



The Approximate Solution for Generalized Proximal Contractions in Complete Metric Spaces

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Abstract : In this paper, we introduce new notions of F_p -contractive mapping is the set of non-self mappings and F_p -proximal contractive mappings of first and second kind. Then, we generalize the best proximity point theorems and show the existence of p -best proximity points and their uniqueness by the help F_p -contractions in complete metric spaces endowed with P_p -property. Moreover, we present some examples to support our results.

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1 Introduction

In 1922, Banach contraction principle [3] opened up a new way in nonlinear analysis, upon which, various applications in a variety of sciences were appeared. After this interesting principle, several authors generalized this principle by introducing the various contractions on metric spaces. Fixed point theory is indispensable for solving various equations of the form $Fx = x$ for self-mappings F defined on subsets of metric spaces or normed linear spaces. But, it is easy to observe that if F is a non-self mapping, then the equation $Fx = x$ does not necessarily possess a solution, called a fixed point of the mapping F .

Actually, being F as a non-self mapping does not guarantee to have a solution for the equation $Fx = x$. In these cases, one can find those points for which non-self mapping F from A to B has the approximate solution to the equation $Fx = x$. Thus, one can obtain an optimal solution by which, $d(x, Fx) = d(A, B)$ and x is called best proximity point. The best proximity point theorem furnishes sufficient conditions that ascertain the existence of an optimal solution to the problem of globally minimizing the error $d(x, Fx)$. We refer the many authors to studied the best proximity point see [12, 5, 6, 1, 2, 10, 9, 4, 7, 8], for more details.

Motivated by Omidvari et al. [12], the main aims of current research are introducing F_p -contractions and F_p -proximal contractions as new concepts and some related theorems. Taking into account these new results, we will discuss existence of the p -best proximity points for given mappings in metric spaces. Finally, some examples present to show the validity of our results.

2 Preliminaries

Definition 2.1. [1] Let X be a metric space, A and B two nonempty subsets of X . Define

$$\begin{aligned} d(A, B) &= \inf\{d(a, b) : a \in A, b \in B\}, \\ A_0 &= \{a \in A : \text{there exists some } b \in B \text{ such that } d(a, b) = d(A, B)\}, \\ B_0 &= \{b \in B : \text{there exists some } a \in A \text{ such that } d(a, b) = d(A, B)\}. \end{aligned}$$

Definition 2.2. [1] Let $f : A \rightarrow B$, is a no -self mappings. Then an element x^* is called best proximity point, if the following condition holds:

$$d(x^*, fx^*) = d(A, B).$$

Moreover, denote $BPP(f)$ as the set of all best proximity points of f .

Definition 2.3. [11] Let (X, d) be a metric space. Then, a function $p : X \times X \rightarrow [0, \infty)$ is called w -distance on X , if the following hold:

1. $p(x, z) \leq p(x, y) + p(y, z)$, for any $x, y, z \in X$;
2. for any $x \in X$, $p(x, \cdot) : X \rightarrow [0, \infty)$ is lower semi continuous;
3. for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ implies $d(x, y) \leq \epsilon$.

Definition 2.4. [2] A is said to be approximately compact with respect to B if every sequence $\{x_n\} \subset A$ satisfies

$$d(y, x_n) \rightarrow d(y, A)$$

for some $y \in B$, has a convergent subsequence.

It is evident that every set is approximately compact with respect to itself. If A intersects B , then $A \cap B$ is contained in both A_0 and B_0 . Further, it can be seen that if A is compact and B is approximately compact with respect to A , then the sets A_0 and B_0 are non-empty.

Definition 2.5. [2] Let $T : A \rightarrow B$ is a mapping and $g : A \rightarrow A$ is an isometry. Then, the mapping T is said to preserve isometric distance with respect to g if

$$d(Tgx_1, Tgx_2) = d(Tx_1, Tx_2),$$

for all $x_1, x_2 \in A$.

Definition 2.6. [12] Let $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a mapping satisfies the followings:

1. F is strictly increasing, i.e., for all $a, b \in \mathbb{R}^+$ such that $\alpha < \beta \Rightarrow F(\alpha) < F(\beta)$;
2. For each sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ of positive numbers $\lim_{n \rightarrow \infty} \alpha_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$;
3. there exists $k \in (0, 1)$ such that $\lim_{\alpha \rightarrow 0^+} \alpha^k F(\alpha) = 0$.

A mapping $T : A \rightarrow B$ is said to be an F -contraction, if there exists a $\tau > 0$ such that, for all $x, y \in A$,

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)).$$

Definition 2.7. [12] A mapping $T : A \rightarrow B$ is said to be an F -proximal contraction for a non-self mapping of first kind, if there exists $\tau > 0$ such that

1. $d(u_1, Tx_1) = d(A, B)$,
2. $d(u_2, Tx_2) = d(A, B)$,

$$3. d(u_1, u_2), d(x_1, x_2) > 0,$$

implies that

$$\tau + F(d(u_1, u_2)) \leq F(d(x_1, x_2))$$

where $u_1, u_2, x_1, x_2 \in A$.

Definition 2.8. [12] A mapping $T : A \rightarrow B$ is said to be a F -proximal contraction of second kind, if there exists $\tau > 0$ such that

1. $d(u_1, Tx_1) = d(A, B)$,
2. $d(u_2, Tx_2) = d(A, B)$,
3. $d(Tu_1, Tu_2), d(Tx_1, Tx_2) > 0$,

implies that

$$\tau + F(d(Tu_1, Tu_2)) \leq F(d(Tx_1, Tx_2))$$

where $u_1, u_2, x_1, x_2 \in A$.

3 The p -best proximity point for F_p -contractive mapping

Definition 3.1. Let X is a metric space, A and B are two nonempty subsets of X . Define

$$\begin{aligned} p(A, B) &= \inf\{p(a, b) : a \in A, b \in B\}, \\ A_{0,p} &= \{a \in A : \text{there exists some } b \in B \text{ such that } p(a, b) = p(A, B)\}, \\ B_{0,p} &= \{b \in B : \text{there exists some } a \in A \text{ such that } p(a, b) = p(A, B)\}. \end{aligned}$$

Definition 3.2. Let (X, d) is a metric space. Then, a function $p : X \times X \rightarrow [0, \infty)$ is called w_s -distance on X , if the following holds:

1. $p(x, z) \leq p(x, y) + p(y, z)$, for any $x, y, z \in X$;
2. $p(x, y) \geq 0$, for any $x, y \in X$;
3. if $\{x_m\}$ and $\{y_m\}$ are any sequences in X such that $x_n \rightarrow x$, $y_n \rightarrow y$ as $n \rightarrow \infty$, then $p(x_n, y_n) \rightarrow p(x, y)$ as $n \rightarrow \infty$;
4. for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ $d(x, y) \leq \epsilon$.

Definition 3.3. Let (A, B) are nonempty subsets of metric space (X, d) and $A_{0,p} \neq \emptyset$. Then, the pair (A, B) is said to have P_p -property, if and only if, for any $x_1, x_2 \in A_{0,p}$ and $y_1, y_2 \in B_{0,p}$

$$\begin{cases} p(x_1, y_1) = p(A, B) \\ p(x_2, y_2) = p(A, B) \end{cases} \Rightarrow p(x_1, x_2) = p(y_1, y_2).$$

Definition 3.4. For given non-self mapping $f : A \rightarrow B$, an element x^* is called p -best proximity point, if the following holds

$$p(x^*, fx^*) = p(A, B).$$

Taking into account the self mapping f in Definition 3.4, one can get a p -fixed point for $f : A \rightarrow A$.

Definition 3.5. Let $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a mapping satisfies in

1. F is strictly increasing, i.e., for all $a, b \in \mathbb{R}^+$ such that $\alpha < \beta \Rightarrow F(\alpha) < F(\beta)$;
2. For each sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ of positive numbers $\lim_{n \rightarrow \infty} \alpha_n = 0$, if and only if, $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$;
3. there exists $k \in (0, 1)$ such that $\lim_{\alpha \rightarrow 0^+} \alpha^k F(\alpha) = 0$.

A mapping $T : A \rightarrow B$ is said to be an F_p -contraction if there exists $\tau > 0$ such that for all $x, y \in A$,

$$p(Tx, Ty) > 0 \Rightarrow \tau + F(p(Tx, Ty)) \leq F(p(x, y)),$$

where p is w_s -distance.

Theorem 3.6. Let A and B are non-empty, closed subsets of a complete metric space (X, d) such that $A_{0,p}$ is nonempty. Let $T : A \rightarrow B$ is an F_p -contraction such that $T(A_{0,p}) \subseteq B_{0,p}$. Suppose that the pair (A, B) has the P_p -property, where p is the w_s -distance. Then, there exists a unique point x in A such that $p(x, Tx) = p(A, B)$.

Proof. Let us consider an element $x_0 \in A_{0,p}$. Since $Tx_0 \in T(A_{0,p}) \subseteq B_{0,p}$, there exists $x_1 \in A_{0,p}$ such that

$$p(x_1, Tx_0) = p(A, B). \quad (3.1)$$

Also, since $Tx_1 \in T(A_{0,p}) \subseteq B_{0,p}$, we get $x_2 \in A_{0,p}$ such that

$$p(x_2, Tx_1) = p(A, B). \quad (3.2)$$

Inductively, we can find a sequence $\{x_n\}$ in $A_{0,p}$ such that

$$p(x_{n+1}, Tx_n) = p(A, B), \quad (3.3)$$

for all $n \in \mathbb{N}$.

By the fact that, (A, B) satisfies the P_p -property, we have

$$p(x_n, x_{n+1}) = p(Tx_{n-1}, Tx_n), \text{ for all } n \in \mathbb{N}.$$

We divide our proof into four cases:

Case 1: The sequence $\{x_n\}$ converges in $A_{0,p}$.

To see this suppose that there exists $n_0 \in \mathbb{N}$ such that $p(x_{n_0-1}, Tx_{n_0}) = 0$, then by (3.3), we have $p(x_{n_0}, x_{n_0+1}) = 0$ which implies $x_{n_0} = x_{n_0+1}$. Therefore,

$$Tx_{n_0} = Tx_{n_0+1}$$

implies

$$p(Tx_{n_0}, Tx_{n_0+1}) = 0. \tag{3.4}$$

From (3.3) and (3.4) we have

$$p(x_{n_0+2}, x_{n_0+1}) = p(Tx_{n_0+1}, Tx_{n_0}) = 0$$

which implies $x_{n_0+2} = x_{n_0+1}$. Therefore, $x_n = x_{n_0}$, for all $n \geq n_0$ and $\{x_n\}$ is convergent in $A_{0,p}$.

Case 2: The sequence $\{p(x_{n+1}, x_n)\}$ tends to zero.

Let $p(Tx_{n-1}, Tx_n) = 0$, for all $n \in \mathbb{N}$. From the fact that T is a F_p -contraction and (3.3), for any positive integer n we have

$$\tau + F(p(Tx_n, Tx_{n-1})) \leq F(p(x_n, x_{n-1})),$$

which yields

$$F(p(x_{n+1}, x_n)) \leq F(p(x_n, x_{n-1})) - \tau \dots \leq F(p(x_1, x_0)) - n\tau. \tag{3.5}$$

Taking limit on both side of (3.5), one can conclude that

$$\lim_{n \rightarrow \infty} F(p(x_{n+1}, x_n)) = -\infty$$

and applying (2) of Definition 3.5, we conclude

$$\lim_{n \rightarrow \infty} p(x_{n+1}, x_n) = 0. \tag{3.6}$$

Case 3: The sequence $\{x_n\}$ is a p -Cauchy sequence.

To reach this goal, applying (3) of Definition 3.5, one get that there exists $k \in (0, 1)$ such that

$$\lim_{n \rightarrow \infty} (p(x_{n+1}, x_n))^k F(p(x_{n+1}, x_n)) = 0. \tag{3.7}$$

Since (3.5) holds, we have

$$F(p(x_{n+1}, x_n)) - F(p(x_1, x_0)) \leq -n\tau,$$

for all $n \in \mathbb{N}$. Therefore,

$$(p(x_{n+1}, x_n))^k F(p(x_{n+1}, x_n)) - (p(x_{n+1}, x_n))^k F(p(x_1, x_0)) \leq -n(p(x_{n+1}, x_n))^k \tau \leq 0. \tag{3.8}$$

Taking limit on both side of (3.8) and applying (3.6) and (3.7), one can conclude that

$$\lim_{n \rightarrow \infty} n(p(x_{n+1}, x_n))^k = 0.$$

Hence, there exists $n_1 \in \mathbb{N}$ such that $n(p(x_{n+1}, x_n))^k \leq 1$, for all $n \geq n_1$. Hence, for any $n \geq n_1$,

$$p(x_{n+1}, x_n) \leq \frac{1}{n^{\frac{1}{k}}}, \quad (3.9)$$

which implies that the series $\sum_{i=1}^{\infty} p(x_{i+1}, x_i)$ is convergent.

Now let $m \geq n \geq n_1$. Then, by the triangular inequality and (3.9), we have

$$p(x_m, x_n) \leq \sum_{i=n+1}^m p(x_{i+1}, x_i)$$

Note that the term $\sum_{i=n+1}^m p(x_{i+1}, x_i)$ is the tale of convergent series $\sum_{i=1}^{\infty} p(x_{i+1}, x_i)$ and so tends to zero when m, n approach to infinity. Therefore,

$$\lim_{n \rightarrow \infty} \{\sup\{p(x_n, x_m) : m \geq n\}\} = 0,$$

Henceforth, $\{x_n\}$ is a p -Cauchy sequence in A . Since (X, d) is complete and A is a closed subset of X , there exist $x^* \in A$ such that $\lim_{n \rightarrow \infty} x_n = x^*$.

Case 4: $\{x^*\}$ is the unique p -best proximity point of T .

Since T is continuous, we have $\lim_{n \rightarrow \infty} Tx_n = Tx^*$. Hence, $p(x_{n+1}, Tx_n) \rightarrow p(x^*, Tx^*)$. From , $p(x^*, Tx^*) = p(A, B)$. So x^* is a p -best proximity point of T . The uniqueness of the p -best proximity points can be proved because, T is F_p -contraction. Suppose that $x_1, x_2 \in A$ such that $x_1 \neq x_2$ and $p(x_1, Tx_1) = p(x_2, Tx_2) = p(A, B)$. Then by the P_p -property of (A, B) , we have $p(x_1, x_2) = p(Tx_1, Tx_2)$. Also, $x_1 \neq x_2 \Rightarrow p(x_1, x_2) \neq 0$ Thus

$$F(p(x_1, x_2)) = F(p(Tx_1, Tx_2)) \leq F(p(x_1, x_2)) - \tau \dots < (p(x_1, x_2)),$$

which is a contraction. Hence the p -best proximity point is unique. □

By setting $A = B$ we obtained following result which is a special case of Theorem 3.1.

Corollary 3.7. *Let (X, d) be a complete metric space and A be a nonempty closed subset of X , p be a w_s -distance. Let $T : A \rightarrow A$ be an F_p -contractive self-map. Then T has a unique p -fixed point in A .*

4 The p -best proximity point for non-self F_p -proximal contractions

In this section, we will present p -best proximity point for non-self F_p -proximal contraction of the first and second kinds.

Definition 4.1. Let F is the function fulfills in Definition 3.5. A mapping $T : A \rightarrow B$ is said to be an F_p -proximal contraction of the first kind if there exists a w_s -distance p and $\tau > 0$ such that

1. $p(u_1, Tx_1) = p(A, B)$,
2. $p(u_2, Tx_2) = p(A, B)$,
3. $p(u_1, u_2), p(x_1, x_2) > 0$,

implies that $\tau + F(p(u_1, u_2)) \leq F(p(x_1, x_2))$, where $u_1, u_2, x_1, x_2 \in A$.

If $T : A \rightarrow B$ is a F_p -proximal contraction of the first kind and (A, B) has the P_p -property then T is a F_p -contractive non-self mapping, where p is w_s -distance.

Definition 4.2. A mapping $T : A \rightarrow B$ is said to be a F_p -proximal contraction of the second kind, if there exists a w_s -distance p and $\tau > 0$ such that

1. $p(u_1, Tx_1) = p(A, B)$,
2. $p(u_2, Tx_2) = p(A, B)$,
3. $p(Tu_1, Tu_2), p(Tx_1, Tx_2) > 0$,

implies that $\tau + F(p(Tu_1, Tu_2)) \leq F(p(Tx_1, Tx_2))$, where $u_1, u_2, x_1, x_2 \in A$.

Definition 4.3. A is said to be p -approximately compact with respect to B if every sequence $\{x_n\} \subset A$ satisfies in $p(y, x_n) \rightarrow p(y, A)$, for some y in B , has a convergent subsequence where p is a w_s -distance.

Definition 4.4. Let $T : A \rightarrow B$ is a mapping and let $g : A \rightarrow A$ is an isometry. The mapping T is said to preserve isometric distance with respect to g , if

$$p(Tgx_1, Tgx_2) = p(Tx_1, Tx_2),$$

where p is a w_s -distance and $x_1, x_2 \in A$.

Theorem 4.5. Let A and B are non-empty, closed subsets of a complete metric space X such that $A_{0,p}$ is non-empty. Let $T : A \rightarrow B$ is continuous, F_p -proximal contraction of the first kind and $T(A_{0,p}) \subseteq B_{0,p}$ and $g : A \rightarrow A$ is an isometry such that $A_{0,p} \subseteq g(A_{0,p})$. Then, there exists a unique element $x \in A$ such that $p(gx, Tx) = p(A, B)$.

Proof. Let us take an element $x_0 \in A_{0,p}$. Since $Tx_0 \in T(A_{0,p}) \subseteq B_{0,p}$ and $A_{0,p} \subseteq g(A_{0,p})$, there exists $x_1 \in A_{0,p}$ such that $p(gx_1, Tx_0) = p(A, B)$. If $x_0 = x_1$, then put $x_n = x_1$, for all $n \geq 2$. Also, we know that $T(x_1) \in T(A_{0,p}) \subseteq B_{0,p}$ and $A_{0,p} \subseteq g(A_{0,p})$. Thus, there exists $x_2 \in g(A_{0,p})$ such that $p(gx_2, Tx_1) = p(A, B)$. If $x_1 = x_2$ then put $x_n = x_2$ for all $n \geq 3$. Continuing this process, we can find a sequence $\{x_n\}$ in $A_{0,p}$, such that

$$p(gx_{n+1}, Tx_n) = p(A, B), \quad (4.1)$$

for all $n \in \mathbb{N}$.

Now we are ready to prove the convergence of the sequence $\{x_n\}$ in A . If there exists $n_0 \in \mathbb{N}$ such that $p(gx_{n_0}, gx_{n_0+1}) = 0$, then it is clear that the sequence $\{x_n\}$ is convergent. Hence, let $p(gx_n, gx_{n+1}) \neq 0$, for all $n \in \mathbb{N}$. Since T is a F_p -proximal contraction of the first kind and (4.1) holds, for any positive integer n , we have

$$\tau + F(p(gx_n, gx_{n+1})) \leq F(p(x_{n-1}, x_n)),$$

which implies that

$$F(p(x_n, x_{n+1})) \leq F(p(x_{n-1}, x_n)) - \tau \dots \leq F(p(x_0, x_1)) - n\tau.$$

Similar to the argument presented in Theorem 3.1, $\{x_n\}$ is a Cauchy sequence in A .

Since X is complete metric space and A is closed subset of X , there exists $x \in A$ such that $\lim_{n \rightarrow \infty} x_n = x$. Therefore, taking limit on both side of (4.1), we obtain $p(gx, Tx) = p(A, B)$. Now, x^* is in A such that $p(gx^*, Tx^*) = p(A, B)$. We show that $x = x^*$. On the contrary, suppose that $x \neq x^*$. Hence, $p(x, x^*) \neq 0$. Since T is a F_p -proximal contraction of the first kind and g is an isometry,

$$F(p(x, x^*)) = F(p(gx, gx^*)) \leq F(p(x, x^*)) - \tau < F(p(x, x^*)),$$

which is a contraction. Therefore, $x = x^*$ and this completes the proof. \square

If g is the identity mapping in the Theorem (4.1), then we obtain the following corollary as a special case.

Corollary 4.6. *Let A and B are non-empty, closed subsets of a complete metric space X such that A is p -approximatively compact with respect to B . Further, suppose that $A_{0,p}$ is non-empty. Let $T : A \rightarrow B$ is a continuous F_p -proximal contraction of the first kind and $T(A_{0,p}) \subseteq B_{0,p}$. Then T has a unique p -best proximity point in A .*

In the following, the p -best proximity point result for non-self F_p -proximal contraction of the second kind is presented.

Theorem 4.7. *Let A and B are non-empty, closed subsets of a complete metric space X such that A is p -approximatively compact with respect to B . Also, assume that $A_{0,p}$ is non-empty. Let $T : A \rightarrow B$ is a continuous F_p -proximal contraction*

of the second kind with $T(A_{0,p}) \subseteq B_{0,p}$ and $g : A \rightarrow A$ is an isometry satisfies in $A_{0,p} \subseteq g(A_{0,p})$. If T preserves isometric distance with respect to g . Then, there exists an element $x \in A$ such that

$$p(gx, Tx) = p(A, B).$$

Moreover, if x^* is another element of A such that $p(gx^*, Tx^*) = p(A, B)$ then $Tx = Tx^*$.

Proof. Let us take an element $x_0 \in A_{0,p}$. Since $Tx_0 \in T(A_{0,p}) \subseteq B_{0,p}$ and $A_{0,p} \subseteq g(A_{0,p})$, there exists $x_1 \in A_{0,p}$ such that $p(gx_1, Tx_0) = p(A, B)$. If $Tx_0 = Tx_1$ then put $x_n = x_1$, for all $n \geq 2$. Otherwise, since $Tx_1 \in T(A_{0,p}) \subseteq B_{0,p}$ and $A_{0,p} \subseteq g(A_{0,p})$, there exists $x_2 \in A_{0,p}$ such that $p(gx_2, Tx_1) = p(A, B)$. If $Tx_1 = Tx_2$ then put $x_n = x_2$, for all $n \geq 3$. Continuing this process, we can find a sequence $\{x_n\}$ in $A_{0,p}$ such that $p(gx_{n+1}, Tx_n) = p(A, B)$, for all $n \in \mathbb{N}$. We are ready to prove the convergence of the sequence $\{Tx_n\}$ in B . If there exists $n_0 \in \mathbb{N}$ such that $p(Tgx_{n_0}, Tgx_{n_0+1}) = 0$ then it is transparent that the sequence $\{Tx_n\}$ is convergent. Hence, let $p(Tgx_n, Tgx_{n+1}) \neq 0$, for all $n \in \mathbb{N}$. Since T is a F_p -proximal contraction of the second kind and preserves isometric distance with respect to g and (4.1) holds. Hence, for any positive integer n we have

$$\tau + F(p(Tgx_n, Tgx_{n+1})) \leq F(p(Tx_{n-1}, Tx_n)),$$

which implies that

$$F(p(Tx_n, Tx_{n+1})) \leq F(p(Tx_{n-1}, Tx_n)) - \tau \dots \leq F(p(Tx_0, Tx_1)) - n\tau.$$

Analogous the proof of Theorem 3.1, $\{Tx_n\}$ is a Cauchy sequence in B .

Since X is complete metric space and B is closed subset of X , there exists $y \in B$ such that $\lim_{n \rightarrow \infty} Tx_n = y$. Applying triangular inequality, we have

$$\begin{aligned} p(y, A) &\leq p(y, gx_n) \\ &\leq p(y, Tx_{n-1}) + p(Tx_{n-1}, gx_n) \\ &= p(y, Tx_{n-1}) + p(A, B) \\ &\leq p(y, Tx_{n-1}) + p(y, A). \end{aligned} \tag{4.2}$$

Taking limit on both side of (4.2), we obtain $\lim_{n \rightarrow \infty} p(y, gx_n) = p(y, A)$. Since A is p -approximatively compact with respect to B , there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ which converges to some $z \in A$. Therefore,

$$p(z, y) = \lim_{k \rightarrow \infty} p(gx_{n_k}, Tx_{n_k-1}) = p(A, B).$$

This implies that $z \in A_{0,p}$. Since $A_{0,p} \subseteq g(A_{0,p})$, there exists $x \in A_{0,p}$ such that $z = gx$. Since $\lim_{n \rightarrow \infty} g(x_{n_k}) = g(x)$ and g is an isometry, we have

$$\lim_{n \rightarrow \infty} x_{n_k} = x.$$

Also, T is continuous so $\{Tx_n\}$ is convergent to y . Therefore, $\lim_{n \rightarrow \infty} Tx_{n_k} = Tx = y$. Thus, it yields

$$p(gx, Tx) = \lim_{n \rightarrow \infty} p(gx_{n_k}, Tx_{n_k}) = p(A, B).$$

Now, x^* is in A such that $p(gx^*, Tx^*) = p(A, B)$ and so $Tx = Tx^*$. On the contrary, suppose that $Tx \neq Tx^*$. Hence $p(Tx, Tx^*) \neq 0$. Since T is a F_p -proximal contraction of the second kind and preserves isometric distance with respect to g where g is an isometry,

$$F(p(Tx, Tx^*)) = F(p(Tgx, Tgx^*)) \leq F(p(Tx, Tx^*)) - \tau < F(p(Tx, Tx^*)),$$

which is a contraction. Therefore, $Tx = Tx^*$ and this completes the proof. \square

The next corollary is obtained by taking g as identity mapping in Theorem 4.2.

Corollary 4.8. *Let A and B are non-empty, closed subsets of a complete metric space X such that A is p -approximatively compact with respect to B . Further, suppose that $A_{0,p}$ is non-empty. Let $T : A \rightarrow B$ is a continuous F_p -proximal contraction of the second kind and $T(A_{0,p}) \subseteq B_{0,p}$. Then T has a p -best proximity point in A . Moreover, if x^* is another p -best proximity point of T then $Tx = Tx^*$.*

Example 4.9. *Let $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ is defined by $F(\alpha) = \ln \alpha$. One can easily check that F satisfies axioms 1-3 of Definition [3.5]. So, each mapping $T : A \rightarrow B$ satisfying Definition [3.5] is an F_p -contraction such that $p(Tx, Ty) \leq e^{-\tau} p(x, y)$ for all $x, y \in X, Tx \neq Ty$. As by Definition of F_p -contraction:*

- 1: If $\alpha_1 < \alpha_2$ implies that $\ln(\alpha_1) < \ln(\alpha_2)$;
- 2: For each sequence $\{\alpha_n\}$ of positive numbers $\lim_{n \rightarrow \infty} \alpha_n = 0$ iff $\lim_{n \rightarrow \infty} \ln(\alpha_n) = -\infty$;

$$\begin{aligned} \lim_{\alpha \rightarrow 0^+} \alpha^k \ln \alpha &= \lim_{\alpha \rightarrow 0^+} \frac{\ln \alpha}{\frac{1}{\alpha^k}} \\ &= \lim_{\alpha \rightarrow 0^+} \frac{\frac{1}{\alpha}}{\frac{k}{\alpha^{k+1}}} \\ &= \lim_{\alpha \rightarrow 0^+} \frac{\alpha^k}{k} = 0 \end{aligned}$$

Open Problem

There is an open problem that whether we can obtain same results for existence of best proximity point theorems by changing or omitting the third axiom of the definition of F_p -contraction.

Competing interests

The authors declared that they have no competing interests.

Author's contributions

All the authors contributed equally in writing this paper. They have also finalized and approved this manuscript.

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