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# Common Best proximity Points for Weakly Proximal Increasing Mappings

**V. Pragadeeswarar**†1

†Department of Mathematics,
Amrita School of Engineering, Coimbatore,
Amrita Vishwa Vidyapeetham
Tamil Nadu, India.

**Abstract :** The aim of this paper is to present the existence and uniqueness of common best proximity point for weakly proximal increasing mappings satisfying certain contractive conditions in a complete ordered metric spaces. Moreover, we furnish suitable examples to demonstrate the validity of the main results.

**Keywords :** Partially ordered set, Optimal approximate solution, Proximally increasing mapping, Fixed point, Best proximity point.
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## 1 Introduction

A study of best proximity point theory is an useful tool for providing optimal approximate solutions when a mapping does not have a fixed point. In other words, optimization problems can be converted to the problem of finding best proximity points. Hence, the existence of best proximity points develops the theory of optimization. For more details, one can go through [1, 2, 3, 4, 5, 6, 7, 8, 9, 10,

[1]Corresponding author email: pragadeeswarar@gmail.com

11, 12]. Also one can find the existence of best proximity points in the setting of partially order metric spaces in [13, 14, 15, 16, 17].

In this article, we introduce a new class of mappings, called weakly proximal increasing, which extends the class of weakly increasing mappings to the class of non-self mappings, and also establish the common best proximity point theorems for this class in the setting of partially ordered metric spaces. Moreover, we give some suitable examples to illustrate our main results. Also, our results extend and generalize the corresponding results given by Radenović and Kadelburg [18] and some authors in the literature.

## 2 Preliminaries

In this section, we give some basic definitions and notions that will be used frequently.

**Definition 2.1.** *([19]). A function $\psi : [0, \infty) \to [0, \infty)$ is said to be an altering distance function or control functions if it satisfies the following conditions.*

*(i)* $\psi$ *is continuous and non-decreasing.*

*(ii)* $\psi(t) = 0$ *if and only if* $t = 0$.

Let $X$ be a non-empty set such that $(X, d, \preceq)$ is a partially ordered metric space. Consider $A$ and $B$ are non-empty subsets of the metric space $(X, d)$. Now, we recall the following notions:

$$d(A, B) := \inf\{d(a, b) : a \in A \text{ and } b \in B\},$$

$$A_0 = \{a \in A : d(a, b) = d(A, B) \text{ for some } b \in B\},$$

$$B_0 = \{b \in B : d(a, b) = d(A, B) \text{ for some } a \in A\}.$$

**Definition 2.2.** *A point $a \in A$ is called a best proximity point of the mapping $T : A \to B$ if $d(a, Ta) = d(A, B)$.*

**Definition 2.3.** *A point $a \in A$ is called a common best proximity point of the mappings $T : A \to B$ and $S : A \to B$ if $d(a, Ta) = d(a, Sa) = d(A, B)$.*

For the case of self mapping, the notion of a best proximity point and a common best proximity point are reduced to a fixed point and a common fixed point respectively.

**Definition 2.4.** *([8]). Let $(A, B)$ be a pair of non-empty subsets of a metric space $(X, d)$ with $A_0 \neq \emptyset$. Then the pair $(A, B)$ is said to have the $P$-property if and only if*

$$\left.\begin{aligned} d(a_1, b_1) = d(A, B) \\ d(a_2, b_2) = d(A, B) \end{aligned}\right\} \implies d(a_1, a_2) = d(b_1, b_2)$$

*where $a_1, a_2 \in A_0$ and $b_1, b_2 \in B_0$.*

**Definition 2.5.** *([13]). A mapping $T : A \to B$ is said to be proximally increasing if it satisfies the condition that*

$$\left.\begin{aligned} b_1 \preceq b_2 \\ d(a_1, Tb_1) = d(A, B) \\ d(a_2, Tb_2) = d(A, B) \end{aligned}\right\} \implies a_1 \preceq a_2$$

*where $a_1, a_2, b_1, b_2 \in A$.*

It is easy to observe that, for a self-mapping, the notion of a proximally increasing mapping reduces to that of increasing mapping.

**Definition 2.6.** *([18]). A pair $(T, S)$ of mappings $T, S : A \to A$ is said to be weakly increasing if $Ta \preceq STa$ and $Sa \preceq TSa$ for all $a \in A$.*

Let us define the new notion called weakly proximal increasing as follows.

**Definition 2.7.** *A pair $(T, S)$ of mappings $T, S : A \to B$ is said to be weakly proximal increasing if it satisfies the following conditions:*

*(i)* $\forall a \in A$, $\exists u_1, u_2 \in A$ *such that* $d(u_1, Ta) = d(A, B)$, $d(u_2, Su_1) = d(A, B)$ *and* $u_1 \preceq u_2$;

*(ii)* $\forall a \in A$, $\exists v_1, v_2 \in A$ *such that* $d(v_1, Sa) = d(A, B)$, $d(v_2, Tv_1) = d(A, B)$ *and* $v_1 \preceq v_2$.

One can see that, for a self-mapping, the notion of weakly proximal increasing mapping reduces to that of a weakly increasing mapping.

Note that weakly proximal increasing mappings need not be proximally increasing.

**Example 2.1.** *Let $X = \{(0, 1), (\frac{1}{2}, \frac{1}{2}), (1, 0), (1, 1), (2, 0)\} \subset \mathbb{R}^2$ and consider the order*
*$(a, b) \preceq (z, t) \Leftrightarrow a \preceq z$ and $b \preceq t$, where $\preceq$ is the usual order on $\mathbb{R}$.*

*Thus,* $(X, \preceq)$ *is a partially ordered set. Besides,* $(X, d_1)$ *is a complete metric space where the metric is defined as* $d_1((a_1, b_1), (a_2, b_2)) = |a_1 - a_2| + |b_1 - b_2|$. *Let* $A = \{(0, 1), (1, 0), (\frac{1}{2}, \frac{1}{2})\}$ *and* $B = \{(2, 0), (1, 1)\}$ *be two closed subsets of* $X$. *Then,* $d(A, B) = 1, A = A_0$ *and* $B = B_0$. *Let* $T, S : A \to B$ *be defined by* $Ta = (1, 1), \forall a \in A$ *and* $S(0, 1) = (1, 1), S(1, 0) = (2, 0)$, $S(\frac{1}{2}, \frac{1}{2}) = (1, 1)$.

*Then, a pair* $(T, S)$ *is weakly proximal increasing mappings. However, both* $T$ *and* $S$ *are not proximally increasing.*

# 3 Main results

Now, let us state our main result.

**Theorem 3.1.** *Let* $A$ *and* $B$ *be two non-empty closed subsets of a partially ordered complete metric space* $(X, \preceq, d)$ *such that* $A_0 \neq \emptyset$ *and the pair* $(A, B)$ *has the* $P-$ *property. Let* $T, S : A \to B$ *be two non-self mappings satisfies the following conditions.*

*(i)* $(T, S)$ *is weakly proximal increasing;*

*(ii)* $T$ *or* $S$ *is continuous;*

*(iii) for every two comparable elements* $a, b \in A$,

$$\psi(d(T(a), S(b))) \leq \psi(m(a, b)) - \phi(m(a, b)), \tag{3.1}$$

*where*

$$m(a, b) = \max\{d(a, b), d(a, Ta) - d(A, B), d(b, Sb) - d(A, B), \frac{d(a, Sb) + d(b, Ta)}{2} - d(A, B)\},$$

$\psi$ *is an altering distance function,* $\phi$ *is a nondecreasing function also* $\phi(t) = 0$ *iff* $t = 0$ *and* $\psi - \phi$ *is a nondecreasing function.*

*Then, there exists at least one element* $u$ *in* $A$ *such that* $d(u, Tu) = d(u, Su) = d(A, B)$.

*Proof.* Since the subset $A \neq \emptyset$, we can take $a_0 \in A$, by using $(T, S)$ is weakly proximal increasing, $\exists\ a_1, a_2$ in $A$ such that $d(a_1, Ta_0) = d(A, B)$, $d(a_2, Sa_1) = d(A, B)$ and $a_1 \preceq a_2$.

For $a_1 \in A$, again by using $(T, S)$ is weakly proximal increasing, $\exists\ a_2^*, a_3$ in $A$ such that $d(a_2^*, Sa_1) = d(A, B)$, $d(a_3, Ta_2^*) = d(A, B)$ and $a_2^* \preceq a_3$.

Using $P$-property for $d(a_2, Sa_1) = d(A, B)$ and $d(a_2^*, Sa_1) = d(A, B)$, we get $a_2 = a_2^*$. Hence, $d(a_2, Sa_1) = d(A, B)$, $d(a_3, Ta_2) = d(A, B)$ and $a_2 \preceq a_3$.

Now, take $a_2 \in A$ and using $(T, S)$ is weakly proximal increasing, $\exists\ a_3^*, a_4$ in $A$ such that $d(a_3^*, Ta_2) = d(A, B)$, $d(a_4, Sa_3^*) = d(A, B)$ and $a_3^* \preceq a_4$. Again, by using the $P$-property for $d(a_3, Ta_2) = d(A, B)$ and $d(a_3^*, Ta_2) = d(A, B)$, we get $a_3 = a_3^*$. Hence, $a_3 \preceq a_4$. Continuing this process, we can construct a sequence $\{a_n\}$ in $A_0$ such that

$$d(a_{2n+1}, Ta_{2n}) = d(A, B) \text{ and } d(a_{2n+2}, Sa_{2n+1}) = d(A, B) \text{ for all } n \geq 0 \tag{3.2}$$

$$\text{with } a_1 \preceq a_2 \preceq \cdots a_n \preceq a_{n+1} \cdots .$$

If there exists $n_0$ such that $a_{n_0} = a_{n_0+1}$. Then, the sequence $\{a_n\}$ is a constant for $n \leq n_0$. Indeed, let $n_0 = 2k$. Then $a_{2k} = a_{2k+1}$ and we obtain from (3.1) that

$$\psi(d(a_{2k+1}, a_{2k+2})) = \psi(d(Ta_{2k}, Sa_{2k+1})) \leq \psi(m(a_{2k}, a_{2k+1})) - \phi(m(a_{2k}, a_{2k+1})) \tag{3.3}$$

where

$$\begin{aligned} m(a_{2k}, a_{2k+1}) &= \max\{d(a_{2k}, a_{2k+1}), d(a_{2k}, Ta_{2k}) - d(A, B), d(a_{2k+1}, Sa_{2k+1}) - d(A, B), \\ &\quad \frac{d(a_{2k}, Sa_{2k+1}) + d(a_{2k+1}, Ta_{2k})}{2} - d(A, B)\} \\ &\leq \max\{d(a_{2k}, a_{2k+1}), d(a_{2k}, a_{2k+1}) + d(a_{2k+1}, Ta_{2k}) - d(A, B), \\ &\quad d(a_{2k+1}, a_{2k+2}) + d(a_{2k+2}, Sa_{2k+1}) - d(A, B), \\ &\quad \frac{d(a_{2k}, a_{2k+1}) + d(a_{2k+1}, a_{2k+2}) + d(a_{2k+2}, Sa_{2k+1}) + d(a_{2k+1}, Ta_{2k})}{2} \\ &\quad - d(A, B)\} \\ &= \max\{d(a_{2k}, a_{2k+1}), d(a_{2k}, a_{2k+1}), d(a_{2k+1}, a_{2k+2}), \\ &\quad \frac{d(a_{2k}, a_{2k+1}) + d(a_{2k+1}, a_{2k+2})}{2}\} \\ &= \max\{0, 0, d(a_{2k+1}, a_{2k+2}), \frac{d(a_{2k+1}, a_{2k+2})}{2}\}. \end{aligned}$$

Since $\psi - \phi$ is a nondecreasing function and using (3.3), we get

$$\begin{aligned} \psi(d(a_{2k+1}, a_{2k+2})) &= \psi(d(Ta_{2k}, Sa_{2k+1})) \\ &\leq \psi(d(a_{2k+1}, a_{2k+2})) - \phi(d(a_{2k+1}, a_{2k+2})) \end{aligned}$$

and so $\phi(d(a_{2k+1}, a_{2k+2})) \leq 0$ and $a_{2k+1} = a_{2k+2}$. Similarly, if $n_0 = 2k + 1$ one can easily see that $a_{2k+2} = a_{2k+3}$ and so the sequence $\{a_n\}$ is constant (starting from some $n_0$ ) and $a_{n_0}$ is a common best proximity point of $T$ and $S$.

Suppose now $d(a_n, a_{n+1}) > 0$ for each $n \in \mathbb{N} \cup \{0\}$. We shall prove that for each $n \in \mathbb{N} \cup \{0\}$

$$d(a_{n+1}, a_{n+2}) \leq m(a_n, a_{n+1}) \leq d(a_n, a_{n+1}). \tag{3.4}$$

Since $a_{2n} \preceq a_{2n+1}$, by (3.1), we obtain that

$$\psi(d(a_{2n+1}, a_{2n+2})) = \psi(d(Ta_{2n}, Sa_{2n+1})) \leq \psi(m(a_{2n}, a_{2n+1})) - \phi(m(a_{2n}, a_{2n+1})) \tag{3.5}$$
$$\leq \psi(m(a_{2n}, a_{2n+1}))$$

and since $\psi$ is nondecreasing, it follows that

$$d(a_{2n+1}, a_{2n+2}) \leq m(a_{2n}, a_{2n+1}) \tag{3.6}$$

where

$$\begin{aligned} m(a_{2n}, a_{2n+1}) &= \max\{d(a_{2n}, a_{2n+1}), d(a_{2n}, Ta_{2n}) - d(A, B), d(a_{2n+1}, Sa_{2n+1}) - d(A, B), \\ &\quad \frac{d(a_{2n}, Sa_{2n+1}) + d(a_{2n+1}, Ta_{2n})}{2} - d(A, B)\} \\ &\leq \max\{d(a_{2n}, a_{2n+1}), d(a_{2n}, a_{2n+1}) + d(a_{2n+1}, Ta_{2n}) - d(A, B), \\ &\quad d(a_{2n+1}, a_{2n+2}) + d(a_{2n+2}, Sa_{2n+1}) - d(A, B), \\ &\quad \frac{d(a_{2n}, a_{2n+1}) + d(a_{2n+1}, a_{2n+2}) + d(a_{2n+2}, Sa_{2n+1}) + d(a_{2n+1}, Ta_{2n})}{2} \\ &\quad - d(A, B)\} \\ &= \max\{d(a_{2n}, a_{2n+1}), d(a_{2n}, a_{2n+1}), d(a_{2n+1}, a_{2n+2}), \\ &\quad \frac{d(a_{2n}, a_{2n+1}) + d(a_{2n+1}, a_{2n+2})}{2}\} \\ &= \max\{d(a_{2n}, a_{2n+1}), d(a_{2n+1}, a_{2n+2})\}. \end{aligned}$$

Since $\psi - \phi$ is a nondecreasing function and using (3.5), we get

$$\begin{aligned} \psi(d(a_{2n+1}, a_{2n+2})) &= \psi(d(Ta_{2n}, Sa_{2n+1})) \\ &\leq \psi(\max\{d(a_{2n}, a_{2n+1}), d(a_{2n+1}, a_{2n+2})) \\ &\quad - \phi(\max\{d(a_{2n}, a_{2n+1}), d(a_{2n+1}, a_{2n+2})). \end{aligned} \tag{3.7}$$

If $d(a_{2n+1}, a_{2n+2}) \geq d(a_{2n}, a_{2n+1}) > 0$, then

$$\psi(d(a_{2n+1}, a_{2n+2})) \leq \psi(d(a_{2n+1}, a_{2n+2})) - \phi(d(a_{2n+1}, a_{2n+2}))$$

which is a contradiction. So, we have $d(a_{2n+1}, a_{2n+2}) \leq d(a_{2n}, a_{2n+1})$ and $m(a_{2n}, a_{2n+1}) \leq d(a_{2n}, a_{2n+1})$. Since, from (3.7) and $m(a_{2n}, a_{2n+1}) \leq d(a_{2n}, a_{2n+1})$, (3.4) is proved for $d(a_{2n+2}, a_{2n+1})$. In a similar way one can obtain that

$$d(a_{2n+3}, a_{2n+2}) \leq m(a_{2n+2}, a_{2n+1}) \leq d(a_{2n+2}, a_{2n+1}). \tag{3.8}$$

Hence, (3.4) holds for each $n \in \mathbb{N} \cup \{0\}$. Hence, the sequence $\{d(a_n, a_{n+1})\}$ is non-increasing and bounded below. Thus, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(a_n, a_{n+1}) = \lim_{n\to\infty} m(a_n, a_{n+1}) = r \geq 0. \tag{3.9}$$

Suppose that $\lim_{n\to\infty} d(a_n, a_{n+1}) = \lim_{n\to\infty} m(a_n, a_{n+1}) = r > 0$. Then the inequality

$$\psi(d(a_{n+1}, a_{n+2})) \leq \psi(m(a_n, a_{n+1})) - \phi(m(a_n, a_{n+1})) \leq \psi(m(a_n, a_{n+1}))$$

implies that

$$\lim_{n\to\infty} \phi(m(a_n, a_{n+1})) = 0. \tag{3.10}$$

But, as $0 < r \leq d(a_{n+1}, a_{n+2}) \leq m(a_n, a_{n+1})$ and $\phi$ is nondecreasing function,

$$0 < \phi(r) \leq \phi(m(a_n, a_{n+1})),$$

and this gives us $\lim_{n\to\infty} \phi(m(a_n, a_{n+1})) \geq \phi(r) > 0$ which contradicts (3.10). Hence,

$$\lim_{n\to\infty} m(a_n, a_{n+1}) = 0 = \lim_{n\to\infty} d(a_{n+1}, a_{n+2}). \tag{3.11}$$

Now to prove that $\{a_n\}$ is a Cauchy sequence. In order to prove that $\{a_n\}$ is a Cauchy sequence in $X$. It is enough to prove that $\{a_{2n}\}$ is a Cauchy sequence. In contrary case, suppose that $\{a_{2n}\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find subsequences $\{a_{2m(k)}\}$ and $\{a_{2n(k)}\}$ of $\{a_{2n}\}$ such that $n(k)$ is smallest index for which $n(k) > m(k) > k, \quad d(a_{2m(k)}, a_{2n(k)}) \geq \epsilon$. This means that

$$d(a_{2m(k)}, a_{2n(k)-1}) < \epsilon. \tag{3.12}$$

$$\begin{aligned}
\epsilon &\leq d(a_{2m(k)}, a_{2n(k)}) \\
&\leq d(a_{2m(k)}, a_{2n(k)-1}) + d(a_{2n(k)-1}, a_{2n(k)}) \\
&< \epsilon + d(a_{2n(k)-1}, a_{2n(k)}).
\end{aligned}$$

Letting $k \to \infty$ and using (3.11) we can conclude that

$$\lim_{k\to\infty} d(a_{2m(k)}, a_{2n(k)}) = \epsilon. \tag{3.13}$$

Again,

$$d(a_{2m(k)}, a_{2n(k)-1}) \leq d(a_{2m(k)}, a_{2n(k)}) + d(a_{2n(k)}, a_{2n(k)-1})$$

and

$$d(a_{2m(k)}, a_{2n(k)}) \leq d(a_{2m(k)}, a_{2n(k)-1}) + d(a_{2n(k)}, a_{2n(k)-1})$$

Therefore,

$$|d(a_{2m(k)}, a_{2n(k)-1}) - d(a_{2m(k)}, a_{2n(k)})| \leq d(a_{2n(k)}, a_{2n(k)-1}).$$

Taking $k \to \infty$ and using (3.13) and (3.11), it follows that

$$\lim_{k\to\infty} d(a_{2m(k)}, a_{2n(k)-1}) = \epsilon. \tag{3.14}$$

Similarly, we can prove that

$$\lim_{k\to\infty} d(a_{2m(k)-1}, a_{2n(k)}) = \lim_{k\to\infty} d(a_{2m(k)-1}, a_{2n(k)-1}) = \lim_{k\to\infty} d(a_{2m(k)+1}, a_{2n(k)})$$
$$= \lim_{k\to\infty} d(a_{2m(k)}, a_{2n(k)+1}) = \lim_{k\to\infty} d(a_{2m(k)-1}, a_{2n(k)+1}) = \epsilon. \tag{3.15}$$

Then we have

$$\lim_{k\to\infty} m(a_{2n(k)}, a_{2m(k)-1}) = \epsilon. \tag{3.16}$$

Since $2m(k) \leq 2n(k)+1$, $a_{2m(k)-1} \preceq a_{2n(k)}$ and using $P-$ property for (3.2) and from (3.1), we have

$$0 < \psi(\epsilon) \leq \psi(d(a_{2m(k)}, a_{2n(k)+1})) \leq \psi(m(a_{2m(k)-1}, a_{2n(k)}) - \phi(m(a_{2m(k)-1}, a_{2n(k)})$$
$$\leq \psi(m(a_{2m(k)-1}, a_{2n(k)}).$$

Using (3.16) and continuity of $\psi$ in the above inequality we can obtain

$$\lim_{k\to\infty} \phi(m(a_{2m(k)-1}, a_{2n(k)})) = 0 \tag{3.17}$$

But, from $\lim_{k\to\infty} m(a_{2m(k)-1}, a_{2n(k)}) = \epsilon$ we can find $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$

$$\frac{\epsilon}{2} \leq m(a_{2m(k)-1}, a_{2n(k)})$$

and consequently,

$$0 < \phi(\frac{\epsilon}{2}) \leq \phi(m(a_{2m(k)-1}, a_{2n(k)})) \text{ for each } k \geq k_0.$$

Therefore, $0 < \phi(\frac{\epsilon}{2}) \leq \phi(m(a_{2m(k)-1}, a_{2n(k)}))$ and this contradicts (3.17). Thus, $\{a_n\}$ is a Cauchy sequence in $A$ and hence it converges to an element in $A$, named $u$.

**Step 3.** We have to prove that $u$ is a common best proximity point of $T$ and $S$.

Without loss of generality, assume that the mapping $T$ is continuous. Since $a_{2n} \to u$, we obtain that $Ta_{2n} \to Tu$. On the other hand, $a_{2n+1} \to u$.

Hence the continuity of the metric function $d$ implies that $d(a_{2n+1}, Ta_{2n}) \to d(u, Tu)$. But (3.2) shows that the sequence $d(a_{2n+1}, Ta_{2n})$ is a constant sequence with the value $d(A, B)$. Therefore, $d(u, Tu) = d(A, B)$.

To prove that $d(u, Su) = d(A, B)$, using $u \preceq u$, we can put $a = b = u$ in (3.1) and obtain that

$$\psi(d(T(u), S(u))) \leq \psi(m(u, u)) - \phi(m(u, u)), \tag{3.18}$$

$$\begin{aligned} \text{where } m(u,u) &= \max\{d(u,u), d(u,Tu) - d(A,B), d(u,Su) - d(A,B), \\ &\quad \frac{d(u,Su) + d(u,Tu)}{2} - d(A,B)\} \\ &= \max\{0, 0, d(u,Su) - d(A,B), \frac{d(u,Su) + d(u,Tu)}{2} - d(A,B)\} \\ &= d(u,Su) - d(A,B). \end{aligned} \tag{3.19}$$

By using the triangle inequality, we get $d(u, Su) \leq d(u, Tu) + d(Tu, Su)$. Since $u$ is a best proximity point for $T$ and $\psi$ is increasing, we obtain that

$$\psi(d(u, Su) - d(A, B)) \leq \psi(d(Tu, Su)). \tag{3.20}$$

Now from (3.18), (3.19) and (3.20) we get that

$$\begin{aligned} \psi(d(u,Su) - d(A,B)) &\leq \psi(d(Tu,Su)) \\ &\leq \psi(d(u,Su) - d(A,B)) - \phi(d(u,Su) - d(A,B)) \end{aligned}$$

i.e., $\phi(d(u, Su) - d(A, B)) \leq 0$. By using the property $\phi(t) = 0$ iff $t = 0$, we get that $d(u, Su) - d(A, B) = 0$. Hence $u$ is a common best proximity point of $T$ and $S$. □

Next, we prove that Theorem 3.1 is still valid for ($T$ or $S$) eventhough, the mappings are not continuous, assuming the following hypothesis in $A$.

$$\{a_n\} \text{ is a nondecreasing sequence in } A \text{ such that } a_n \to a, \text{ then } a_n \preceq a. \tag{3.21}$$

**Theorem 3.2.** *Assume the condition (3.21) instead of continuity of ($T$ or $S$) in Theorem 3.1.*

*Proof.* Following the proof of Theorem 3.1, there exists a sequence $\{a_n\}$ in $A$ satisfying the following condition

$$d(a_{2n+1}, Ta_{2n}) = d(A, B) \text{ and } d(a_{2n+2}, Sa_{2n+1}) = d(A, B) \text{ for all } n \geq 0 \tag{3.22}$$

$$\text{with } a_0 \preceq a_1 \preceq a_2 \preceq \cdots a_n \preceq a_{n+1} \cdots .$$

Since $\{a_n\}$ is nondecreasing in $A$ and $a_n \to u$ then by condition (3.21), we get $a_n \preceq u$. Take $a = a_{2n}$ and $b = u$ in (3.1). Then we obtain that

$$\psi(d(Ta_{2n}, Su)) \leq \psi(m(a_{2n}, u)) - \phi(m(a_{2n}, u)), \tag{3.23}$$

$$\begin{aligned}
\text{where } m(a_{2n}, u) &= \max\{d(a_{2n}, u), d(a_{2n}, Ta_{2n}) - d(A, B), d(u, Su) - d(A, B),\\
&\quad \frac{d(a_{2n}, Su) + d(u, Ta_{2n})}{2} - d(A, B)\}\\
&\leq \max\{d(a_{2n}, u), d(a_{2n}, a_{2n+1}) + d(a_{2n+1}, Ta_{2n}) - d(A, B),\\
&\quad d(u, Su) - d(A, B),\\
&\quad \frac{d(a_{2n}, Su) + d(u, a_{2n+1}) + d(a_{2n+1}, Ta_{2n})}{2} - d(A, B)\}\\
&= \max\{d(a_{2n}, u), d(a_{2n}, a_{2n+1}), d(u, Su) - d(A, B),\\
&\quad \frac{d(a_{2n}, Su) + d(u, a_{2n+1}) - d(A, B)}{2}\}.
\end{aligned}$$

From the above inequality, we obtain that

$$\begin{aligned}
d(u, Su) - d(A, B) &\leq m(a_{2n}, u)\\
&\leq \max\{d(a_{2n}, u), d(a_{2n}, a_{2n+1}), d(u, Su) - d(A, B),\\
&\quad \frac{d(a_{2n}, Su) + d(u, a_{2n+1}) - d(A, B)}{2}\}.
\end{aligned}$$

Now using $a_n \to u$, we get

$$\lim_{n\to\infty} m(a_{2n}, u) = d(u, Su) - d(A, B). \tag{3.24}$$

Hence, we can find $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$,
$\frac{d(u,Su)-d(A,B)}{2} \leq m(a_{2n}, u)$ and consequently, since $\phi$ is nondecreasing,
we obtain that

$$\phi(\frac{d(u, Su) - d(A, B)}{2}) \leq \phi(m(a_{2n}, u)) \text{ for all } n \geq n_0. \tag{3.25}$$

By using the triangle inequality, we get

$$\begin{aligned}
d(a_{2n+1}, Su) &\leq d(a_{2n+1}, Ta_{2n}) + d(Ta_{2n}, Su)\\
d(a_{2n+1}, Su) - d(A, B) &\leq d(Ta_{2n}, Su).
\end{aligned}$$

Now, using $\psi$ is increasing from the above inequality and from (3.23), we obtain that

$$\begin{aligned}
\psi(d(a_{2n+1}, Su) - d(A, B)) \leq \psi(d(Ta_{2n}, Su)) &\leq \psi(m(a_{2n}, u)) - \phi(m(a_{2n}, u))\\
&\leq \psi(m(a_{2n}, u)).
\end{aligned} \tag{3.26}$$

Using (3.24), $a_n \to u$ and continuity of $\psi$ in (3.26), we can obtain that

$$\lim_{n\to\infty} \phi(m(a_{2n}, u)) = 0. \tag{3.27}$$

From the property of $\phi$, (3.25) and (3.27), we get $d(u, Su) - d(A, B) = 0$. Hence $u$ is a common best proximity point of $T$ and $S$. This completes the proof of the theorem. □

Putting $T = S$ in Theorem 3.1 and Theorem 3.2, we get

**Corollary 3.1.** *Let $A$ and $B$ be two non-empty closed subsets of a partially ordered complete metric space $(X, \preceq, d)$ such that $A_0 \neq \emptyset$ and the pair $(A, B)$ has the $P-$property. Let $S : A \to B$ be a non-self mapping satisfies the following conditions.*

*(i)* *$(S, S)$ is weakly proximal increasing;*

*(ii)* *$S$ is continuous (or) $\{a_n\}$ is a nondecreasing sequence in $A$ such that $a_n \to a$, then $a_n \preceq a$;*

*(iii)* *for every two comparable elements $a, b \in A$,*

$$\psi(d(S(a), S(b))) \leq \psi(m(a, b)) - \phi(m(a, b)), \text{ where} \tag{3.28}$$

$m(a, b) = \max\{d(a, b), d(a, Sa) - d(A, B), d(b, Sb) - d(A, B), \frac{d(a,Sb)+d(b,Sa)}{2} - d(A, B)\}$, *$\psi$ is altering distance function, $\phi$ is nondecreasing function also $\phi(t) = 0$ iff $t = 0$ and $\psi - \phi$ is nondecreasing function.*

*Then, there exists at least one element $u$ in $A$ such that $d(u, Su) = d(A, B)$.*

This corollary can be proved in a similar way as Theorem 3.1 and Theorem 3.2.

The following simple example shows that conditions of theorems in the previous section are not sufficient for the uniqueness of a best proximity points (resp. common best proximity points).

Let us illustrate the Theorem 3.1 with the following example.

**Example 3.1.** *Let $X = \{(0, 1), (0, -1), (1, 0), (-1, 0)\} \subset \mathbb{R}^2$ and consider the order*
*$(a, b) \preceq (z, t) \Leftrightarrow a \preceq z$ and $b \preceq t$, where $\preceq$ is the usual order on $\mathbb{R}$.*

*Thus, $(X, \preceq)$ is a partially ordered set. Besides $(X, d_2)$ is a complete metric space, considering $d_2$ as the euclidean metric. Let $A = \{(0, 1), (1, 0)\}$ and $B = \{(-1, 0), (0, -1)\}$ be two closed subsets of $X$. Then, $d(A, B) = 1, A = A_0$ and $B = B_0$. Let $S : A \to B$ be defined by $S(x, y) = (-y, -x)$. The only comparable* *pair of points in $A$ is $a \preceq a$ for $a \in A$, hence the inequality (3.28) is fulfilled for arbitrary control functions. Also, it satisfies the condition (3.21). It is easy to see that $S$ is continuous and $(S, S)$ is weakly proximal increasing. Therefore, all the hypotheses of the Theorem 3.1 are satisfied. Also, it can be observed that $S$ has two best proximity points. i.e.,* $(0, 1)$ *and* $(1, 0)$.

Since, for any nonempty subset $A$ of $X$, the pair $(A, A)$ has the $P-$ property, also one can see that, for a self-mapping, the notion of a proximally increasing mapping reduces to an increasing mapping and a weakly proximal increasing mapping becomes a weakly increasing mapping, we can deduce the following result, due to Radenović and Kadelburg ([18], Theorem 3.1.), as a corollary of Theorem 3.1 and Theorem 3.2, by taking $A = B$.

**Corollary 3.2.** *Let $(A, \preceq, d)$ be an ordered complete metric space.*

*(i)* *$(T, S)$ is a pair of weakly increasing mappings;*

*(ii)* *either $T$ or $S$ is continuous (or) $\{a_n\}$ is a nondecreasing sequence in $A$ such that $a_n \to a$, then $a_n \preceq a$;*

*(iii)* *for every two comparable elements $a, b \in A$,*

$$\psi(d(T(a), S(b))) \leq \psi(m(a, b)) - \phi(m(a, b)) \tag{3.29}$$

*where $m(a, b) = \max\{d(a, b), d(a, Ta), d(b, Sb), \frac{d(a,Sb)+d(b,Ta)}{2}\}$, $\psi$ is an altering distance function, $\phi$ is a nondecreasing function also $\phi(t) = 0$ iff $t = 0$ and $\psi - \phi$ is a nondecreasing function.*

*Then, there exists at least one element $u$ in $A$ such that $d(u, Tu) = d(u, Su) = 0$.*

**Theorem 3.3.** *In addition to the hypotheses of Theorem 3.1, assume that $T$ and $S$ are proximally increasing and*

$$\textit{for every } a, b \in A, \textit{ there exists } z \in A \textit{ that is comparable to } a \textit{ and } b \tag{3.30}$$

*then $T$ and $S$ have a unique common best proximity point.*

*Proof.* From Theorem 3.1, the set of common best proximity points of $T$ and $S$ is non-empty. Suppose that there exist $u, v$ in $A$ which are common best proximity points for $T$ and $S$. We distinguish two cases:
**Case:1** If $u$ and $v$ are comparable.
Since $d(u, Tu) = d(u, Su) = d(A, B)$ and $d(v, Tv) = d(v, Sv) = d(A, B)$. By the assumption of $P-$ property and (3.28), we get

$$\psi(d(u, v)) = \psi(d(Tu, Sv)) \leq \psi(m(u, v)) - \phi(m(u, v)), \text{where} \tag{3.31}$$

$$m(u,v) = \max\{d(u,v), d(u,Tu) - d(A,B), d(v,Sv) - d(A,B), \frac{d(v,Tu) + d(u,Sv)}{2} - d(A,B)\} = d(u,v).$$

From (3.31), we obtain $\psi(d(u,v)) \leq \psi(d(u,v)) - \phi(d(u,v))$, which implies $\phi(d(u,v)) = 0$, and by the assumption about $\phi$, we get $d(u,v) = 0$, or equivalently, $u = v$.

**Case:2** If $u$ is not comparable to $v$.
By the condition (3.30) there exists $a_0 \in A$ comparable to $u$ and $v$. By condition (i) of $(T,S)$ is weakly proximal increasing, for $a_0 \in A$, there exist elements $a_1, a_2$ in $A$ such that

$$d(a_1, Ta_0) = d(A,B), d(a_2, Sa_1) = d(A,B), \text{ and } a_1 \preceq a_2.$$

Since $d(u,Tu) = d(A,B)$, $d(a_1,Ta_0) = d(A,B)$ also $u$ and $a_0$ are comparable. Hence, by $T$ is proximally increasing, we get $u$ and $a_1$ are comparable. Now, by condition (ii) of $(T,S)$ is weakly proximal increasing, for $a_1 \in A$, there exist elements $a_2^*, a_3$ in $A$ such that

$$d(a_2^*, Sa_1) = d(A,B), d(a_3, Ta_2^*) = d(A,B), \text{ and } a_2^* \preceq a_3.$$

By using the $P-$ property for $d(a_2,Sa_1) = d(A,B)$ and $d(a_2^*,Sa_1) = d(A,B)$, we get $a_2^* = a_2$. Hence, we have

$$d(a_2, Sa_1) = d(A,B), d(a_3, Ta_2) = d(A,B), \text{ and } a_2 \preceq a_3.$$

Since $d(u,Su) = d(A,B)$, $d(a_2,Sa_1) = d(A,B)$ also $u$ and $a_1$ are comparable. Hence, by $S$ is proximally increasing, we get $u$ and $a_2$ are comparable.

Continuing this process, we get $u$ and $a_n$ are comparable, $d(a_{2n+1}, Ta_{2n}) = d(A,B), d(a_{2n+2}, Sa_{2n+1}) = d(A,B)$ and $a_1 \preceq \cdots \preceq a_n \cdots$ for all $n \geq 0$.

We shall prove that for each $n = 0, 1, 2 \cdots$

$$d(a_{n+1}, u) \leq m(a_n, u) = d(a_n, u). \tag{3.32}$$

Using the facts that $d(a_{2n+1}, Ta_{2n}) = d(A,B)$, $d(u,Su) = d(A,B)$ and from the condition (3.1), we get

$$\psi(d(a_{2n+1}, u)) = \psi(d(Ta_{2n}, Su)) \leq \psi(m(a_{2n}, u)) - \phi(a_{2n}, u)) \leq \psi(m(a_{2n}, u)). \tag{3.33}$$

Since $\psi$ is nondecreasing, we get $d(a_{2n+1}, u) \leq m(a_{2n}, u)$, where

$$\begin{aligned} m(a_{2n}, u) &= \max\{d(a_{2n}, u), d(a_{2n}, Ta_{2n}) - d(A, B), d(u, Tu) - d(A, B), \\ &\qquad \frac{d(a_{2n}, Su) + d(u, Ta_{2n})}{2} - d(A, B)\} \\ &\leq \max\{d(a_{2n}, u), d(a_{2n+1}, a_{2n}), 0, \frac{d(a_{2n}, u) + d(u, a_{2n+1})}{2}\} \\ &\leq \max\{d(a_{2n}, u), d(a_{2n+1}, u)\} \end{aligned}$$

for sufficiently large $n$, because $d(a_{2n+1}, a_{2n}) \to 0$ when $n \to \infty$. Similarly as in the proof of Theorem 3.1, it can be shown that $d(a_{2n+1}, u) \leq m(a_{2n}, u) = d(a_{2n}, u)$ for all $n \geq 0$. Also, it is easy to prove that $d(a_{2n+2}, u) \leq m(a_{2n+1}, u) = d(a_{2n+1}, u)$ for all $n \geq 0$ by using the facts that $d(a_{2n+2}, Sa_{2n+1}) = d(A, B)$, and $d(u, Tu) = d(A, B)$ in condition (3.1). Hence, (3.32) holds for all $n \geq 0$. Hence, the sequence $\{d(a_n, u)\}$ is monotone non-increasing and bounded. Thus, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(a_n, u) = \lim_{n\to\infty} m(a_{n-1}, u) = r \geq 0. \tag{3.34}$$

Suppose that $\lim_{n\to\infty} d(a_n, u) = \lim_{n\to\infty} m(a_{n-1}, u) = r > 0$. Then the inequality

$$\psi(d(a_n, u)) \leq \psi(m(a_{n-1}, u)) - \phi(m(a_{n-1}, u)) \leq \psi(m(a_{n-1}, u))$$

implies that

$$\lim_{n\to\infty} \phi(m(a_{n-1}, u)) = 0. \tag{3.35}$$

But, as $0 < r \leq d(a_n, u) \leq m(a_{n-1}, u)$ and $\phi$ is a nondecreasing function,

$$0 < \phi(r) \leq \phi(m(a_{n-1}, u)),$$

and this gives us $\lim_{n\to\infty} \phi(m(a_{n-1}, u)) \geq \phi(r) > 0$ which contradicts to (3.35). Hence,

$$\lim_{n\to\infty} d(a_n, u) = \lim_{n\to\infty} m(a_{n-1}, u) = 0.$$

Analogously, it can be proved that

$$\lim_{n\to\infty} d(a_n, v) = \lim_{n\to\infty} m(a_{n-1}, v) = 0.$$

Finally, the uniqueness of the limit gives us $u = v$. □

**Example 3.2.** *Let $X = \mathbb{R}^2$ and consider the order $(x, y) \preceq (z, t) \Leftrightarrow x \preceq z$ and $y \preceq t$, where $\preceq$ is the usual order on $\mathbb{R}$.*

*Thus, $(X, \preceq)$ is a partially ordered set. Besides, $(X, d_1)$ is a complete metric space where the metric is defined as $d_1((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$.*

*Let $A = \{(0,x) : x \in \mathbb{R}\}$ and $B = \{(1,x) : x \in \mathbb{R}\}$ be two nonempty closed subsets of $X$. Then, $d(A,B) = 1, A = A_0$ and $B = B_0$. Let $T, S : A \to B$ be defined as $T(0,x) = (1, \frac{-x}{2})$ and $S(0,y) = (1, \frac{-y}{3})$. Then, it can be seen that $T$ and $S$ are continuous, proximally increasing mappings and weakly proximal increasing. Now, we have to prove condition (3.1).*

*Now, assuming $(0,0) \preceq (0,x) \preceq (0,y)$, we discuss the following cases.*

*(i) If $\frac{y}{3} \geq \frac{x}{2}$, then $d(T(0,x), S(0,y)) = \frac{y}{3} - \frac{x}{2}$, and*

$$m((0,x),(0,y)) = \max\{d((0,x),(0,y)), d((0,x),T(0,x)) - d(A,B),$$

$$d((0,y),S(0,y)) - d(A,B), \frac{d((0,y),T(0,x)) + d((0,x),S(0,y))}{2} - d(A,B)\} = \frac{4y}{3}.$$

*(ii) If $\frac{y}{3} \leq \frac{x}{2}$ and $9x \leq 8y$, then $d(T(0,x), S(0,y)) = \frac{x}{2} - \frac{y}{3}$, and*

$$m((0,x),(0,y)) = \max\{d((0,x),(0,y)), d((0,x),T(0,x)) - d(A,B),$$

$$d((0,y),S(0,y)) - d(A,B), \frac{d((0,y),T(0,x)) + d((0,x),S(0,y))}{2} - d(A,B)\} = \frac{4y}{3}.$$

*(iii) If $\frac{y}{3} \leq \frac{x}{2}$ and $9x \geq 8y$, then $d(T(0,x), S(0,y)) = \frac{x}{2} - \frac{y}{3}$, and*

$$m((0,x),(0,y)) = \max\{d((0,x),(0,y)), d((0,x),T(0,x)) - d(A,B),$$

$$d((0,y),S(0,y)) - d(A,B), \frac{d((0,y),T(0,x)) + d((0,x),S(0,y))}{2} - d(A,B)\} = \frac{3x}{2}.$$

*Now, assuming $(0,-x) \preceq (0,-y) \preceq (0,0)$, we discuss the following cases.*

*(i) If $\frac{y}{3} \geq \frac{x}{2}$, then $d(T(0,-x), S(0,-y)) = \frac{y}{3} - \frac{x}{2}$, and*

$$m((0,-x),(0,-y)) = \max\{d((0,-x),(0,-y)), d((0,-x),T(0,-x)) - d(A,B),$$

$$d((0,-y),S(0,-y)) - d(A,B), \frac{d((0,-y),T(0,-x)) + d((0,-x),S(0,-y))}{2} - d(A,B)\} = \frac{4y}{3}.$$

*(ii) If $\frac{y}{3} \leq \frac{x}{2}$, then $d(T(0,-x), S(0,-y)) = \frac{x}{2} - \frac{y}{3}$, and*

$$m((0,-x),(0,-y)) = \max\{d((0,-x),(0,-y)), d((0,-x),T(0,-x)) - d(A,B),$$

$$d((0,-y),S(0,-y)) - d(A,B), \frac{d((0,-y),T(0,-x)) + d((0,-x),S(0,-y))}{2} - d(A,B)\} = \frac{3x}{2}.$$

*Now, assuming $(0,-x) \preceq (0,0) \preceq (0,y)$, we discuss the following cases.*

*(i) If $a \leq y$ and $\frac{y}{3} \geq x$, then $d(T(0,-x), S(0,y)) = \frac{x}{2} + \frac{y}{3}$, and*

$$m((0,-x),(0,y)) = \max\{d((0,-x),(0,y)), d((0,-x),T(0,-x)) - d(A,B),$$

$$d((0,y),S(0,y)) - d(A,B), \frac{d((0,y),T(0,-x)) + d((0,-x),S(0,y))}{2} - d(A,B)\} = \frac{4y}{3}.$$

*(ii) If* $a \leq y$ *and* $\frac{y}{3} \leq x$, *then* $d(T(0,-x), S(0,y)) = \frac{x}{2} + \frac{y}{3}$, *and*

$$m((0,-x),(0,y)) = \max\{d((0,-x),(0,y)), d((0,-x),T(0,-x)) - d(A,B),$$

$$d((0,y),S(0,y)) - d(A,B), \frac{d((0,y),T(0,-x)) + d((0,-x),S(0,y))}{2} - d(A,B)\} = y+x.$$

*(iii) If* $a \geq y$ *and* $\frac{x}{2} \geq y$, *then* $d(T(0,-x), S(0,y)) = \frac{x}{2} + \frac{y}{3}$, *and*

$$m((0,-x),(0,y)) = \max\{d((0,-x),(0,y)), d((0,-x),T(0,-x)) - d(A,B),$$

$$d((0,y),S(0,y)) - d(A,B), \frac{d((0,y),T(0,-x)) + d((0,-x),S(0,y))}{2} - d(A,B)\} = \frac{3x}{2}.$$

*(iv) If* $a \geq y$ *and* $\frac{x}{2} \leq y$, *then* $d(T(0,-x), S(0,y)) = \frac{x}{2} + \frac{y}{3}$, *and*

$$m((0,-x),(0,y)) = \max\{d((0,-x),(0,y)), d((0,-x),T(0,-x)) - d(A,B),$$

$$d((0,y),S(0,y)) - d(A,B), \frac{d((0,y),T(0,-x)) + d((0,-x),S(0,y))}{2} - d(A,B)\} = y+x.$$

*By assume that* $\psi, \phi : [0,\infty] \to [0,\infty]$ *such that* $\psi(t) = t$ *and* $\phi(t) = \frac{t}{4}$, *we get*

$$\psi(d(T(0,x), S(0,y))) \leq \psi(m(0,x),(0,y)) - \phi(m(0,x),(0,y)), \ \ \forall\, (0,x) \preceq (0,y) \in A.$$

*Hence all the hypotheses of the Theorem are satisfied. Also, it can be observed that* $(0,0)$ *is the unique common best proximity point of the mapping* $T$ *and* $S$.

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