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On Solving the Equilibrium Problem and Fixed Point Problem for Nonspreading Mappings and Lipschitzian Mappings in Hilbert Spaces

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Abstract : The purpose of this research is to introduce and study the S-mapping generated by a finite family of nonspreading mappings and Lipschitzian mapping in a Hilbert space. Then, we prove a strong convergence theorem for finding a common element of the set of fixed points of a finite family of nonspreading mappings and Lipschitzian mappings and a common solution of a finite family of equilibrium problems. Futhermore, applying our main theorem, the additional results for equilibrium problem are obtained.

Keywords : nonspreading mapping, Lipschitzian mapping, S-mapping, equilibrium problem.
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1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. The fixed point problem for the mapping $T : C \to H$ is to find $x \in C$ such that

$$x = Tx. \tag{1.1}$$

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We denote the set of solutions of (1.1) by $Fix(T)$. It is well known that $Fix(T)$ is closed and convex and $P_{Fix(T)}$ is well-defined.

In 2008, Kohsaka and Takahashi [6] introduced *the nonspreading mapping* T in Hilbert space H as follows:

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2, \forall x, y \in C. \tag{1.2}$$

In 2009, it is shown by Iemoto and Takahashi [7] that (1.2) is equivalent to the following equation.

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \text{ for all } x, y \in C.$$

Many researcher proved the strong convergence theorem for nonspreading mapping and some related mappings in Hilbert space, see for example [8, 9, 10, 11].

We now recall some well-known concepts and results as follows:

Definition 1.1. *Let $T : C \to C$ be a mapping. Then T is called*

(i) *μ-Lipschitz continuous if there exists a nonnegative real number $\mu \geq 0$ such that*
$$\|Tx - Ty\| \leq \mu \|x - y\|, \forall x, y \in C.$$

(ii) quasi-nonexpansive *if*

$$\|Tx - p\| \leq \|x - p\|, \textit{ for every } x \in C \textit{ and } p \in Fix(T).$$

Remark 1.2. *[15] If $T : C \to C$ is nonspreading with $Fix(T) \neq \emptyset$, then T is quasi-nonexpansive.*

In 2009, Kangtunyakarn and Suantai [12] proposed an S-mapping for nonlinear mappings as follows:

$$\begin{aligned}
U_0 &= I\\
U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I\\
U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I\\
U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I\\
&\vdots\\
U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I\\
S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I,
\end{aligned}$$

where $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. This mapping is called *S-mapping* generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. They proved that $Fix(S) = \bigcap_{i=1}^N Fix(T_i)$.

Later, in 2012, Kangtunyakarn [13] studied the S-mapping for a finite family of nonspreading mappings and obtained that $Fix(S) = \bigcap_{i=1}^N Fix(T_i)$ and S is quasi-nonexpansive mapping.

Let $F : C \times C \to \mathbb{R}$ be bifunction. *The classical equilibrium problem* is to find $x \in C$ such that

$$F(x, y) \geq 0, \forall y \in C, \tag{1.3}$$

which was first considered and investigated by Blum and Oettli [1] in 1994. The set of solutions of (1.3) is denoted by $EP(F)$.

The equilibrium problem provides a general framework to study a wide class of problems arising in economics, finance, network analysis, transportation, elasticity and optimization. The theory of equilibrium problems has become an explosive growth in theoretical advances and applications across all disciplines of pure and applied sciences, see [1, 5, 17].

In 2013, Suwannaut and Kangtunyakarn [14] introduced *the combination of equilibrium problem* which is to find $x \in C$ such that

$$\sum_{i=1}^{N} a_i F_i\left(x, y\right) \geq 0, \forall y \in C, \tag{1.4}$$

where $F_i : C \times C \to \mathbb{R}$ be bifunctions and $a_i \in (0, 1)$ with $\sum_{i=1}^{N} a_i = 1$, for every $i = 1, 2, \ldots, N$. The set of solution (1.4) is denoted by

$$EP\left(\sum_{i=1}^{N} a_i F_i\right) = \left\{x \in C : \left(\sum_{i=1}^{N} a_i F_i\right)(x, y) \geq 0, \forall y \in C\right\}.$$

Remark 1.3. *Very recently, in the work of Suwannaut and Kangtunyakarn [15], Khuangsatung and Kangtunyakarn [16] and Bnouhachem [17], they give the numerical examples for main theorems and show that their iteration for the combination of equilibrium problem converges faster than their iteration for the classical equilibrium problem.*

Inspired by the related research described above, we introduce and study the S-mapping generated by a finite family of nonspreading mappings and Lipschitzian mapping in a Hilbert space. Then, we prove a strong convergence theorem for finding a common element of the set of fixed points of a finite family of nonspreading mappings and Lipschitzian mappings and a common solution of a finite family of equilibrium problems. Futhermore, applying our main theorem, the additional results for equilibrium problem are obtained.

2 Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. We denote weak convergence and strong convergence by notations "$\rightharpoonup$" and "$\to$", respectively. For every $x \in H$, there is a unique nearest point $P_C x$ in C such that

$$\|x - P_C x\| \leq \|x - y\|, \forall y \in C.$$

Such an operator P_C is called the metric projection of H onto C.

Lemma 2.1 ([3]). *For a given $z \in H$ and $u \in C$,*

$$u = P_C z \Leftrightarrow \langle u - z, v - u \rangle \geq 0, \forall v \in C.$$

Furthermore, P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \forall x, y \in H.$$

Lemma 2.2 ([4]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n) s_n + \delta_n, \forall n \geq 0,$$

where α_n is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

Definition 2.3 ([12]). *Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of mappings of C into itself. For each $j = 1, 2, ...,$ let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S : C \to C$ as follows:*

$$\begin{aligned}
U_0 &= I \\
U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I \\
U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I \\
U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I \\
&\vdots \\
U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I \\
S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned}$$

This mapping is called S-mapping *generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$.*

Lemma 2.4 ([13]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into C with $\bigcap_{i=1}^N Fix(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, \ldots, N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0,1)$ for all $j = 1, 2, \ldots, N-1$ and $\alpha_1^N \in (0,1]$, $\alpha_3^N \in [0,1)$, $\alpha_2^j \in [0,1)$ for all $j = 1, 2, \ldots, N$. Let S be the S-mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Then, the following properties hold:*

(i) $Fix(S) = \bigcap_{i=1}^{N} Fix(T_i)$;

(ii) S *is a quasi-nonexpansive mapping.*

Now, the finite family of nonspreading mapping and Lipschitzian mappings is considered and the following result is proved.

Lemma 2.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonspreading mappings and L_i-Lipschitzian mapping of C into itself with $\bigcap_{i=1}^{N} Fix\,(T_i) \neq \emptyset$. For each $j = 1, 2, \ldots, N$ and $k = 1, 3$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I$, where $I = [0, 1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$, $\alpha_1^{n,p}, \alpha_3^{n,p} \in (0, 1)$ for all $p = 1, 2, \ldots, N-1$ and $\alpha_1^{n,N} \in (0, 1]$, $\alpha_3^{n,N} \in [0, 1)$, $\alpha_2^{n,j} \in [0, 1)$ such that $\alpha_k^{n,j} \to \alpha_k^j$ as $n \to \infty$ and $\sum_{n=1}^{\infty} \left| \alpha_k^{n+1,j} - \alpha_k^{n,j} \right| < \infty$. For every $n \in \mathbb{N}$, let S and S_n be the S-mapping generated by $T_1, T_1, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$ and $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$, respectively. Then, for every bounded sequences $\{x_n\}$ in C, the following statement hold:*

(i) $\lim_{n\to\infty} \|S_n x_n - S x_n\| = 0$;

(ii) $\sum_{n=1}^{\infty} \|S_n x_{n-1} - S_{n-1} x_{n-1}\| < \infty$.

Proof. Let $\{x_n\}$ be a bounded sequence in C and let U_k and $U_{n,k}$ be generated by $T_1, T_1, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$ and $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$, respectively.
First, we shall prove that (i) holds. For each $n \in \mathbb{N}$, we obtain

$$\begin{aligned}
\|U_{n,1}x_n - U_1 x_n\| &= \left\| \alpha_1^{n,1} T_1 x_n + \left(1 - \alpha_1^{n,1}\right) x_n - \alpha_1^1 T_1 x_n - \left(1 - \alpha_1^1\right) x_n \right\| \\
&= \left\| \alpha_1^{n,1} T_1 x_n - \alpha_1^{n,1} x_n - \alpha_1^1 T_1 x_n + \alpha_1^1 x_n \right\| \\
&= \left\| \left(\alpha_1^{n,1} - \alpha_1^1\right) T_1 x_n - \left(\alpha_1^{n,1} - \alpha_1^1\right) x_n \right\| \\
&= \left| \alpha_1^{n,1} - \alpha_1^1 \right| \|T_1 x_n - x_n\| . \qquad (2.1)
\end{aligned}$$

For $k \in \{2, 3, \ldots, N\}$, we derive

$$\begin{aligned}
\|U_{n,k}x_n - U_k x_n\| = &\Big\| \alpha_1^{n,k} T_k U_{n,k-1} x_n + \alpha_2^{n,k} U_{n,k-1} x_n + \alpha_3^{n,k} x_n - \alpha_1^k T_k U_{k-1} x_n \\
&- \alpha_2^k U_{k-1} x_n - \alpha_3^k x_n \Big\|
\end{aligned}$$

$$
\begin{aligned}
&= \Big\| \alpha_1^{n,k} T_k U_{n,k-1} x_n - \alpha_1^{n,k} T_k U_{k-1} x_n + \alpha_1^{n,k} T_k U_{k-1} x_n - \alpha_1^{k} T_k U_{k-1} x_n \\
&\quad + \alpha_2^{n,k} U_{n,k-1} x_n - \alpha_2^{n,k} U_{k-1} x_n + \alpha_2^{n,k} U_{k-1} x_n - \alpha_2^{k} U_{k-1} x_n \\
&\quad + \left(\alpha_3^{n,k} - \alpha_3^{k}\right) x_n \Big\| \\
&= \Big\| \alpha_1^{n,k} \left(T_k U_{n,k-1} x_n - T_k U_{k-1} x_n\right) + \left(\alpha_1^{n,k} - \alpha_1^{k}\right) T_k U_{k-1} x_n \\
&\quad + \alpha_2^{n,k} \left(U_{n,k-1} x_n - U_{k-1} x_n\right) + \left(\alpha_2^{n,k} - \alpha_2^{k}\right) U_{k-1} x_n \\
&\quad + \left(\alpha_3^{n,k} - \alpha_3^{k}\right) x_n \Big\| \\
&\le \alpha_1^{n,k} \left\| T_k U_{n,k-1} x_n - T_k U_{k-1} x_n \right\| + \left|\alpha_1^{n,k} - \alpha_1^{k}\right| \left\| T_k U_{k-1} x_n \right\| \\
&\quad + \alpha_2^{n,k} \left\| U_{n,k-1} x_n - U_{k-1} x_n \right\| + \left|\alpha_2^{n,k} - \alpha_2^{k}\right| \left\| U_{k-1} x_n \right\| \\
&\quad + \left|\alpha_3^{n,k} - \alpha_3^{k}\right| \left\| x_n \right\| \\
&\le \alpha_1^{n,k} L_k \left\| U_{n,k-1} x_n - U_{k-1} x_n \right\| + \left|\alpha_1^{n,k} - \alpha_1^{k}\right| \left\| T_k U_{k-1} x_n \right\| \\
&\quad + \alpha_2^{n,k} \left\| U_{n,k-1} x_n - U_{k-1} x_n \right\| \\
&\quad + \left|1 - \alpha_1^{n,k} - \alpha_3^{n,k} - 1 + \alpha_1^{k} + \alpha_3^{k}\right| \left\| U_{k-1} x_n \right\| + \left|\alpha_3^{n,k} - \alpha_3^{k}\right| \left\| x_n \right\| \\
&\le (L_k + 1) \left\| U_{n,k-1} x_n - U_{k-1} x_n \right\| + \left|\alpha_1^{n,k} - \alpha_1^{k}\right| \left\| T_k U_{k-1} x_n \right\| \\
&\quad + \left(\left|\alpha_1^{n,k} - \alpha_1^{k}\right| + \left|\alpha_3^{n,k} - \alpha_3^{k}\right|\right) \left\| U_{k-1} x_n \right\| + \left|\alpha_3^{n,k} - \alpha_3^{k}\right| \left\| x_n \right\| \\
&= (L_k + 1) \left\| U_{n,k-1} x_n - U_{k-1} x_n \right\| \\
&\quad + \left|\alpha_1^{n,k} - \alpha_1^{k}\right| \left(\left\| T_k U_{k-1} x_n \right\| + \left\| U_{k-1} x_n \right\|\right) \\
&\quad + \left|\alpha_3^{n,k} - \alpha_3^{k}\right| \left(\left\| U_{k-1} x_n \right\| + \left\| x_n \right\|\right). \qquad (2.2)
\end{aligned}
$$

From (2.1) and (2.2), we get

$$
\begin{aligned}
\left\| S_n x_n - S x_n \right\| &= \left\| U_{n,N} x_n - U_N x_n \right\| \\
&\le (L_N + 1) \left\| U_{n,N-1} x_n - U_{N-1} x_n \right\| \\
&\quad + \left|\alpha_1^{n,N} - \alpha_1^{N}\right| \left(\left\| T_N U_{N-1} x_n \right\| + \left\| U_{N-1} x_n \right\|\right) \\
&\quad + \left|\alpha_3^{n,N} - \alpha_3^{N}\right| \left(\left\| U_{N-1} x_n \right\| + \left\| x_n \right\|\right) \\
&\le (L_N + 1) \Big((L_{N-1} + 1) \left\| U_{n,N-2} x_n - U_{N-2} x_n \right\| \\
&\quad + \left|\alpha_1^{n,N-1} - \alpha_1^{N-1}\right| \left(\left\| T_{N-1} U_{N-2} x_n \right\| + \left\| U_{N-2} x_n \right\|\right) \\
&\quad + \left|\alpha_3^{n,N-1} - \alpha_3^{N-1}\right| \left(\left\| U_{N-2} x_n \right\| + \left\| x_n \right\|\right) \Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\alpha_1^{n,N}-\alpha_1^N\right|\left(\|T_N U_{N-1}x_n\|+\|U_{N-1}x_n\|\right)\\
&+\left|\alpha_3^{n,N}-\alpha_3^N\right|\left(\|U_{N-1}x_n\|+\|x_n\|\right)\\
=&\prod_{j=N-1}^{N}(L_j+1)\|U_{n,N-2}x_n-U_{N-2}x_n\|\\
&+\sum_{j=N-1}^{N}(L_j+1)^{N-j}+\left|\alpha_1^{n,j}-\alpha_1^j\right|\left(\|T_jU_{j-1}x_n\|+\|U_{j-1}x_n\|\right)\\
&+\sum_{j=N-1}^{N}(L_j+1)^{N-j}\left|\alpha_3^{n,j}-\alpha_3^j\right|\left(\|U_{j-1}x_n\|+\|x_n\|\right)\\
\vdots&\\
\leq&\prod_{j=2}^{N}(L_j+1)\|U_{n,1}x_n-U_1x_n\|\\
&+\sum_{j=2}^{N}(L_j+1)^{N-j}+\left|\alpha_1^{n,j}-\alpha_1^j\right|\left(\|T_jU_{j-1}x_n\|+\|U_{j-1}x_n\|\right)\\
&+\sum_{j=2}^{N}(L_j+1)^{N-j}\left|\alpha_3^{n,j}-\alpha_3^j\right|\left(\|U_{j-1}x_n\|+\|x_n\|\right)\\
=&\prod_{j=2}^{N}(L_j+1)\left|\alpha_1^{n,1}-\alpha_1^1\right|\|T_1x_n-x_n\|\\
&+\sum_{j=2}^{N}(L_j+1)^{N-j}+\left|\alpha_1^{n,j}-\alpha_1^j\right|\left(\|T_jU_{j-1}x_n\|+\|U_{j-1}x_n\|\right)\\
&+\sum_{j=2}^{N}(L_j+1)^{N-j}\left|\alpha_3^{n,j}-\alpha_3^j\right|\left(\|U_{j-1}x_n\|+\|x_n\|\right).
\end{aligned}
$$

From the condition $\alpha_k^{n,j}\to\alpha_k^j$ as $n\to\infty$, where $k=1,3, j=1,2,\ldots,N$, we can conclude that

$$\lim_{n\to\infty}\|S_nx_n-Sx_n\|=0.$$

Next, we will claim that (i) holds. For any $n\in\mathbb{N}$, we have

$$
\begin{aligned}
&\|U_{n,1}x_{n-1}-U_{n-1,1}x_{n-1}\|\\
&=\left\|\alpha_1^{n,1}T_1x_{n-1}+\left(1-\alpha_1^{n,1}\right)x_{n-1}-\alpha_1^{n-1,1}T_1x_{n-1}-\left(1-\alpha_1^{n-1,1}\right)x_{n-1}\right\|\\
&=\left\|\alpha_1^{n,1}T_1x_{n-1}-\alpha_1^{n,1}x_{n-1}-\alpha_1^{n-1,1}T_1x_{n-1}+\alpha_1^{n-1,1}x_{n-1}\right\|
\end{aligned}
$$

$$
\begin{aligned}
&= \left\| \left(\alpha_1^{n,1} - \alpha_1^{n-1,1}\right) T_1 x_{n-1} - \left(\alpha_1^{n,1} - \alpha_1^{n-1,1}\right) x_{n-1} \right\| \\
&= \left|\alpha_1^{n,1} - \alpha_1^{n-1,1}\right| \left\| T_1 x_{n-1} - x_{n-1} \right\|. \qquad (2.3)
\end{aligned}
$$

For $k \in \{2, 3, \ldots, N\}$, we obtain

$$
\begin{aligned}
&\left\| U_{n,k} x_{n-1} - U_{n-1,k} x_{n-1} \right\| \\
=&\Big\| \alpha_1^{n,k} T_k U_{n,k-1} x_{n-1} + \alpha_2^{n,k} U_{n,k-1} x_{n-1} + \alpha_3^{n,k} x_{n-1} - \alpha_1^{n-1,k} T_k U_{n-1,k-1} x_{n-1} \\
&- \alpha_2^{n-1,k} U_{n-1,k-1} x_{n-1} - \alpha_3^{n-1,k} x_{n-1} \Big\| \\
=&\Big\| \alpha_1^{n,k} T_k U_{n,k-1} x_{n-1} - \alpha_1^{n,k} T_k U_{n-1,k-1} x_{n-1} + \alpha_1^{n,k} T_k U_{n-1,k-1} x_{n-1} \\
&- \alpha_1^{n-1,k} T_k U_{n-1,k-1} x_{n-1} + \alpha_2^{n,k} U_{n,k-1} x_{n-1} - \alpha_2^{n,k} U_{n-1,k-1} x_{n-1} \\
&+ \alpha_2^{n,k} U_{n-1,k-1} x_{n-1} - \alpha_2^{n-1,k} U_{n-1,k-1} x_{n-1} + \left(\alpha_3^{n,k} - \alpha_3^{n-1,k}\right) x_{n-1} \Big\| \\
=&\Big\| \alpha_1^{n,k} \left(T_k U_{n,k-1} x_{n-1} - T_k U_{n-1,k-1} x_{n-1}\right) + \left(\alpha_1^{n,k} - \alpha_{n-1,1}^{k}\right) T_k U_{n-1,k-1} x_{n-1} \\
&+ \alpha_2^{n,k} \left(U_{n,k-1} x_{n-1} - U_{n-1,k-1} x_{n-1}\right) + \left(\alpha_2^{n,k} - \alpha_2^{n-1,k}\right) U_{n-1,k-1} x_{n-1} \\
&+ \left(\alpha_3^{n,k} - \alpha_3^{n-1,k}\right) x_{n-1} \Big\| \\
\leq&\ \alpha_1^{n,k} \left\| T_k U_{n,k-1} x_{n-1} - T_k U_{n-1,k-1} x_{n-1} \right\| + \left|\alpha_1^{n,k} - \alpha_1^{n-1,k}\right| \left\| T_k U_{n-1,k-1} x_{n-1} \right\| \\
&+ \alpha_2^{n,k} \left\| U_{n,k-1} x_{n-1} - U_{n-1,k-1} x_{n-1} \right\| + \left|\alpha_2^{n,k} - \alpha_2^{n-1,k}\right| \left\| U_{n-1,k-1} x_{n-1} \right\| \\
&+ \left|\alpha_3^{n,k} - \alpha_3^{n-1,k}\right| \left\| x_{n-1} \right\| \\
\leq&\ \alpha_1^{n,k} L_k \left\| U_{n,k-1} x_{n-1} - U_{n-1,k-1} x_{n-1} \right\| + \left|\alpha_1^{n,k} - \alpha_1^{n-1,k}\right| \left\| T_k U_{n-1,k-1} x_{n-1} \right\| \\
&+ \alpha_2^{n,k} \left\| U_{n,k-1} x_{n-1} - U_{n-1,k-1} x_{n-1} \right\| \\
&+ \left|1 - \alpha_1^{n,k} - \alpha_3^{n,k} - 1 + \alpha_1^{n-1,k} + \alpha_3^{n-1,k}\right| \left\| U_{n-1,k-1} x_{n-1} \right\| \\
&+ \left|\alpha_3^{n,k} - \alpha_3^{n-1,k}\right| \left\| x_{n-1} \right\| \\
\leq&\ (L_k + 1) \left\| U_{n,k-1} x_{n-1} - U_{n-1,k-1} x_{n-1} \right\| + \left|\alpha_1^{n,k} - \alpha_1^{n-1,k}\right| \left\| T_k U_{n-1,k-1} x_{n-1} \right\| \\
&+ \left(\left|\alpha_1^{n,k} - \alpha_1^{n-1,k}\right| + \left|\alpha_3^{n,k} - \alpha_3^{n-1,k}\right| \right) \left\| U_{n-1,k-1} x_{n-1} \right\| \\
&+ \left|\alpha_3^{n,k} - \alpha_3^{n-1,k}\right| \left\| x_{n-1} \right\| \\
=&\ (L_k + 1) \left\| U_{n,k-1} x_{n-1} - U_{n-1,k-1} x_{n-1} \right\| \\
&+ \left|\alpha_1^{n,k} - \alpha_1^{n-1,k}\right| \left(\left\| T_k U_{n-1,k-1} x_{n-1} \right\| + \left\| U_{n-1,k-1} x_{n-1} \right\| \right) \\
&+ \left|\alpha_3^{n,k} - \alpha_3^{n-1,k}\right| \left(\left\| U_{n-1,k-1} x_{n-1} \right\| + \left\| x_{n-1} \right\| \right). \qquad (2.4)
\end{aligned}
$$

From (2.3) and (2.4), we get

$$
\begin{aligned}
&\|S_n x_{n-1} - S_{n-1} x_{n-1}\| \\
&= \|U_{n,N} x_{n-1} - U_{n-1,N} x_{n-1}\| \\
&\leq (L_N + 1) \|U_{n,N-1} x_{n-1} - U_{n-1,N-1} x_{n-1}\| \\
&\quad + \left|\alpha_1^{n,N} - \alpha_1^{n-1,N}\right| (\|T_N U_{n-1,N-1} x_{n-1}\| + \|U_{n-1,N-1} x_{n-1}\|) \\
&\quad + \left|\alpha_3^{n,N} - \alpha_3^{n-1,N}\right| (\|U_{n-1,N-1} x_{n-1}\| + \|x_{n-1}\|) \\
&\leq (L_N + 1) \Big((L_{N-1} + 1) \|U_{n,N-2} x_{n-1} - U_{n-1,N-2} x_{n-1}\| \\
&\quad + \left|\alpha_1^{n,N-1} - \alpha_1^{n-1,N-1}\right| (\|T_{N-1} U_{n-1,N-2} x_{n-1}\| + \|U_{n-1,N-2} x_{n-1}\|) \\
&\quad + \left|\alpha_3^{n,N-1} - \alpha_3^{n-1,N-1}\right| (\|U_{n-1,N-2} x_{n-1}\| + \|x_{n-1}\|) \Big) \\
&\quad + \left|\alpha_1^{n,N} - \alpha_1^{n-1,N}\right| (\|T_N U_{n-1,N-1} x_{n-1}\| + \|U_{n-1,N-1} x_{n-1}\|) \\
&\quad + \left|\alpha_3^{n,N} - \alpha_3^{n-1,N}\right| (\|U_{n-1,N-1} x_{n-1}\| + \|x_{n-1}\|) \\
&= \prod_{j=N-1}^{N} (L_j + 1) \|U_{n,N-2} x_{n-1} - U_{n-1,N-2} x_{n-1}\| \\
&\quad + \sum_{j=N-1}^{N} (L_j + 1)^{N-j} \left|\alpha_1^{n,j} - \alpha_1^{n-1,j}\right| (\|T_j U_{n-1,j-1} x_{n-1}\| + \|U_{n-1,j-1} x_{n-1}\|) \\
&\quad + \sum_{j=N-1}^{N} (L_j + 1)^{N-j} \left|\alpha_3^{n,j} - \alpha_3^{n-1,j}\right| (\|U_{n-1,j-1} x_{n-1}\| + \|x_{n-1}\|) \\
&\ \vdots \\
&\leq \prod_{j=2}^{N} (L_j + 1) \|U_{n,1} x_{n-1} - U_{n-1,1} x_{n-1}\| \\
&\quad + \sum_{j=2}^{N} (L_j + 1)^{N-j} \left|\alpha_1^{n,j} - \alpha_1^{n-1,j}\right| (\|T_j U_{n-1,j-1} x_{n-1}\| + \|U_{n-1,j-1} x_{n-1}\|) \\
&\quad + \sum_{j=2}^{N} (L_j + 1)^{N-j} \left|\alpha_3^{n,j} - \alpha_3^{n-1,j}\right| (\|U_{n-1,j-1} x_{n-1}\| + \|x_{n-1}\|) \\
&= \prod_{j=2}^{N} (L_j + 1) \left|\alpha_1^{n,1} - \alpha_1^{n-1,1}\right| \|T_1 x_{n-1} - x_{n-1}\| \\
&\quad + \sum_{j=2}^{N} (L_j + 1)^{N-j} \left|\alpha_1^{n,j} - \alpha_1^{n-1,j}\right| (\|T_j U_{n-1,j-1} x_{n-1}\| + \|U_{n-1,j-1} x_{n-1}\|) \\
&\quad + \sum_{j=2}^{N} (L_j + 1)^{N-j} \left|\alpha_3^{n,j} - \alpha_3^{n-1,j}\right| (\|U_{n-1,j-1} x_{n-1}\| + \|x_{n-1}\|).
\end{aligned}
$$

From the condition $\sum_{n=1}^{\infty}\left|\alpha_k^{n+1,j}-\alpha_k^{n,j}\right|<\infty$, where $k=1,3, j=1,2,\ldots,N$, hence we get

$$\sum_{n=1}^{\infty}\|S_nx_{n-1}-S_{n-1}x_{n-1}\|<\infty.$$

□

For solving the equilibrium problem for a bifunction $F: C\times C\to\mathbb{R}$, let us assume that F and C satisfy the following conditions:
(A1) $F(x,x)=0$ for all $x\in C$;
(A2) F is monotone, i.e., $F(x,y)+F(y,x)\leq 0$ for all $x,y\in C$;
(A3) For each $x,y,z\in C$,

$$\lim_{t\to 0^+}F(tz+(1-t)x,y)\leq F(x,y);$$

(A4) For each $x\in C, y\mapsto F(x,y)$ is convex and lower semicontinuous.

Lemma 2.6 ([14])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i=1,2,\ldots,N$, let $F_i: C\times C\to\mathbb{R}$ be bifunctions satisfying $(A1)-(A4)$ with $\bigcap_{i=1}^{N}EP(F_i)\neq\emptyset$. Then,*

$$EP\left(\sum_{i=1}^{N}a_iF_i\right)=\bigcap_{i=1}^{N}EP(F_i),$$

where $a_i\in(0,1)$ for every $i=1,2,\ldots,N$ and $\sum_{i=1}^{N}a_i=1$.

Lemma 2.7 ([1])**.** *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C\times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r>0$ and $x\in H$. Then, there exists $z\in C$ such that*

$$F(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\forall y\in C.$$

Lemma 2.8 ([5])**.** *Assume that $F: C\times C\to\mathbb{R}$ satisfies $(A1)-(A4)$. For $r>0$, define a mapping $T_r: H\to C$ as follows:*

$$T_r(x)=\left\{z\in C: F(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\forall y\in C\right\}$$

for all $x\in H$. Then, the following hold:

(i) T_r is single-valued;

(ii) T_r is firmly nonexpansive, i.e., for any $x,y\in H$,

$$\|T_r(x)-T_r(y)\|^2\leq\langle T_r(x)-T_r(y),x-y\rangle;$$

(iii) $Fix\,(T_r) = EP(F)$;

(iv) $EP(F)$ *is closed and convex.*

Remark 2.9 ([14])**.** *Since* $\sum_{i=1}^{N} a_i F_i$ *satisfies (A1)-(A4), by Lemma 2.6 and Lemma 2.8, we obtain*

$$Fix(T_r) = EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP\,(F_i)\,,$$

where $a_i \in (0,1)$, *for each* $i = 1, 2, \ldots, N$, *and* $\sum_{i=1}^{N} a_i = 1$.

3 Strong convergence theorem

Theorem 3.1. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *For* $i = 1, 2, \ldots, N$, *let* $F_i : C \times C \to \mathbb{R}$ *be a bifunction satisfying* $(A1)-(A4)$. *For every* $j = 1, 2, \ldots, N$, *let* $\{T_j\}_{j=1}^{N}$ *be a finite family of nonspreading mappings and* L_i*-Lipschitzian mappings of* C *into itself with* $\bigcap_{j=1}^{N} Fix(T_j) \neq \emptyset$. *Suppose that* $\Omega := \bigcap_{j=1}^{N} Fix(T_j) \cap \bigcap_{i=1}^{N} EP\,(F_i) \neq \emptyset$. *For each* $j = 1, 2, \ldots, N$ *and* $k = 1, 3$, *let* $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I$, *where* $I = [0,1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$, $\alpha_1^{n,p}, \alpha_3^{n,p} \in (0,1)$ *for all* $p = 1, 2, \ldots, N-1$ *and* $\alpha_1^{n,N} \in (0,1]$, $\alpha_3^{n,N} \in [0,1)$, $\alpha_2^{n,j} \in [0,1)$. *For each* $n \in \mathbb{N}$, *let* S_n *be the* S*-mapping generated by* $T_1, T_2, \ldots, T_N$ *and* $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$. *Let* $\{x_n\}$ *be the sequence generated by* $x_1 \in C$ *and*

$$\begin{cases} \displaystyle\sum_{i=1}^{N} a_i F_i\,(u_n, y) + \frac{1}{r_n}\,\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n f\,(x_n) + \beta_n u_n + \delta_n S_n x_n, \forall n \geq 1, \end{cases} \tag{3.1}$$

where $f : C \to C$ *be a contraction mapping with a constant* ξ *and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\} \subseteq (0,1)$ *with* $\alpha_n + \beta_n + \delta_n = 1$, $\forall n \geq 1$. *Suppose the following statement are true:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \tau \leq \beta_n, \delta_n \leq \upsilon < 1$ *for some* $\tau, \upsilon > 0$;

(iii) $\sum_{i=1}^{N} a_i = 1$;

(iv) $0 < \epsilon \leq r_n \leq \rho < 1$, *for some* $\epsilon, \rho > 0$;

(v) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$, $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$,
$\sum_{n=1}^{\infty}|\delta_{n+1}-\delta_n|<\infty$, $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$,
$\sum_{n=1}^{\infty}\left|\alpha_k^{n+1}-\alpha_k^n\right|<\infty$, *for all* $k=1,3$;

Then $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $q=P_\Omega f(q)$.

Proof. The proof shall be divided into seven steps.
Step 1. We will prove that $\{x_n\}$ is bounded.
Since $\sum_{i=1}^{N}a_iF_i$ satisfies (A1)-(A4) and

$$\sum_{i=1}^{N}a_iF_i\left(u_n,y\right)+\frac{1}{r_n}\left\langle y-u_n,u_n-x_n\right\rangle\geq 0,\forall y\in C,$$

by Lemma 2.8 and Remark 2.9, we have $u_n=T_{r_n}x_n$ and $Fix(T_{r_n})=\bigcap_{i=1}^{N}EP\left(F_i\right)$.
Let $z\in\Omega$. By Lemma 2.4, we have

$$\begin{aligned}
&\left\|x_{n+1}-z\right\|\\
\leq&\alpha_n\left\|f\left(x_n\right)-z\right\|+\beta_n\left\|u_n-z\right\|+\delta_n\left\|S_nx_n-z\right\|\\
\leq&\alpha_n\left\|f\left(x_n\right)-f(z)+f(z)-z\right\|+\beta_n\left\|T_{r_n}x_n-z\right\|+\delta_n\left\|x_n-z\right\|\\
\leq&\alpha_n\left(\left\|f\left(x_n\right)-f(z)\right\|+\left\|f(z)-z\right\|\right)+\left(1-\alpha_n\right)\left\|x_n-z\right\|\\
\leq&\alpha_n\left(\xi\left\|x_n-z\right\|+\left\|f(z)-z\right\|\right)+\left(1-\alpha_n\right)\left\|x_n-z\right\|\\
=&\left(1-\alpha_n\left(1-\xi\right)\right)\left\|x_n-z\right\|+\alpha_n\left\|f(z)-z\right\|\\
\leq&\max\left\{\left\|x_1-z\right\|,\frac{\left\|f(z)-z\right\|}{1-\xi}\right\}.
\end{aligned}$$

By induction, we have $\|x_n-z\|\leq\max\left\{\|x_1-z\|,\frac{\|f(z)-z\|}{1-\xi}\right\},\forall n\in\mathbb{N}$. It follows that $\{x_n\}$ is bounded so is $\{u_n\}$.
Step 2. We will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.
By the definition of x_n, we obtain

$$\begin{aligned}
&\left\|x_{n+1}-x_n\right\|\\
=&\left\|\alpha_nf\left(x_n\right)+\beta_nu_n+\delta_nS_nx_n-\left(\alpha_{n-1}f\left(x_{n-1}\right)+\beta_{n-1}u_{n-1}+\delta_{n-1}S_{n-1}x_{n-1}\right)\right\|\\
\leq&\alpha_n\left\|f\left(x_n\right)-f\left(x_{n-1}\right)\right\|+\left|\alpha_n-\alpha_{n-1}\right|\left\|f\left(x_{n-1}\right)\right\|+\beta_n\left\|u_n-u_{n-1}\right\|\\
&+\left|\beta_n-\beta_{n-1}\right|\left\|u_{n-1}\right\|+\delta_n\left\|S_nx_n-S_nx_{n-1}\right\|+\delta_n\left\|S_nx_{n-1}-S_{n-1}x_{n-1}\right\|\\
&+\left|\delta_n-\delta_{n-1}\right|\left\|S_{n-1}x_{n-1}\right\|\\
\leq&\alpha_n\xi\left\|x_n-x_{n-1}\right\|+\left|\alpha_n-\alpha_{n-1}\right|\left\|f\left(x_{n-1}\right)\right\|+\beta_n\left\|u_n-u_{n-1}\right\|\\
&+\left|\beta_n-\beta_{n-1}\right|\left\|u_{n-1}\right\|+\delta_n\left\|x_n-x_{n-1}\right\|\\
&+\delta_n\left\|S_nx_{n-1}-S_{n-1}x_{n-1}\right\|+\left|\delta_n-\delta_{n-1}\right|\left\|S_{n-1}x_{n-1}\right\|.
\end{aligned}\tag{3.2}$$

Since $u_n = T_{r_n}x_n$, by utilizing the definition of T_{r_n}, we obtain

$$\sum_{i=1}^{N} a_i F_i \left(T_{r_n}x_n, y\right) + \frac{1}{r_n}\left\langle y - T_{r_n}x_n, T_{r_n}x_n - x_n\right\rangle \geq 0, \forall y \in C, \tag{3.3}$$

and

$$\sum_{i=1}^{N} a_i F_i \left(T_{r_{n+1}}x_{n+1}, y\right) + \frac{1}{r_{n+1}}\left\langle y - T_{r_{n+1}}x_{n+1}, T_{r_{n+1}}x_{n+1} - x_{n+1}\right\rangle \geq 0, \forall y \in C. \tag{3.4}$$

From (3.3) and (3.4), it follows that

$$\sum_{i=1}^{N} a_i F_i \left(T_{r_n}x_n, T_{r_{n+1}}x_{n+1}\right) + \frac{1}{r_n}\left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_n}x_n - x_n\right\rangle \geq 0, \tag{3.5}$$

and

$$\sum_{i=1}^{N} a_i F_i \left(T_{r_{n+1}}x_{n+1}, T_{r_n}x_n\right) + \frac{1}{r_{n+1}}\left\langle T_{r_n}x_n - T_{r_{n+1}}x_{n+1}, T_{r_{n+1}}x_{n+1} - x_{n+1}\right\rangle \geq 0. \tag{3.6}$$

From (3.5) and (3.6) and the fact that $\sum_{i=1}^{N} a_i F_i$ satisfies (A2), we have

$$\begin{aligned} &\frac{1}{r_n}\left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_n}x_n - x_n\right\rangle \\ &+ \frac{1}{r_{n+1}}\left\langle T_{r_n}x_n - T_{r_{n+1}}x_{n+1}, T_{r_{n+1}}x_{n+1} - x_{n+1}\right\rangle \geq 0, \end{aligned}$$

which implies that

$$\left\langle T_{r_n}x_n - T_{r_{n+1}}x_{n+1}, \frac{T_{r_{n+1}}x_{n+1} - x_{n+1}}{r_{n+1}} - \frac{T_{r_n}x_n - x_n}{r_n}\right\rangle \geq 0.$$

It follows that

$$\begin{aligned} &\Big\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_n}x_n - T_{r_{n+1}}x_{n+1} + T_{r_{n+1}}x_{n+1} - x_n \\ &- \frac{r_n}{r_{n+1}}\left(T_{r_{n+1}}x_{n+1} - x_{n+1}\right)\Big\rangle \geq 0. \end{aligned} \tag{3.7}$$

From (3.7), we obtain

$$\begin{aligned}
&\left\|T_{r_{n+1}}x_{n+1}-T_{r_n}x_n\right\|^2\\
\leq&\left\langle T_{r_{n+1}}x_{n+1}-T_{r_n}x_n, T_{r_{n+1}}x_{n+1}-x_n-\frac{r_n}{r_{n+1}}\left(T_{r_{n+1}}x_{n+1}-x_{n+1}\right)\right\rangle\\
=&\left\langle T_{r_{n+1}}x_{n+1}-T_{r_n}x_n, x_{n+1}-x_n+\left(1-\frac{r_n}{r_{n+1}}\right)\left(T_{r_{n+1}}x_{n+1}-x_{n+1}\right)\right\rangle\\
\leq&\left\|T_{r_{n+1}}x_{n+1}-T_{r_n}x_n\right\|\left\|x_{n+1}-x_n+\left(1-\frac{r_n}{r_{n+1}}\right)\left(T_{r_{n+1}}x_{n+1}-x_{n+1}\right)\right\|\\
\leq&\left\|T_{r_{n+1}}x_{n+1}-T_{r_n}x_n\right\|\left[\|x_{n+1}-x_n\|+\left|1-\frac{r_n}{r_{n+1}}\right|\left\|T_{r_{n+1}}x_{n+1}-x_{n+1}\right\|\right]\\
=&\left\|T_{r_{n+1}}x_{n+1}-T_{r_n}x_n\right\|\left[\|x_{n+1}-x_n\|+\frac{1}{r_{n+1}}\left|r_{n+1}-r_n\right|\left\|T_{r_{n+1}}x_{n+1}-x_{n+1}\right\|\right]\\
\leq&\left\|T_{r_{n+1}}x_{n+1}-T_{r_n}x_n\right\|\left[\|x_{n+1}-x_n\|+\frac{1}{\rho}\left|r_{n+1}-r_n\right|\left\|T_{r_{n+1}}x_{n+1}-x_{n+1}\right\|\right],
\end{aligned}$$

which follows that

$$\|u_{n+1}-u_n\|\leq\|x_{n+1}-x_n\|+\frac{1}{\rho}\left|r_{n+1}-r_n\right|\|u_{n+1}-x_{n+1}\|. \tag{3.8}$$

From (3.8), we have

$$\|u_n-u_{n-1}\|\leq\|x_n-x_{n-1}\|+\frac{1}{\rho}\left|r_n-r_{n-1}\right|\|u_n-x_n\|. \tag{3.9}$$

Substituting (3.9) into (3.2), we derive

$$\begin{aligned}
&\|x_{n+1}-x_n\|\\
\leq&\alpha_n\xi\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\|+\beta_n\Big(\|x_n-x_{n-1}\|\\
&+\frac{1}{\rho}|r_n-r_{n-1}|\|u_n-x_n\|\Big)+|\beta_n-\beta_{n-1}|\|u_{n-1}\|+\delta_n\|x_n-x_{n-1}\|\\
&+\delta_n\|S_nx_{n-1}-S_{n-1}x_{n-1}\|+|\delta_n-\delta_{n-1}|\|S_{n-1}x_{n-1}\|\\
\leq&(1-\alpha_n(1-\xi))\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\|\\
&+\frac{1}{\rho}|r_n-r_{n-1}|\|u_n-x_n\|+|\beta_n-\beta_{n-1}|\|u_{n-1}\|+\delta_n\|S_nx_{n-1}-S_{n-1}x_{n-1}\|\\
&+|\delta_n-\delta_{n-1}|\|S_{n-1}x_{n-1}\|.
\end{aligned}$$

From the conditions (i), (v), Lemma 2.2 and Lemma 2.5(ii), we obtain

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.10}$$

Step 3. We will show that $\lim_{n\to\infty}\|u_n-x_n\|=0$ and $\lim_{n\to\infty}\|S_nx_n-x_n\|=0$. To show this, let $z\in\Omega$. Since $u_n=T_{r_n}x_n$ and T_{r_n} is firmly nonexpansive mapping, then we obtain

$$\begin{aligned}\|z - T_{r_n} x_n\|^2 &= \|T_{r_n} z - T_{r_n} x_n\|^2 \\ &\leq \langle T_{r_n} z - T_{r_n} x_n, z - x_n \rangle \\ &= \frac{1}{2}\left(\|T_{r_n} x_n - z\|^2 + \|x_n - z\|^2 - \|T_{r_n} x_n - x_n\|^2\right),\end{aligned}$$

which follows that

$$\|u_n - z\|^2 \leq \|x_n - z\|^2 - \|u_n - x_n\|^2. \tag{3.11}$$

From (3.1) and (3.11), we get

$$\begin{aligned}&\|x_{n+1} - z\|^2 \\ \leq&\alpha_n \|f(x_n) - z\|^2 + \beta_n \|u_n - z\|^2 + \delta_n \|Sx_n - z\|^2 \\ \leq&\alpha_n \|f(x_n) - z\|^2 + \beta_n \Big(\|x_n - z\|^2 - \|u_n - x_n\|^2 \Big) + \delta_n \|x_n - z\|^2 \\ \leq&\alpha_n \|f(x_n) - z\|^2 + \|x_n - z\|^2 - \beta_n \|u_n - x_n\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}\beta_n \|u_n - x_n\|^2 &\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \alpha_n \|f(x_n) - z\|^2 \\ &\leq (\|x_n - z\| + \|x_{n+1} - z\|) \|x_{n+1} - x_n\| + \alpha_n \|f(x_n) - z\|^2.\end{aligned} \tag{3.12}$$

From (3.10) and the conditions (i), (ii), it implies by (3.12) that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.13}$$

By the definition of x_n, we obtain

$$\begin{aligned}x_{n+1} - x_n &= \alpha_n f(x_n) + \beta_n u_n + \delta_n S_n x_n - x_n \\ &= \alpha_n (f(x_n) - x_n) + \beta_n (u_n - x_n) + \delta_n (S_n x_n - x_n).\end{aligned}$$

This follows that

$$\delta_n \|S_n x_n - x_n\| \leq \alpha_n \|f(x_n) - x_n\| + \beta_n \|u_n - x_n\| + \|x_{n+1} - x_n\|.$$

From (3.10), (3.13) and the conditions (i) and (ii), we obtain

$$\lim_{n\to\infty} \|S_n x_n - x_n\| = 0. \tag{3.14}$$

Next, we will derive that

$$\lim_{n\to\infty} \|T_i U_{i-1} x_n - U_{i-1} x_n\| = 0, \forall i = 1, 2, \ldots, N.$$

From (3.1), we obtain

$$
\begin{aligned}
&\|x_{n+1}-z\|^2\\
\leq&\alpha_n\|f(x_n)-z\|^2+\beta_n\|u_n-z\|^2+\delta_n\|S_nx_n-z\|^2\\
\leq&\alpha_n\|f(x_n)-z\|^2+\beta_n\|x_n-z\|^2+\delta_n\big\|\alpha_1^N(T_NU_{N-1}x_n-z)\\
&+\alpha_2^N(U_{N-1}x_n-z)+\alpha_3^N(x_n-z)\big\|^2\\
\leq&\alpha_n\|f(x_n)-z\|^2+\beta_n\|x_n-z\|^2+\delta_n\big(\alpha_1^N\|T_NU_{N-1}x_n-z\|^2\\
&+\alpha_2^N\|U_{N-1}x_n-z\|^2+\alpha_3^N\|x_n-z\|^2-\alpha_1^N\alpha_2^N\|T_NU_{N-1}x_n-U_{N-1}x_n\|^2\big)\\
\leq&\alpha_n\|f(x_n)-z\|^2+\beta_n\|x_n-z\|^2+\delta_n\big(\big(1-\alpha_3^N\big)\|U_{N-1}x_n-z\|^2\\
&+\alpha_3^N\|x_n-z\|^2-\alpha_1^N\alpha_2^N\|T_NU_{N-1}x_n-U_{N-1}x_n\|^2\big)\\
=&\alpha_n\|f(x_n)-z\|^2+\beta_n\|x_n-z\|^2+\delta_n\big(\big(1-\alpha_3^N\big)\big\|\alpha_1^{N-1}(T_{N-1}U_{N-2}x_n-z)\\
&+\alpha_2^{N-1}(U_{N-2}x_n-z)+\alpha_3^{N-1}(x_n-z)\big\|^2\\
&+\alpha_3^N\|x_n-z\|^2-\alpha_1^N\alpha_2^N\|T_NU_{N-1}x_n-U_{N-1}x_n\|^2\big)\\
\leq&\alpha_n\|f(x_n)-z\|^2+\beta_n\|x_n-z\|^2+\delta_n\big(\big(1-\alpha_3^N\big)\big(\alpha_1^{N-1}\|T_{N-1}U_{N-2}x_n-z\|^2\\
&+\alpha_2^{N-1}\|U_{N-2}x_n-z\|^2+\alpha_3^{N-1}\|x_n-z\|^2\\
&-\alpha_1^{N-1}\alpha_2^{N-1}\|T_{N-1}U_{N-2}x_n-U_{N-2}x_n\|^2\big)\\
&+\alpha_3^N\|x_n-z\|^2-\alpha_1^N\alpha_2^N\|T_NU_{N-1}x_n-U_{N-1}x_n\|^2\big)\\
\leq&\alpha_n\|f(x_n)-z\|^2+\beta_n\|x_n-z\|^2+\delta_n\big(\big(1-\alpha_3^N\big)\big(\big(1-\alpha_3^{N-1}\big)\|U_{N-2}x_n-z\|^2\\
&+\alpha_3^{N-1}\|x_n-z\|^2-\alpha_1^{N-1}\alpha_2^{N-1}\|T_{N-1}U_{N-2}x_n-U_{N-2}x_n\|^2\big)\\
&+\alpha_3^N\|x_n-z\|^2-\alpha_1^N\alpha_2^N\|T_NU_{N-1}x_n-U_{N-1}x_n\|^2\big)\\
=&\alpha_n\|f(x_n)-z\|^2+\beta_n\|x_n-z\|^2+\delta_n\big(\big(1-\alpha_3^N\big)\big(1-\alpha_3^{N-1}\big)\|U_{N-2}x_n-z\|^2\\
&+\big(1-\alpha_3^N\big)\alpha_3^{N-1}\|x_n-z\|^2-\alpha_1^{N-1}\alpha_2^{N-1}\big(1-\alpha_3^N\big)\|T_{N-1}U_{N-2}x_n-U_{N-2}x_n\|^2\\
&+\alpha_3^N\|x_n-z\|^2-\alpha_1^N\alpha_2^N\|T_NU_{N-1}x_n-U_{N-1}x_n\|^2\big)\\
=&\alpha_n\|f(x_n)-z\|^2+\beta_n\|x_n-z\|^2+\delta_n\left(\prod_{j=N-1}^{N}\left(1-\alpha_3^j\right)\|U_{N-2}x_n-z\|^2\right.\\
&+\left(1-\prod_{j=N-1}^{N}\left(1-\alpha_3^j\right)\right)\|x_n-z\|^2\\
&-\alpha_1^{N-1}\alpha_2^{N-1}\big(1-\alpha_3^N\big)\|T_{N-1}U_{N-2}x_n-U_{N-2}x_n\|^2\\
&\left.-\alpha_1^N\alpha_2^N\|T_NU_{N-1}x_n-U_{N-1}x_n\|^2\right)\\
\leq&\alpha_n\|f(x_n)-z\|^2+\beta_n\|x_n-z\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\delta_n\Bigg(\prod_{j=N-2}^{N}\left(1-\alpha_3^j\right)\|U_{N-3}x_n-z\|^2+\left(1-\prod_{j=N-2}^{N}\left(1-\alpha_3^j\right)\right)\|x_n-z\|^2\\
&-\alpha_1^{N-2}\alpha_2^{N-2}\prod_{j=N-1}^{N}\left(1-\alpha_3^j\right)\|T_{N-2}U_{N-3}x_n-U_{N-3}x_n\|^2\\
&-\alpha_1^{N-1}\alpha_2^{N-1}\left(1-\alpha_3^N\right)\|T_{N-1}U_{N-2}x_n-U_{N-2}x_n\|^2\\
&-\alpha_1^{N}\alpha_2^{N}\|T_NU_{N-1}x_n-U_{N-1}x_n\|^2\Bigg)\\
&\vdots\\
\leq&\alpha_n\|f(x_n)-z\|^2+\beta_n\|x_n-z\|^2\\
&+\delta_n\Bigg(\prod_{j=1}^{N}\left(1-\alpha_3^j\right)\|x_n-z\|^2+\left(1-\prod_{j=1}^{N}\left(1-\alpha_3^j\right)\right)\|x_n-z\|^2\\
&-\alpha_1^{1}\alpha_2^{1}\prod_{j=2}^{N}\left(1-\alpha_3^j\right)\|T_1x_n-x_n\|^2\\
&\vdots\\
&-\alpha_1^{N-2}\alpha_2^{N-2}\prod_{j=N-1}^{N}\left(1-\alpha_3^j\right)\|T_{N-2}U_{N-3}x_n-U_{N-3}x_n\|^2\\
&-\alpha_1^{N-1}\alpha_2^{N-1}\left(1-\alpha_3^N\right)\|T_{N-1}U_{N-2}x_n-U_{N-2}x_n\|^2\\
&-\alpha_1^{N}\alpha_2^{N}\|T_NU_{N-1}x_n-U_{N-1}x_n\|^2\Bigg)\\
\leq&\alpha_n\|f(x_n)-z\|^2+\|x_n-z\|^2\\
&-\delta_n\alpha_1^{1}\alpha_2^{1}\prod_{j=2}^{N}\left(1-\alpha_3^j\right)\|T_1x_n-x_n\|^2\\
&\vdots\\
&-\delta_n\alpha_1^{N-2}\alpha_2^{N-2}\prod_{j=N-1}^{N}\left(1-\alpha_3^j\right)\|T_{N-2}U_{N-3}x_n-U_{N-3}x_n\|^2\\
&-\delta_n\alpha_1^{N-1}\alpha_2^{N-1}\left(1-\alpha_3^N\right)\|T_{N-1}U_{N-2}x_n-U_{N-2}x_n\|^2\\
&-\delta_n\alpha_1^{N}\alpha_2^{N}\|T_NU_{N-1}x_n-U_{N-1}x_n\|^2\Bigg).
\end{aligned}
\tag{3.15}
$$

From (3.15), we get

$$\begin{aligned}
&\delta_n \alpha_1^1 \alpha_2^1 \prod_{j=2}^{N} \left(1-\alpha_3^j\right) \|T_1 x_n - x_n\|^2 \\
&\leq \alpha_n \|f(x_n) - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
&\leq \alpha_n \|f(x_n) - z\|^2 + (\|x_n - z\| - \|x_{n+1} - z\|) \|x_{n+1} - x_n\| .
\end{aligned}$$

It implies by (3.10) and the conditions (i) and (ii), that

$$\lim_{n\to\infty} \|T_1 x_n - x_n\| = 0. \tag{3.16}$$

Applying the same method as (3.16), we also obtain

$$\lim_{n\to\infty} \|T_i U_{i-1} x_n - U_{i-1} x_n\| = 0, \forall i = 1, 2, \ldots, N. \tag{3.17}$$

Step 4. We will claim that $\limsup_{n\to\infty} \langle f(q) - q, x_n - q\rangle \leq 0$, where $q = P_\Omega f(q)$. To show this, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle f(q) - q, x_n - q\rangle = \lim_{k\to\infty} \langle f(q) - q, x_{n_k} - q\rangle .$$

Without loss of generality, we can assume that $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$. From (3.13), we obtain $u_{n_k} \rightharpoonup \omega$ as $k \to \infty$.
Since $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in (I \times I \times I)$, where $I = [0,1]$, for $j = 1, 2, \ldots, N$, without loss of generality, we may assume that

$$\alpha_j^{(n_k)} = (\alpha_1^{n_k,j}, \alpha_2^{n_k,j}, \alpha_3^{n_k,j}) \to \alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in (I \times I \times I) \text{ as } k \to \infty,$$

for every $j = 1, 2, \ldots, N$. Let S be the S-mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1^j, \alpha_2^j, \ldots, \alpha_N^j$. By Lemma 2.4, we have S is quasi-nonexpansive and $Fix(S) = \bigcap_{i=1}^N Fix(T_i)$.
From Lemma 2.5(i), we obtain

$$\lim_{k\to\infty} \|S_{n_k} x_{n_k} - S x_{n_k}\| = 0. \tag{3.18}$$

Since

$$\|x_{n_k} - S x_{n_k}\| \leq \|x_{n_k} - S_{n_k} x_{n_k}\| + \|S_{n_k} x_{n_k} - S x_{n_k}\| ,$$

by (3.14) and (3.18), we have

$$\lim_{k\to\infty} \|x_{n_k} - S x_{n_k}\| = 0. \tag{3.19}$$

Next, we will claim that $\omega \in Fix(S)$. Suppose that $\omega \neq S\omega$. From the Opial's condition (3.17) and (3.19), we derive

$$
\begin{aligned}
&\liminf_{k\to\infty} \|x_{n_k}-\omega\|^2 \\
<&\liminf_{k\to\infty} \|x_{n_k}-S\omega\|^2 \\
=&\liminf_{k\to\infty} \left(\|x_{n_k}-Sx_{n_k}\|^2+\|Sx_{n_k}-S\omega\|^2+2\left\langle x_{n_k}-Sx_{n_k}, Sx_{n_k}-S\omega\right\rangle\right) \\
=&\liminf_{k\to\infty} \|Sx_{n_k}-S\omega\|^2 \\
=&\liminf_{k\to\infty} \left\|\alpha_1^N\left(T_NU_{N-1}x_{n_k}-T_NU_{N-1}\omega\right)+\alpha_2^N\left(U_{N-1}x_{n_k}-U_{N-1}\omega\right)\right. \\
&\left.+\alpha_3^N\left(x_{n_k}-\omega\right)\right\|^2 \\
\le&\liminf_{k\to\infty} \left(\alpha_1^N\|T_NU_{N-1}x_{n_k}-T_NU_{N-1}\omega\|^2+\alpha_2^N\|U_{N-1}x_{n_k}-U_{N-1}\omega\|^2\right. \\
&\left.+\alpha_3^N\|x_{n_k}-\omega\|^2\right) \\
\le&\liminf_{k\to\infty} \left(\alpha_1^N\|U_{N-1}x_{n_k}-U_{N-1}\omega\|^2\right. \\
&+2\left\langle U_{N-1}x_{n_k}-T_NU_{N-1}x_{n_k}, U_{N-1}\omega-T_NU_{N-1}\omega\right\rangle \\
&\left.+\alpha_2^N\|U_{N-1}x_{n_k}-U_{N-1}\omega\|^2+\alpha_3^N\|x_{n_k}-\omega\|^2\right) \\
=&\liminf_{k\to\infty} \left(\left(1-\alpha_3^N\right)\|U_{N-1}x_{n_k}-U_{N-1}\omega\|^2+\alpha_3^N\|x_{n_k}-\omega\|^2\right) \\
=&\liminf_{k\to\infty} \left(\left(1-\alpha_3^N\right)\left(\alpha_1^{N-1}\|T_{N-1}U_{N-2}x_{n_k}-T_{N-1}U_{N-2}\omega\|^2\right.\right. \\
&\left.\left.+\alpha_2^{N-1}\|U_{N-2}x_{n_k}-U_{N-2}\omega\|^2+\alpha_3^{N-1}\|x_{n_k}-\omega\|^2\right)+\alpha_3^N\|x_{n_k}-\omega\|^2\right) \\
\le&\liminf_{k\to\infty} \left(\left(1-\alpha_3^N\right)\left(\alpha_1^{N-1}\|U_{N-2}x_{n_k}-U_{N-2}\omega\|^2\right.\right. \\
&+2\left\langle U_{N-2}x_{n_k}-T_{N-1}U_{N-2}x_{n_k}, U_{N-2}\omega-T_{N-1}U_{N-2}\omega\right\rangle \\
&\left.\left.+\alpha_2^{N-1}\|U_{N-2}x_{n_k}-U_{N-2}\omega\|^2+\alpha_3^{N-1}\|x_{n_k}-\omega\|^2\right)+\alpha_3^N\|x_{n_k}-\omega\|^2\right) \\
=&\liminf_{k\to\infty} \left(\left(1-\alpha_3^N\right)\left(\left(1-\alpha_3^{N-1}\right)\|U_{N-2}x_{n_k}-U_{N-2}\omega\|^2\right.\right. \\
&\left.\left.+\alpha_3^{N-1}\|x_{n_k}-\omega\|^2\right)+\alpha_3^N\|x_{n_k}-\omega\|^2\right) \\
=&\liminf_{k\to\infty} \left(\prod_{j=N-1}^{N}\left(1-\alpha_3^j\right)\|U_{N-2}x_{n_k}-U_{N-2}\omega\|^2\right. \\
&\left.+\left(1-\prod_{j=N-1}^{N}\left(1-\alpha_3^j\right)\right)\|x_{n_k}-\omega\|^2\right) \\
&\vdots \\
\le&\liminf_{k\to\infty} \left(\prod_{j=1}^{N}\left(1-\alpha_3^j\right)\|U_0x_{n_k}-U_0\omega\|^2\right. \\
&\left.+\left(1-\prod_{j=1}^{N}\left(1-\alpha_3^j\right)\right)\|x_{n_k}-\omega\|^2\right)
\end{aligned}
$$

$$\leq \liminf_{k\to\infty} \left(\prod_{j=1}^{N} \left(1-\alpha_3^j\right) \|x_{n_k}-\omega\|^2 + \left(1-\prod_{j=1}^{N}\left(1-\alpha_3^j\right)\right) \|x_{n_k}-\omega\|^2 \right)$$
$$= \liminf_{k\to\infty} \|x_{n_k}-\omega\|^2 .$$

This is a contradiction. Thus, we obtain $q \in Fix(S)$. From Lemma 2.4, hence we get

$$\omega \in \bigcap_{i=1}^{N} Fix(T_i). \tag{3.20}$$

Next, we will show that $\omega \in \bigcap_{i=1}^{N} EP\left(F_i\right)$.
Since

$$\sum_{i=1}^{N} a_i F_i\left(u_n, y\right) + \frac{1}{r_n}\left\langle y-u_n, u_n-x_n\right\rangle \geq 0, \forall y \in C,$$

and $\sum_{i=1}^{N} a_i F_i$ satisfies the conditions (A1)-(A4), we obtain

$$\frac{1}{r_n}\left\langle y-u_n, u_n-x_n\right\rangle \geq \sum_{i=1}^{N} a_i F_i\left(y, u_n\right), \forall y \in C.$$

In particular, it follows that

$$\left\langle y-u_{n_k}, \frac{u_{n_k}-x_{n_k}}{r_{n_k}}\right\rangle \geq \sum_{i=1}^{N} a_i F_i\left(y, u_{n_k}\right), \forall y \in C. \tag{3.21}$$

From (3.13), (3.21) and (A4), we have

$$\sum_{i=1}^{N} a_i F_i\left(y, \omega\right) \leq 0,\ \forall y \in C. \tag{3.22}$$

Put $y_t := ty + (1-t)\omega$, $t \in (0, 1]$, we have $y_t \in C$. By using (A1), (A4) and (3.22), we have

$$\begin{aligned}0 &= \sum_{i=1}^{N} a_i F_i \left(y_t, y_t\right) \\ &= \sum_{i=1}^{N} a_i F_i \left(y_t, ty + (1-t)\omega\right) \\ &\leq t \sum_{i=1}^{N} a_i F_i \left(y_t, y\right) + (1-t) \sum_{i=1}^{N} a_i F_i \left(y_t, \omega\right) \\ &\leq t \sum_{i=1}^{N} a_i F_i \left(y_t, y\right).\end{aligned}$$

It implies that

$$\sum_{i=1}^{N} a_i F_i \left(ty + (1-t)\omega, y\right) \geq 0, \forall t \in (0,1] \text{ and } \forall y \in C. \tag{3.23}$$

From (3.23), taking $t \to 0^+$ and using (A3), we can conclude that

$$0 \leq \sum_{i=1}^{N} a_i F_i \left(\omega, y\right), \ \forall y \in C.$$

Therefore, $\omega \in EP\left(\sum_{i=1}^{N} a_i F_i\right)$. By Lemma 2.6, we obtain $EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP\left(F_i\right)$. It follows that

$$\omega \in \bigcap_{i=1}^{N} EP\left(F_i\right). \tag{3.24}$$

By (3.20) and (3.24), we obtain

$$\omega \in \Omega. \tag{3.25}$$

Since $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$, (3.25) and Lemma 2.1, it yields that

$$\limsup_{n\to\infty} \langle f(q) - q, x_n - q\rangle = \lim_{k\to\infty} \langle f(q) - q, x_{n_k} - q\rangle = \langle f(q) - q, \omega - q\rangle \leq 0. \tag{3.26}$$

Step 7. Finally, we will prove that $\{x_n\}$ and $\{u_n\}$ converges strongly to $q =$

$P_{\Omega f(q)}$, for every $i = 1, 2, \ldots, N$. From (3.1), we have

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
=& \|\alpha_n (f(x_n) - q) + \beta_n (u_n - q) + \delta_n (S_n x_n - q)\|^2 \\
\le& \|\beta_n (u_n - q) + \delta_n (S_n x_n - q)\|^2 + 2\alpha_n \langle f(x_n) - q, x_{n+1} - q\rangle \\
\le& (\beta_n \|u_n - q)\| + \delta_n \|S_n x_n - q\|)^2 \\
&+ 2\alpha_n \langle f(x_n) - f(q), x_{n+1} - q\rangle + 2\alpha_n \langle f(q) - q, x_{n+1} - q\rangle \\
\le& ((1 - \alpha_n) \|x_n - q\|)^2 + 2\alpha_n \|f(x_n) - f(q)\| \|x_{n+1} - q\| \\
&+ 2\alpha_n \langle f(q) - q, x_{n+1} - q\rangle \\
\le& (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \xi \|x_n - q\| \|x_{n+1} - q\| \\
&+ 2\alpha_n \langle f(q) - q, x_{n+1} - q\rangle \\
\le& (1 - \alpha_n)^2 \|x_n - q\|^2 + \alpha_n \xi \left(\|x_n - q\|^2 + \|x_{n+1} - q\|^2\right) \\
&+ 2\alpha_n \langle f(q) - q, x_{n+1} - q\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
\le& \frac{(1 - \alpha_n)^2 + \alpha_n \xi}{1 - \alpha_n \xi} \|x_n - q\|^2 + \frac{2\alpha_n}{1 - \alpha_n \xi} \langle f(q) - q, x_{n+1} - q\rangle \\
=& \frac{1 - \alpha_n \xi - 2\alpha_n(1 - \xi)}{1 - \alpha_n \xi} \|x_n - q\|^2 + \frac{\alpha_n^2}{1 - \alpha_n \xi} \|x_n - q\|^2 \\
&+ \frac{2\alpha_n}{1 - \alpha_n \xi} \langle f(q) - q, x_{n+1} - q\rangle \\
=& \left(1 - \frac{2\alpha_n(1 - \xi)}{1 - \alpha_n \xi}\right) \|x_n - q\|^2 + \frac{\alpha_n^2}{1 - \alpha_n \xi} \|x_n - q\|^2 \\
&+ \frac{2\alpha_n}{1 - \alpha_n \xi} \langle f(q) - q, x_{n+1} - q\rangle \\
=& \left(1 - \frac{2\alpha_n(1 - \xi)}{1 - \alpha_n \xi}\right) \|x_n - q\|^2 + \frac{2\alpha_n(1 - \xi)}{1 - \alpha_n \xi} \left(\frac{\alpha_n}{2(1 - \xi)} \|x_n - q\|^2 \right. \\
&\left. + \frac{1}{1 - \xi} \langle f(q) - q, x_{n+1} - q\rangle \right).
\end{aligned}$$

Applying the condition (i), (3.26) and Lemma 2.2, we have the sequence $\{x_n\}$ converges strongly to $q = P_\Omega f(q)$. From (3.13), we also obtain $\{u_n\}$ converges strongly to $q = P_{\mathcal{F}} f(q)$. This completes the proof. □

The following results are direct consequences of Theorem 3.1.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, \ldots, N$, let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$.*

For every $j = 1, 2, \ldots, N$, let $\{T_j\}_{j=1}^{N}$ be a finite family of nonspreading mappings and L_i-Lipschitzian mappings of C into itself with $\bigcap_{j=1}^{N} Fix(T_j) \neq \emptyset$. Suppose that $\Omega := \bigcap_{j=1}^{N} Fix(T_j) \cap EP(F) \neq \emptyset$. For each $j = 1, 2, \ldots, N$ and $k = 1, 3$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I$, where $I = [0, 1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$, $\alpha_1^{n,p}, \alpha_3^{n,p} \in (0, 1)$ for all $p = 1, 2, \ldots, N-1$ and $\alpha_1^{n,N} \in (0, 1]$, $\alpha_3^{n,N} \in [0, 1)$, $\alpha_2^{n,j} \in [0, 1)$. For each $n \in \mathbb{N}$, let S_n be the S-mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$. Let $\{x_n\}$ be the sequence generated by $x_1 \in C$ and

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n u_n + \delta_n S_n x_n, \forall n \geq 1, \end{cases} \tag{3.27}$$

where $f : C \to C$ be a contraction mapping with a constant ξ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\} \subseteq (0, 1)$ with $\alpha_n + \beta_n + \delta_n = 1$, $\forall n \geq 1$. Suppose the following statement are true:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \tau \leq \beta_n, \delta_n \leq \upsilon < 1$ *for some* $\tau, \upsilon > 0$;

(iii) $0 < \epsilon \leq r_n \leq \rho < 1$, *for some* $\epsilon, \rho > 0$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,
$\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$,
$\sum_{n=1}^{\infty} \left|\alpha_k^{n+1} - \alpha_k^{n}\right| < \infty$, *for all* $k = 1, 3$;

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $q = P_\Omega f(q)$.

Proof. Put $F_i = F, \forall i = 1, 2, \ldots, N$, we obtain the desired result. □

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. For every $j = 1, 2, \ldots, N$, let $\{T_j\}_{j=1}^{N}$ be a finite family of nonspreading mappings and L_i-Lipschitzian mappings of C into itself with $\bigcap_{j=1}^{N} Fix(T_j) \neq \emptyset$. For each $j = 1, 2, \ldots, N$ and $k = 1, 3$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I$, where $I = [0, 1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$, $\alpha_1^{n,p}, \alpha_3^{n,p} \in (0, 1)$ for all $p = 1, 2, \ldots, N-1$ and $\alpha_1^{n,N} \in (0, 1]$, $\alpha_3^{n,N} \in [0, 1)$, $\alpha_2^{n,j} \in [0, 1)$. For each $n \in \mathbb{N}$, let S_n be the S-mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$. Let $\{x_n\}$ be the sequence generated by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \delta_n S_n x_n, \forall n \geq 1, \tag{3.28}$$

where $f : C \to C$ *be a contraction mapping with a constant* ξ *and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\} \subseteq (0,1)$ *with* $\alpha_n + \beta_n + \delta_n = 1$, $\forall n \geq 1$. *Suppose the following statement are true:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \tau \leq \beta_n, \delta_n \leq \upsilon < 1$ *for some* $\tau, \upsilon > 0$;

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,
$\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} \left|\alpha_k^{n+1} - \alpha_k^n\right| < \infty$, *for all* $k = 1, 3$;

Then $\{x_n\}$ *converges strongly to* $q = P_{\bigcap_{j=1}^{N} Fix(T_j)} f(q)$.

Proof. Put $F_i = 0, \forall i = 1, 2, \ldots, N$. Thus, we have $u_n = P_C x_n = x_n$. Hence, using Theorem 3.1, the desired result can be proved. □

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