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Space-Fractional Telegraph Equations

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Abstract : In this paper, we use the invariant subspace method to solve the space-fractional telegraph equation, in which fractional derivative is considered in the Caputo sense. We classify invariant subspaces for the space-fractional telegraph equation. By choosing an appropriate invariant subspace, the space-fractional telegraph equation is reduced to a system of fractional ordinary differential equations. Finally, finding the solutions of the system yields the solution of space-fractional telegraph equation.

Keywords : Space-fractional telegraph equation, Invariant subspace method, Caputo fractional derivative, Laplace transform method.
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1 Introduction

A fractional partial differential equation is a general form of a partial differential equation by replacing the integer order derivatives with the fractional order. Due to the extensive application of fractional differential equations in various fields of engineering and science, many researchers have paid attention to find the solutions of them.

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Fractional telegraph equation is a simple example of fractional partial differential equation. Many analytical and approximate techniques have been developed to derive their solutions. For example, Momani [1] derived the analytical and approximate solutions to the space and time fractional telegraph equations by using Adomian decomposition method. Chen et al. [2] used the method of separation of variable to solve time fractional telegraph equation with certain nonhomogeneous boundary conditions. Srivastava et al. [3] derived the approximate solutions of time fractional telegraph equation by using the reduced differential transformation method. Recently, Kumar [4]employed the homotopy analysis method and Laplace transform to approximate the solutions of the space fractional telegraph equation.

In recent decades, the invariant subspace method, initially developed by Galaktionov and Svirshchevskii [5] has been used as an effective and practical method for constructing exact solutions to nonlinear partial differential equations. Later on, it has been extended to fractional partial differential equations by many authors in [6, 7, 8, 9].

In this paper, we apply the invariant subspace method to obtain exact solutions for space-fractional telegraph equations. We classify invariant subspaces for the space-fractional telegraph equation. By choosing an appropriate invariant subspace, the space-fractional telegraph equation can be reduced to a system of fractional ordinary differential equations subject to the boundary conditions. The obtained reduced system of fractional ordinary differential equations can be solved by using the Laplace transform method.

The organization of the article is as follows: in section 2, we recall some basic definitions of fractional integrals and derivatives and show the idea of the invariant subspace method. In section 3, we explain how the invariant subspace method can be extended to find a solution of space-fractional telegraph equation. Finally, in section 4, we apply the invariant subspace method to solve some examples.

2 Preliminaries

In this section, we introduce some definitions and present properties that are used in the paper.

2.1 Fractional integral and derivatives

Definition 2.1. *Suppose that α and t are positive real numbers. Then the Riemann-Liouville fractional integral is defined by*

$$J_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(x)}{(t-x)^{1-\alpha}} dx,$$

where

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt,$$

is the Gamma function.

Definition 2.2. *The Riemann-Liouville fractional derivative of order $\alpha > 0$ of the function f is given by*

$$D_t^\alpha f(t) = \begin{cases} \frac{d^n}{dt^n} J_t^{n-\alpha} f(t), & n-1 < \alpha < n, \quad n \in \mathbb{N}, \\ \frac{d^n}{dt^n} f(t), & \alpha = n, \quad n \in \mathbb{N}. \end{cases}$$

Definition 2.3. *The Caputo fractional derivative of order $\alpha > 0$ of the function f is given by*

$$\frac{d^\alpha f}{dt^\alpha} = \begin{cases} J_t^{n-\alpha} \frac{d^n}{dt^n} f(t), & n-1 < \alpha < n, \quad n \in \mathbb{N}, \\ \frac{d^n}{dt^n} f(t), & \alpha = n, \quad n \in \mathbb{N}. \end{cases}$$

In particular, we denote by $\dfrac{\partial^\alpha}{\partial x^\alpha}$, the Caputo fraction partial derivative with respect to x of order α.

Definition 2.4. *[9, 10] Two-parameter function of Mittag-Leffler type is defined as*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad \alpha > 0, \beta > 0, \tag{2.1}$$

e.g. $E_{1,1}(z) = e^z, E_{2,1}(z^2) = \cosh(z), E_{2,1}(-z^2) = \cos(z), E_{2,2}(z^2) = \frac{\sinh(z)}{z}$, $E_{2,2}(-z^2) = \frac{\sin z}{z}$, *and* $z^2 E_{2,3}(z^2) = E_{2,1}(z^2) - 1$.

Proposition 2.1. *[10] Let $n - 1 < \alpha \le n, n \in \mathbb{N}$. The Laplace transform of the Caputo derivative of order α is defined as*

$$\mathcal{L}\left\{\frac{d^\alpha f(x)}{dx^\alpha}; s\right\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0),$$

where $F(s)$ is the Laplace transform of $f(x)$.
Let $\alpha, \beta, \lambda \in \mathbb{R}, \ \alpha, \beta > 0$. Then the Laplace transform of the two-parameter function of Mittag-Leffler type (2.1) is given by

$$\mathcal{L}\{z^{\beta-1} E_{\alpha,\beta}(\pm\lambda z^\alpha); s\} = \frac{s^{\alpha-\beta}}{s^\alpha \mp \lambda}, \quad Re(s) > |\lambda|^{1/\alpha}. \tag{2.2}$$

2.2 Invariant subspace method

Consider the evolution of partial differential equation of the form

$$u_t = F[u], \quad u = u(x,t), \tag{2.3}$$

where F is a nonlinear differential operator of order k, that is,

$$F[u] = F(x, u, u_x, ..., \partial_x^k u), \quad \partial_x^k u = \frac{\partial^k u}{\partial x^k}.$$

Let W_n be a finite dimensional linear space spanned by linearly independent functions $f_1(x), f_2(x), ..., f_n(x)$, that is,

$$\begin{aligned} W_n &= L\{f_1(x), ..., f_n(x)\} \\ &= \left\{ \sum_{i=1}^{n} c_i f_i(x) | \ c_i = constants, \quad i = 1, 2, ..., n \right\}. \end{aligned}$$

Definition 2.5. *A finite dimensional linear space W_n is said to be invariant with respect to a differential operator F if $F[W_n] \subseteq W_n$, that is, $F[u] \in W_n$, for all $u \in W_n$.*

In order to solve the equation (2.3), we suppose that W_n is an invariant subspace with respect to a given operator F if $F[W_n] \subseteq W_n$ and then operator F is said to preserve or admit W_n, which means:

$$F[u] = F\left[\sum_{i=1}^{n} c_i(t) f_i(x)\right] = \sum_{i=1}^{n} \Psi_i\big(c_1(t), c_2(t), ..., c_n(t)\big) f_i(x), \tag{2.4}$$

where $\{\Psi_i\}$ are the expansion coefficients of $F[u] \in W_n$ on the basis $\{f_i\}$.
We assume the solution of (2.3) by

$$u(x,t) = \sum_{i=1}^{n} c_i(t) f_i(x), \tag{2.5}$$

where $f_i(x) \in W_n, i = 1, ..., n$.
Since W_n is an invariant subspace under F, we obtain equation (2.4).
By substituting (2.4) and (2.5) into (2.3), we get

$$\sum_{i=1}^{n} c_i'(t) f_i(x) = \sum_{i=1}^{n} \Psi_i(c_1(t), c_2(t), ..., c_n(t)) f_i(x),$$

and

$$\sum_{i=1}^{n} \left[c_i'(t) - \Psi_i(c_1(t), c_2(t), ..., c_n(t) \right] f_i(x) = 0.$$

Since $f_1(x), f_2(x), ..., f_n(x)$ are linearly independent functions, we obtain a system of ODEs

$$c_i'(t) = \Psi_i(c_1(t), c_2(t), ..., c_n(t)), \quad i = 1, 2, ..., n.$$

Finally, by solving this system, we obtain the desired solution (2.5).

3 Explicit solution of space-fractional telegraph equations

In this section, we will apply the invariant subspace method to classify some invariant subspaces for the space-fractional telegraph equation and finally find a solution of this equation.

Consider the space-fractional telegraph equation

$$\frac{\partial^\alpha u}{\partial x^\alpha} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u, \quad 1 < \alpha \leq 2, \tag{3.1}$$

where $\frac{\partial^\alpha}{\partial x^\alpha}$ is a space-fractional partial derivative in the Caputo sense. Now, we denote the left side of equation (3.1) by

$$F[u] = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u. \tag{3.2}$$

The following theorem shows an exact solution to the space-fractional telegraph equation (3.1) by using the invariant subspace method.

Theorem 3.1. *The space-fractional telegraph equation (3.1) admits a solution of the form*

$$u(x,t) = c_1(x) + c_2(x)t + \cdots + c_{n+1}(x)t^n,$$

where $c_i(x)$ are solutions of the following system of fractional ODEs for $i = 1, 2, \cdots, n+1$.

$$\begin{cases} \frac{d^\alpha c_1(x)}{dx^\alpha} & = 2c_3(x) + c_2(x) + c_1(x), \\ \frac{d^\alpha c_2(x)}{dx^\alpha} & = 6c_4(x) + 2c_3(x) + c_2(x), \\ & \vdots \\ \frac{d^\alpha c_{n+1}(x)}{dx^\alpha} & = c_{n+1}(x). \end{cases} \tag{3.3}$$

Proof. The operator $F[.]$ defined by (3.2) is invariant under $W_n = L\{1, t, \cdots, t^n\}$ because

$$F(c_1 + c_2 t + c_3 t^2 + \cdots + c_{n+1}t^n) = (2c_3 + c_2 + c_1) + (6c_4 + 2c_3 + c_2)t + \cdots + c_{n+1}t^n \in W_n.$$

Assume the solution $u(x,t)$ as a linear combination of the elements in the invariant subspace, that is,

$$u(x,t) = c_1(x) + c_2(x)t + \cdots + c_{n+1}(x)t^n. \tag{3.4}$$

Then we have

$$F[u(x,t)] = [2c_3(x) + c_2(x) + c_1(x)] + [6c_4(x) + 2c_3(x) + c_2(x)]t + \cdots + c_{n+1}(x)t^n. \tag{3.5}$$

Taking the fractional derivative of order α to both sides of equation (3.4), we obtain

$$\frac{d^{\alpha}u(x,t)}{dx^{\alpha}} = \frac{d^{\alpha}c_1(x)}{dx^{\alpha}} + \frac{d^{\alpha}c_2(x)}{dx^{\alpha}}t + \cdots + \frac{d^{\alpha}c_{n+1}(x)}{dx^{\alpha}}t^n. \tag{3.6}$$

Substituting (3.6) and (3.5) in (3.1), we get

$$\begin{aligned}&\frac{d^{\alpha}c_1(x)}{dx^{\alpha}} + \frac{d^{\alpha}c_2(x)}{dx^{\alpha}}t + \cdots + \frac{d^{\alpha}c_{n+1}(x)}{dx^{\alpha}}t^n \\ &\quad = \big[2c_3(x) + c_2(x) + c_1(x)\big] + \big[6c_4(x) + 2c_3(x) + c_2(x)\big]t + \cdots + c_{n+1}(x)t^n.\end{aligned}$$

Since $1, t, \cdots, t^n$ are linearly independent, we obtain a system of fractional ordinary differential equations (3.3). □

Remark 3.1. *Under the operator (3.2), there are several invariant subspaces which can be proved in a similar way. In below, we classify some invariant subspaces with respect to the operator (3.2).*

1. *The subspace* $W_2 = L\{1, e^{at}\}$, $a \neq 0$ *is invariant under* F *because*

$$F\big[c_1 + c_2e^{at}\big] = c_1 + (a^2c_2 + ac_2 + c_2)e^{at} \in W_2.$$

2. *The subspace* $W_3^1 = L\{1, \sin(at), \cos(at)\}$, $a \neq 0$ *is invariant under* F *because*

$$\begin{aligned}F\big[c_1 + c_2\sin(at) + c_3\cos(at)\big] &= c_1 + [c_2 - ac_3 - a^2c_2]\sin(at) \\ &\quad + [c_3 + ac_2 - a^2c_3]\cos(at) \in W_3^1.\end{aligned}$$

3. *The subspace* $W_3^2 = L\{1, \sinh(at), \cosh(at)\}$, $a \neq 0$ *is invariant under* F *because*

$$\begin{aligned}F\big[c_1 + c_2\sinh(at) + c_3\cosh(at)\big] &= c_1 + [c_2 + ac_3 + a^2c_2]\sinh(at) \\ &\quad + [c_3 + ac_2 + a^2c_3]\cosh(at) \in W_3^2.\end{aligned}$$

4. *The subspace* $W_3^3 = L\{1, e^{at}, te^{at}\}$, $a \neq 0$ *is invariant under* F *because*

$$\begin{aligned}F\big[c_1 + c_2e^{at} + c_3te^{at}\big] &= c_1 + [(1 + a + a^2)c_2 + (1 + 2a)c_3]e^{at} \\ &\quad + c_3(1 + a + a^2)te^{at} \in W_3^3.\end{aligned}$$

5. *The subspace* $W_3^4 = L\{1, e^{at}\cos bt, e^{at}\sin bt\}$, $a, b \neq 0$ *is invariant under* F *because*

$$\begin{aligned}&F\big[c_1 + c_2e^{at}\cos bt + c_3e^{at}\sin bt\big] \\ &\qquad = c_1 + [c_2 + (ac_2 + bc_3) + (a^2c_2 + b^2c_2)]e^{at}\cos bt \\ &\qquad + [c_3 + (ac_3 - bc_2) - (abc_2 + b^2c_3) \\ &\qquad + (a^2c_3 - abc_2)]e^{at}\sin bt \in W_3^4.\end{aligned}$$

The advantage of these different invariant subspaces is that, by choosing an appropriate invariant subspace, we can solve the space fractional telegraph equation subject to different boundary conditions.

4 Illustrative Examples

In this section, we will apply the invariant subspace method to solve some examples in [1], in which the author used the Adomian decomposition method in deriving the exact solutions to the fractional Telegraph equations.

Example 4.1. *Consider the space-fractional telegraph equation with* $1 < \alpha \leq 2$

$$\frac{\partial^\alpha u}{\partial x^\alpha} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u, \quad x > 0, \quad t > 0, \tag{4.1}$$

subject to the boundary conditions $u(0,t) = e^{-t}$, $\frac{\partial u(0,t)}{\partial x} = e^{-t}$.
Under the operator

$$F[u] = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u,$$

we choose the invariant subspace

$$W_2 = L\{1, e^{-t}\}.$$

Assume the solution $u(x,t)$ *as a linear combination of the elements in the invariant subspace, that is,*

$$u(x,t) = a(x) + b(x)e^{-t}.$$

It follows from the boundary conditions that

$$a(0) = 0, \quad b(0) = 1, \quad a'(0) = 0, b'(0) = 1.$$

Substituting $u(x,t)$ *into the equation (4.1)*

$$\frac{d^\alpha a(x)}{dx^\alpha} + e^{-t}\frac{d^\alpha b(x)}{dx^\alpha} = a(x) + b(x)e^{-t},$$

and using the facts that 1 *and* e^{-t} *are linearly independent, we obtain the system of space-fractional ODEs and the corresponding boundary conditions*

$$\frac{d^\alpha}{dx^\alpha}a(x) = a(x), \quad a(0) = 0, a'(0) = 0, \tag{4.2}$$

$$\frac{d^\alpha}{dx^\alpha}b(x) = b(x), \quad b(0) = 1, b'(0) = 1. \tag{4.3}$$

In order to solve the above space-fraction ODEs, we use the Laplace transform technique as shown in Proposition 2.1 when $n = 2$*, which turns the space-fraction ODE (4.2) into*

$$s^\alpha A(s) - s^{\alpha-1}a(0) - s^{\alpha-2}a'(0) = A(s)$$
$$A(s) = 0.$$

Taking the inverse Laplace transform, we get $a(x) = 0$.
Applying the Laplace transform to both sides of (4.3) yields

$$\begin{aligned} s^{\alpha}B(s) - s^{\alpha-1}b(0) - s^{\alpha-2}b'(0) &= B(s) \\ B(s) &= \frac{s^{\alpha-1}}{s^{\alpha}-1} + \frac{s^{\alpha-2}}{s^{\alpha}-1} \\ &= \mathcal{L}\{E_{\alpha,1}(x^{\alpha}); s\} + \mathcal{L}\{xE_{\alpha,2}(x^{\alpha}); s\}. \end{aligned}$$

Taking the inverse Laplace transform gives

$$b(x) = E_{\alpha,1}(x^{\alpha}) + xE_{\alpha,2}(x^{\alpha}).$$

Therefore, the exact solution of equation (4.1) is

$$u(x,t) = e^{-t}\big[E_{\alpha,1}(x^{\alpha}) + xE_{\alpha,2}(x^{\alpha})\big],$$

which is the same solution obtained by the Adomian decomposition method by Momani [1]. In particular, if $\alpha = 2$, *the solution of (4.1) is*

$$u(x,t) = e^{-t}\big[E_{2,1}(x^2) + xE_{2,2}(x^2)\big] = e^{-t}\big[\cosh x + \sinh x\big] = e^{x-t},$$

which is the same as the exact solution of the classical telegraph equation.

Example 4.2. *Consider the nonhomogeneous space-fractional telegraph equation*

$$\frac{\partial^{\alpha}u}{\partial x^{\alpha}} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u - x^2 - t + 1, \quad x > 0, \quad t > 0, \tag{4.4}$$

subject to the boundary conditions $u(0,t) = t$, $\frac{\partial u(0,t)}{\partial x} = 0$.
In this case, we choose the invariant subspace

$$W_2' = L\{1, t\}$$

and search for a solution of the form

$$u(x,t) = a(x) + b(x)t.$$

By substituting $u(x,t)$ *into the equation (4.4), we obtain the system of space-fractional equation with the corresponding boundary conditions:*

$$\frac{d^{\alpha}a(x)}{dx^{\alpha}} = a(x) + b(x) - x^2 + 1, \quad a(0) = a'(0) = 0, \tag{4.5}$$

$$\frac{d^{\alpha}b(x)}{dx^{\alpha}} = b(x) + 1, \quad b(0) = 1, \quad b'(0) = 0. \tag{4.6}$$

Applying the Laplace transform to (4.6), we get

$$s^{\alpha}B(s) - s^{\alpha-1}b(0) - s^{\alpha-2}b'(0) = B(s) - \frac{1}{s},$$

which implies

$$B(s) = \frac{s^{\alpha-1}}{s^\alpha - 1} - \frac{1}{s(s^\alpha - 1)} = \frac{s^{\alpha-1}}{s^\alpha - 1} - \left[\frac{s^{\alpha-1}}{s^\alpha - 1} - \frac{1}{s}\right] = \frac{1}{s}.$$

Taking the inverse Laplace transform gives $b(x) = 1$. Now applying the Laplace transform technique to (4.5), we get

$$\begin{aligned} A(s) &= \frac{2}{s(s^\alpha - 1)} - \frac{2}{s^3(s^\alpha - 1)} \\ &= 2\left[\frac{s^{\alpha-1}}{s^\alpha - 1} - \frac{1}{s} - \frac{s^{\alpha-3}}{s^\alpha - 1} + \frac{1}{s^3}\right] \\ &= 2\mathcal{L}\{E_{\alpha,1}(x^\alpha); s\} - 2\mathcal{L}\{1\} - 2\mathcal{L}\{x^2 E_{\alpha,3}(x^\alpha); s\} + \mathcal{L}\{x^2\}. \end{aligned}$$

Taking the inverse Laplace transform, we obtain

$$a(x) = 2E_{\alpha,1}(x^\alpha) - 2 - 2x^2 E_{\alpha,3}(x^\alpha) + x^2.$$

Then the exact solution of the space-fractional nonhomogeneous telegraph equation (4.4) is given by

$$u(x,t) = 2E_{\alpha,1}(x^\alpha) - 2 - 2x^2 E_{\alpha,3}(x^\alpha) + x^2 + t.$$

This is the same result as discussed in [1] but simply to solve. For $\alpha = 2$, we obtain the solution of a traditional nonhomogeneous telegraph equation

$$u(x,t) = 2E_{2,1}(x^2) - 2 - 2\big[E_{2,1}(x^2) - 1\big] + x^2 + t = x^2 + t.$$

5 Conclusions

In this paper, we employ the invariant subspace method to obtain exact solutions of the space-fractional telegraph equations. According to the linearity of telegraph equations, they admit several invariant subspaces. However, by choosing an appropriate invariant subspace, the fractional telegraph equation can easily be reduced to a system of space-fractional ordinary differential equations, subject to certain boundary conditions. Then, the Laplace transform method is applied to solve this reduced system of fractional differential equations. Finally, the obtained solutions are represented in terms of the Mittag-Leffer functions and approach the solutions of traditional telegraph equations with integer order.

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References

[1] S. Momani, Analytic and approximate solutions of the space-and time-fractional telegraph equations. Applied Mathematics and Computation. 2005 Nov 15;170(2):1126-34.

[2] J. Chen, F. Liu, V. Anh, Analytical solution for the time-fractional telegraph equation by the method of separating variables. Journal of Mathematical Analysis and Applications. 2008 Feb 15;338(2):1364-77.

[3] V.K. Srivastava, M.K. Awasthi, M. Tamsir, RDTM solution of Caputo time fractional-order hyperbolic telegraph equation. AIP Advances. 2013 Mar;3(3):032142.

[4] S. Kumar, A new analytical modelling for fractional telegraph equation via Laplace transform. Applied Mathematical Modelling. 2014 Jul 1;38(13):3154-63.

[5] V.A. Galaktionov, S.R. Svirshchevskii, Exact solutions and invariant subspaces of nonlinear partial differential equations in mechanics and physics. CRC Press; 2006 Nov 2.

[6] R.K. Gazizov, A.A. Kasatkin, Construction of exact solutions for fractional order differential equations by the invariant subspace method. Computers and Mathematics with Applications. 2013 Sep 1;66(5):576-84.

[7] R. Sahadevan, T. Bakkyaraj, Invariant subspace method and exact solutions of certain nonlinear time fractional partial differential equations. Fractional Calculus and Applied Analysis. 2015 Feb 1;18(1):146-62.

[8] R. Sahadevan, P. Prakash, Exact solution of certain time fractional nonlinear partial differential equations. Nonlinear Dynamics. 2016 Jul 1;85(1):659-73.

[9] P. Choudhary, V. Daftardar-Gejji, Invariant subspace method: a tool for solving fractional partial differential equations. Fractional Calculus and Applied Analysis. 2017 Apr 25;20(2):477-93.

[10] M.K. Ishteva, Properties and applications of the Caputo fractional operator [master thesis]. Bulgaria: Department of Mathematics, Universitt Karlsruhe. 2005.

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