then $D(gx,gy) < \infty$, we have

$$\begin{aligned} D(gx,gy) &= D(fx,fy) \\ &\leq \alpha(gx,gy)D(fx,fy) \\ &\leq \lambda \max\{2D(gx,gy), D(gx,fx)+D(gy,fy), D(gx,fy)+D(gy,fx)\}. \end{aligned}$$

Now, we will consider in 3 case.

Case (1); If $\max\{2D(gx,gy), D(gx,fx)+D(gy,fy), D(gx,fy)+D(gy,fx)\} = 2D(gx,gy)$, we have

$$D(gx,gy) \leq 2\lambda D(gx,gy).$$

Since $2\lambda \in (0,1)$, thus $gx = gy$.

Case (2); If $\max\{2D(gx,gy), D(gx,fx)+D(gy,fy), D(gx,fy)+D(gy,fx)\} = D(gx,fx)+D(gy,fy)$, we have

$$\begin{aligned} D(gx,gy) &\leq \lambda\{D(gx,gx)+D(gy,gy)\} \\ &= 0. \end{aligned}$$

This mean that $gx = gy$.

Case (3); If $\max\{2D(gx,gy), D(gx,fx)+D(gy,fy), D(gx,fy)+D(gy,fx)\} = D(gx,fy)+D(gy,fx)$, we have

$$\begin{aligned} D(gx,gy) &\leq \lambda\{D(gx,fy)+D(gy,fx)\} \\ &= \lambda\{D(gx,gy)+D(gy,gx)\} \\ &= 2\lambda D(gx,gy). \end{aligned}$$

Since $\lambda < 1/2$, we obtain that $D(gx,gy) = 0$, then $gx = gy$.

Now, let $x_0 \in X$ be such that $x_0 \in A(f,g)$. Then we have $\alpha(gx_0, fx_0) \geq 1$ and $\beta(D,f,x_0) < \infty$. By the assumption that $f(X) \subseteq g(X)$ and $f(x_0) \in X$, it is easy to construct a sequence $\{x_n\}$ in X for which

$$gx_n = fx_{n-1}$$

for all $n \in \mathbb{N}$. If $gx_{n_0} = gx_{n_0-1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0-1} is a coincidence point of f and g. Therefore, we will only consider the case that $gx_n \neq gx_{n-1}$ is satisfied for each $n \in \mathbb{N}$.

Since $\alpha(gx_0, fx_0) = (gx_0, gx_1) \geq 1$ and f is triangular-(α, D)-admissible with respect to g, we obtain $\alpha(fx_0, fx_1) = \alpha(gx_1, gx_2) \geq 1$. Continuing this process inductively, we get that

$$\alpha(gx_n, gx_{n+1}) \geq 1 \quad \text{for each} \quad n \in \mathbb{N}. \tag{2.2}$$

Moreover, since f is triangular-(α, D)-admissible with respect to g, we have

$$\alpha(gx_k, gx_l) \geq 1 \quad \text{for each} \quad k, l \in \mathbb{N} \;\text{ such that }\; k < l. \tag{2.3}$$

Next, let $n \in \mathbb{N}$ with $n \geq 2$. Then, for all $i, j \in \mathbb{N}$, we have

$$\begin{aligned} &D(gx_{n+i+1}, gx_{n+j+1}) = D(fx_{n+i}, fx_{n+j}) \\ &\leq \alpha(gx_{n+i}, gx_{n+j}) D(fx_{n+i}, fx_{n+j}) \\ &\leq \lambda \max\{2D(gx_{n+i}, gx_{n+j}), D(gx_{n+i}, fx_{n+i}) + D(gx_{n+j}, fx_{n+j}), D(gx_{n+j}, fx_{n+i}) \\ &\quad + D(gx_{n+i}, fx_{n+j})\} \\ &\leq 2\lambda\beta(D, f, x_{n-1}) \end{aligned}$$

which implies that

$$\beta(D, f, x_n) \leq 2\lambda\beta(D, f, x_{n-1}).$$

Consequently, we have

$$\beta(D, f, x_n) \leq (2\lambda)^n \beta(D, f, x_0)$$

and

$$D(gx_n, gx_m) = D(fx_{n-1}, fx_{m-1}) \leq \beta(D, f, x_{n-2}) \leq (2\lambda)^{n-2}\beta(D, f, x_0)$$

for all integer m such that $m > n$.

Since $\beta(D, f, x_0) < \infty$ and $2\lambda < 1$, we receive

$$\lim_{n,m\to\infty} D(gx_n, gx_m) = 0.$$

As a conclusion, it is proved that $\{gx_n\}$ is a D-Cauchy sequence in (X, D). □

We offer a theorem on the existence of coincidence points and common fixed points of admissible BKC-contractions as follows.

Theorem 2.5. *Let (X, D) be a D-complete JS-metric space, $\alpha : X \times X \to [0,\infty)$ and let $f, g : X \to X$ be functions. Suppose that:*

(a) $f(X) \subseteq g(X)$*;*

(b) (f, g) *is a admissible* BKC*-contraction;*

(c) There exists $x_0 \in A(f, g)$.

Then there exists $u \in X$ such that the sequence $\{gx_n\}$ (as defined in Lemma 2.4) D-converges to $gu \in X$. Moreover if we assume further that:

(d) f *and* g *are continuous; and*

(e) f *and* g *are commuting, i.e* $fg = gf$.

Then we have $C(f, g) \neq \emptyset$. Moreover, if we get $\alpha(gx, gy) \geq 1$ for any $x, y \in C(f, g)$, then $Cm(f, g) \neq \emptyset$.

Proof. By Lemma 2.4 and the fact that (X, D) is a D-complete JS-metric space, there exists $u \in X$ such that

$$\lim_{n\to\infty} D(gx_n, u) = \lim_{n\to\infty} D(fx_n, u) = 0.$$

Thus,

$$\{gx_n\}, \{fx_n\} \in C(D, X, u).$$

By the G-continuity of f and the continuity of g on (X, D), we get

$$\{fgx_n\} \in C(D, X, fu) \quad \text{and} \quad \{gfx_n\} \in C(D, X, gu).$$

Since f and g are commuting, we have $\{gfx_n\} \in C(D, X, fu)$. Moreover, from Proposition 1.5, we have that $fu = gu$. Hence, u is a coincidence point of f and g. This means that $u \in C(f, g)$.

Next, we will show the last statement. Let $c = gu = fu$. Since f and g are commuting, $gc = gfu = fgu = fc$. Thus, $c \in C(f, g)$. By the assumption, we have $\alpha(gu, gc) \geq 1$. By lemma 2.4, we can conclude that $fc = gc = gu = c$. Hence, $c \in Cm(f, g)$ and the proof is complete. □

We give examples to illustrate Theorems 2.5.

Example 2.6. *Suppose that $X = [0,1]$. Given the generalized metrics D on X defined by*

$$D(x,y) = \begin{cases} x+y, & x \neq 0 \text{ and } y \neq 0, \\ \frac{x}{2}, & y = 0, \\ \frac{y}{2}, & x = 0, \end{cases}$$

where $x, y \in X$.
We have that (X, D) is D-complete. Now, we consider $\alpha(x,y)$ given by

$$\alpha(x,y) = \begin{cases} 1 & x, y \in [0, \frac{1}{4}] \text{ with } x \neq 0 \text{ or } y = 0 \\ \\ 0 & \text{otherwise} \end{cases}$$

Given the self-mappings f and g on X defined by

$$f(x) = x^4 \quad \text{and} \quad g(x) = x^2.$$

Some tedious manipulation yields the assumptions (a), (c), (d) and (e) in Theorem 2.5 . Further, notice that $x_0 = \frac{1}{2} \in X$ such that $\alpha(g\frac{1}{2}, f\frac{1}{2}) = \alpha(\frac{1}{4}, \frac{1}{16}) \geq 1$ and let sequence $\{x_n\} \subseteq X$ and $n \in \mathbb{N} \cup \{0\}$, we have $\beta(D, f, x_0) = \sup\{D(fx_i, fx_j) = D((x_i)^4, (x_j)^4) : i, j \in \mathbb{N})\} < \infty$, then $\frac{1}{2} \in A(f, g)$.

We would like to show that (b) (f, g) is a admissible BKC-contraction.

Claim 1: f is triangular-(α, D)-admissible with respect to g.

Let $x, y, z \in X$. Assume that $\alpha(gx, gy) \geq 1$. Then, $x^2, y^2 \in [0, \frac{1}{4}]$, and $gx = x^2 \neq 0$ or $gy = y^2 = 0$. It follows that $x^4, y^4 \in [0, \frac{1}{16}]$, and $fx = x^4 \neq 0$ or $fy = x^4 = 0$. Therefore, $\alpha(fx, fv) \geq 1$ and it easy to see that $D(gx, gy) < \infty$.

Next, assume that $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$. It can be observed that if $z = 0$, then $y = 0$, and if $z \neq 0$, then $x \neq 0$. That is, $x \neq 0$ or $y = 0$. Therefore, $\alpha(x, y) \geq 1$, this implies, f is triangular-(α, D)-admissible with respect to g.

Claim 2: (f, g) is an admissible BKC-contraction.

Given $x, y \in X$. Assume that $\alpha(gx, gy) \geq 1$, that is, $x^2, y^2 \in [0, \frac{1}{4}]$, and $gx = x^2 \neq 0$ or $gy = y^2 = 0$. Consider the following cases :

Case 1 : $gy = 0$. We have that

$$\begin{aligned} \alpha(gx, gy)D(fx, fy) &= D(x^4, 0) \\ &= \frac{x^4}{2} \\ &\leq \frac{1}{4}\left(\frac{x^2}{2}\right) \\ &\leq \frac{1}{4}\left(\frac{x^2}{2} + \frac{x^4}{2}\right) \\ &= \frac{1}{4}\left(D(gx, fy) + D(gy, fx)\right) \\ &\leq \frac{1}{4}\max\{D(gx, fx) + D(gy, fy), D(gx, fy) + D(gy, fx)\}. \end{aligned}$$

Case 2 : $gy \neq 0$. Then, $gx \neq 0$. Consider

$$\begin{aligned}
\alpha(gx,gy)D(fx,fy) &= D(x^4,y^4) \\
&= x^4+y^4 \\
&\leq \frac{x^2}{4}+\frac{y^2}{4} \\
&\leq \frac{x^2}{4}+\frac{y^4}{4}+\frac{y^2}{4}+\frac{x^4}{4} \\
&\leq \frac{1}{4}[x^2+y^4+y^2+x^4] \\
&\leq \frac{1}{4}\max\{D(gx,fx)+D(gy,fy), D(gx,fy)+D(gy,fx)\}.
\end{aligned}$$

Therefore, (f,g) is an admissible BKC-contraction.
Thus,f and g have a coincidence point, precisely, 0.

To state next theorem, let us introduce some terms we will use throughout our work.

Definition 2.7. *Let (X,D) be a JS-metric space, $\alpha : X\times X \to [0,\infty)$ and let $f,g : X \to X$ be given functions. Then the pair (f,g) will be called a admissible K-contraction if:*

(a) f is triangular-(α,D)-admissible with respect to g; and

(b) There exists $\lambda \in [0,1/2)$ such that for all $x,y \in X$ with $\alpha(gx,gy)\geq 1$, we have

$$\alpha(gx,gy)D(fx,fy) \leq \lambda[D(gx,fx)+D(gy,fy)]. \tag{2.4}$$

Definition 2.8. *Let (X,D) be a JS-metric space, $\alpha : X\times X \to [0,\infty)$ and let $f,g : X \to X$ be given functions. Then the pair (f,g) will be called a admissible C-contraction if:*

(a) f is triangular-(α,D)-admissible with respect to g; and

(b) There exists $\lambda \in [0,1/2)$ such that for all $x,y \in X$ with $\alpha(gx,gy)\geq 1$, we have

$$\alpha(gx,gy)D(fx,fy) \leq \lambda[D(gx,fy)+D(gy,fx)]. \tag{2.5}$$

Remark 2.9. *It easy to see that if (f,g) is a admissible K-contraction or admissible C-contraction, then (f,g) is a admissible BKC-contraction.*

Now, we will show an existence theorem for coincidence points of admissible K-contraction case, as follows.

Theorem 2.10. *Let (X,D) be a D-complete JS-metric space, $\alpha : X\times X \to [0,\infty)$ and let $f,g : X \to X$ be functions. Suppose that:*

(a) $f(X) \subseteq g(X)$ and $(g(X),D)$ is complete;

(b) (f, g) *is a admissible* K*-contraction; and*

(c) *There exists* $x_0 \in A(f, g)$.

Then there exists $u \in X$ *such that the sequence* $\{gx_n\}$ *(as defined in Lemma 2.4)* D*-converges to* $gu \in X$. *Moreover if we assume further that:*

(d) $D(fu, gu) < \infty$;

(e) *for any* $\{x_n\}$ *in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and for all* $u \in X$, *if* $x_n \in C(D, X, u)$, *then* $\alpha(x_n, u) \geq 1$ *for all* n; *and*

(f) $C\lambda < 1$ *whenever there exist* $C > 0$ *and* $\lambda \in [0, 1/2)$ *such that*

$$D(fu, gu) \leq C\lambda \limsup_{n\to\infty}[D(fx_{n-1}, fx_n) + D(gu, fu)].$$

Then we can conclude that $C(f, g) \neq \emptyset$.

Proof. By Lemma 2.4, there exists a sequence $\{gx_n\}$ which is D-Cauchy in (X, D). In addition, by assumption (a), $(g(X), D)$ is a complete JS-metric space. Thus, there exists $u \in X$ satisfying

$$\lim_{n\to\infty} D(gx_n, gu) = \lim_{n\to\infty} D(fx_n, gu) = 0.$$

Moreover, by property of D, there exists $C_X > 0$ such that

$$D(fu, gu) \leq C_X \limsup_{n\to\infty} D(fu, fx_n).$$

By the fact (f, g) is a admissible K-contraction and assumption (e), there is $\lambda \in [0, 1/2)$ such that

$$D(fx_n, fu) \leq \alpha(gx_n, gu)D(fx_n, fu) \leq \lambda[D(gx_n, fx_n) + D(gu, fu)].$$

Moreover, we obtain that

$$D(fu, gu) \leq C\lambda \limsup_{n\to\infty}[D(fx_{n-1}, fx_n) + D(gu, fu)] = C\lambda D(gu, fu).$$

By assumption (d) and (f), we get that $D(fu, gu) = 0$. This implies that $C(f, g) \neq \emptyset$.

□

To obtain an existence theorem for coincidence points of admissible C-contraction case, we start with the following lemma.

Lemma 2.11. *[1] Suppose that* λ *is a real number with* $0 \leq \lambda < 1$, *and* $\{b_n\}$ *is a sequence of positives real numbers with* $\lim_{n\to\infty} b_n = 0$. *Then, for any sequence of positives real numbers* $\{a_n\}$ *such that* $a_{n+1} \leq \lambda a_n + b_n$ *for all* $n \in \mathbb{N}$, *we have* $\lim_{n\to\infty} a_n = 0$.

Theorem 2.12. *Let (X, D) be a D-complete JS-metric space, $\alpha : X \times X \to [0, \infty)$ and let $f, g : X \to X$ be functions. Suppose that:*

(a) $f(X) \subseteq g(X)$ and $(g(X), D)$ is complete;

(b) (f, g) is a admissible C-contraction; and

(c) There exists $x_0 \in A(f, g)$.

Then there exists $u \in X$ such that the sequence $\{gx_n\}$ (as defined in Lemma 2.4) D-converges to $gu \in X$. Moreover if we assume further that:

(d) $D(fx_0, fu) < \infty$; and

(e) for any $\{x_n\}$ in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and for all $u \in X$, if $x_n \in C(D, X, u)$, then $\alpha(x_n, u) \geq 1$ for all n.

Then we can conclude that $C(f, g) \neq \emptyset$.

Proof. By Lemma 2.4, there exists a sequence $\{gx_n\}$ which is D-Cauchy in (X, D). In addition, by assumption (a), $(g(X), D)$ is a complete JS-metric space. Thus, there exists $u \in X$ satisfying

$$\lim_{n\to\infty} D(gx_n, gu) = \lim_{n\to\infty} D(fx_n, gu) = 0.$$

By the fact (f, g) is a admissible C-contraction and assumption (e), we have $\alpha(gx_n, gu) \geq 1$ and there exists $\lambda \in [0, 1/2)$ such that

$$\begin{aligned} D(fx_n, fu) &\leq \alpha(gx_n, gu)D(fx_n, fu) \\ &\leq \lambda[D(gu, fx_n) + D(gx_n, fu)] \\ &= \lambda[D(gu, gx_{n+1}) + D(fx_{n-1}, fu)]. \end{aligned} \tag{2.6}$$

By assumption (e) and the fact that f is triangular-(α, D)-admissible with respect to g, we have $D(gu, gx_{n+1}) = D(gu, fx_n) < \infty$. Continuing the process in (2.6), we have $D(fx_n, fu) < \infty$ by assumption (d). By lemma 2.11 we obtain that

$$\lim_{n\to\infty} D(fx_n, fu) = 0.$$

In conclusion, we have $gu = fu$. This implies that $C(f, g) \neq \emptyset$. □

Example 2.13. *Let $X = [0, 1]$, and let D be a generalized metric such that*

$$D(x, y) = \begin{cases} x + y, & x \neq 0 \text{ and } y \neq 0, \\ \frac{x}{2}, & y = 0, \\ \frac{y}{2}, & x = 0, \end{cases}$$

Then (X, D) is D-complete.

Next, suppose that

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x \neq 0 \text{ or } y = 0 \\ \\ 0 & \text{otherwise} \end{cases}$$

In addition, define self-mappings f and g on X by

$$f(x) = \frac{x}{x+12} \quad \text{and} \quad g(x) = \frac{x}{4}.$$

We will show that $C(f,g) \neq \emptyset$ by using Theorem 2.12.

First, note that $f(X) \subseteq g(X)$ and $g(X)$ is D-complete. Moreover, we have $x_0 = 0 \in X$ such that $\alpha(g(0), f(0)) = \alpha(0,0) \geq 1$ and $\beta(D, f, 0) < \infty$.

Moreover, since $x_0 = 0$, we have $D(fx_0, fu) = \frac{u}{2} < \infty$ for any $u \in X$.

Next, we will prove that following claims:

Claim 1: *f is triangular-(α, D)-admissible with respect to g.*

Let $x, y, z \in X$. Assume that $\alpha(gx, gy) \geq 1$. Then, $gx \neq 0$ or $gy = 0$. That is $x \neq 0$ or $y = 0$. Thus, $fx \neq 0$ or $fy = 0$. Therefore, $\alpha(fx, fy) \geq 1$, and it easy to see that $D(gx, gy) = D(\frac{x}{4}, \frac{y}{4}) < \infty$.

Next, assume that $\alpha(x,z) \geq 1$ and $\alpha(z,y) \geq 1$. It can be observed that if $z = 0$, then $y = 0$, and if $z \neq 0$, then $x \neq 0$. That is, $x \neq 0$ or $y = 0$. Therefore, $\alpha(x,y) \geq 1$. Hence, f is triangular-(α, D)-admissible with respect to g.

Claim 2: *(f,g) is an admissible C-contraction with $\lambda = \frac{1}{3}$.*

Suppose $x, y \in X$. Assume that $\alpha(gx, gy) \geq 1$. Consider the following cases: Case 1: $gy = 0$. Then $fy = 0$ and we have

$$\begin{aligned}
\alpha(gx,gy)D(fx,fy) &= D(fx,fy) \\
&= D(\frac{x}{x+12}, 0) \\
&= \frac{1}{2}\left(\frac{x}{x+12}\right) \\
&= \frac{1}{3}\left(\frac{x}{2(x+12)} + \frac{x}{2(x+12)} + \frac{x}{2(x+12)}\right) \\
&\leq \frac{1}{3}\left(\frac{x}{2(8)} + \frac{x}{2(8)} + \frac{x}{2(x+12)}\right) \\
&= \lambda[D(gx,fy) + D(gy,fx)].
\end{aligned}$$

Case 2: $gy \neq 0$. Then $gx \neq 0$ and

$$\begin{aligned}
\alpha(gx, gy)D(fx, fy) &= D(fx, fy) \\
&= D(\frac{x}{x+12}, \frac{y}{y+12}) \\
&= \frac{x}{x+12} + \frac{y}{y+12} \\
&\leq \frac{1}{3}\left(\frac{x}{4} + \frac{y}{4}\right) \\
&\leq \frac{1}{3}\left(\frac{x}{4} + \frac{x}{x+12} + \frac{y}{4} + \frac{y}{y+12}\right) \\
&= \lambda[D(gx, fy) + D(gy, fx)].
\end{aligned}$$

Therefore, we have Claim 2.

Finally, we have to prove that assumption (e) in Theorem 2.12. let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\{x_n\} \in C(D, X, c)$ for some $c \in X$. We will show that $\alpha(x_n, c) \geq 1$. By the definition of $\alpha(x, y)$,

$$x_n \neq 0 \text{ or } x_{n+1} = 0, \text{ for each } n \in N. \tag{2.7}$$

If $x_n \neq 0$ for each $n \in \mathbb{N}$, then we have $\alpha(x_n, c) \geq 1$ for each $n \in \mathbb{N}$. On the other hand, if there exists $n_0 \in N$ such that $x_{n_0} = 0$, then by (2.7), $x_k = 0$ whenever $k \geq n_0$. Now, we will show that $c = 0$. Suppose on the contrary that $c \neq 0$. Observe that

$$D(x_k, c) = D(0, c) = \frac{c}{2} \neq 0 \quad \text{for all} \quad k \geq n_0$$

which contradicts to the fact that $\{x_n\} \in C(D, X, c)$. Hence, $c = 0$ and we receive that $\alpha(x_n, c) \geq 1$. By Theorem 2.12, there exists a coincidence point of f and g.

3 Application

We wish to apply our finding to the existence problem of a solution to the integral equation. This is one of the crucial uses of fixed point theorems that can be found in the literatures (See [6, 7, 8, 9]).

$$x(t) = \int_0^T F(t, s, x(s))ds + b(t) \tag{3.1}$$

for $t \in [0, T]$, where T is a real number such that $T > 0$.
Suppose that $X = C([0, T], \mathbb{R})$ and

$$D(x, y) = \max_{t\in[0,T]} |x(t)| + \max_{t\in[0,T]} |y(t)|$$

for any $x, y \in C([0, T], \mathbb{R})$. We have that (X, D) is a D-complete JS-metric space.

Theorem 3.1. *According to* (3.1)*, if we suppose that:*

(i) $F : [0, T] \times [0, T] \times \mathbb{R} \to \mathbb{R}$ *is a continuous function;*

(ii) For any $x, y \in \mathbb{R}$*,* $x \leq y$ *implies* $F(t, s, x) \leq F(t, s, y)$ *and*

$$|F(t, s, x)| + |F(t, s, y)| \leq \frac{1}{4T} (|x| + |y|)$$

where $s, t \in [0, T]$*; and*

(iii) There is $x_0 \in X$ *such that* $x_0(t) \geq \int_0^T F(t, s, x_0(s))ds$ *where* $t \in [0, T]$*.*

Then, there is a solution to the integral equation (3.1)*.*

Proof. Let us define functions f and g on X so that

$$fx(t) = \int_0^T F(t, s, x(s))ds,$$

and $gx(t) = x(t)$ for any $x \in X$ and $t \in [0, T]$.

Suppose that $\alpha : X \times X \to [0, \infty)$ is a function defined by

$$\alpha(x, y) = \begin{cases} 1, & x(t) \geq y(t) \quad \text{for any} \quad t \in [0, T], \\ 0, & \text{otherwise.} \end{cases}$$

It easy to see that $f(X) \subseteq g(X)$, and f and g are continuous functions.

In addition, assumption (iii) induces assumption (c) of Theorem 2.5, and $u = x_0$ so $D(fx_0, fu) = 0$, which implies assumption (c) of Theorem 2.5. Moreover, assumption (e) of Theorem 2.5 is clearly satisfied.

Next, we will show that (f, g) is an admissible BKC-contraction for some $\lambda \in [0, 1/2)$.

To begin with, we will prove that f is triangular-(α, D)-admissible with respect to g. Observe that if $\alpha(gx, gy) \geq 1$, then $gx(t) \geq gy(t)$ for any $t \in [0, T]$. In other words, $x(t) \geq y(t)$ for any $t \in [0, T]$. So, by assumption (ii), we have that $F(t, s, x) \geq F(t, s, y)$. This leads to

$$\begin{aligned} fx(t) &= \int_0^T F(t, s, x(s))ds \\ &\geq \int_0^T F(t, s, y(s))ds \\ &= fy(t). \end{aligned}$$

Consequently, $\alpha(fx, fy) \geq 1$. Now let $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$, it easy to see that $\alpha(x, z) \geq 1$, thus f is triangular-(α, D)-admissible with respect to g.

Finally, we show the following.

Given $x(t) \geq y(t)$ for all $t \in [0, T]$, by assumption (ii), we have that for any $t \in [0, \infty)$,

$$\begin{aligned} \alpha(gx, gy)(|fx(t)| + |fy(t)|) &= |fx(t)| + |fy(t)| \\ &\leq \int_0^T |F(t, s, x(s))| + |F(t, s, y(s))|ds \\ &\leq \frac{1}{4T} \int_0^T (|x(s)| + |y(s)|)ds \\ &\leq \frac{1}{4} \left(\max_{t \in [0,T]} |gx(t)| + \max_{t \in [0,T]} |gy(t)| \right). \end{aligned}$$

This implies that (f, g) is an admissible BKC-contraction for $\lambda = \frac{1}{4}$.

By Theorem 2.5, there is a coincidence point of f and g. It is clear that this point is a solution to the integral equation. □

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