Continuity of the Solution Mappings to Primal and Dual Vector Equilibrium Problems

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Abstract: In this paper, we consider strong forms of the primal and dual vector equilibrium problems in Hausdorff topological vector spaces. Under suitable assumptions, some continuity properties of solution mappings to such problems are established. The mains results improve recent existing ones in the literature. Some examples are provided to illustrate our results. Applications to vector optimization problems and vector variational inequalities are also discussed.

Keywords : vector equilibrium problem; dual vector equilibrium problem; vector optimization problem; vector variational inequality; cone-continuity; solution mapping
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1   Introduction

Equilibrium problem plays an important role in nonlinear analysis because it provides a unified model of several important problems in optimization as well as in

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mathematical economics and physics, such as variational inequality, fixed points, saddle points, Nash equilibrium, and complementarity problems. Motivated by the pioneering work of Giannessi [8] which extended classical variational inequality to the case of vector-valued mapping, many researchers have extended the scalar equilibrium problem to the vector case in several difference ways. For further details, we refer the reader to [4, 9, 5, 18, 1, 25] and the references therein.

The theory of existence of solutions to vector equilibrium problems and more general settings has been extensively studied by many researchers; see, e.g., [14, 6, 7, 15, 16, 18, 4]. A relatively new but rapidly developing topic is the stability and sensitivity analysis concerning (semi)continuity properties in the sense of Hausdorff and Berge of solution mappings to such problems (see [2, 3, 11, 10, 26, 19, 28] and the references therein). It is worth noting that to obtain the upper semicontinuity of solution mapping, hypotheses regarding the same property may be imposed on the data of the problem. Unfortunately, this is not the case of the lower semicontinuity, some relatively strict and/or unnatural assumptions need to be attached. Gong [10] established the continuity of the solution mapping to the mixed parametric monotone weak vector equilibrium problems in topological vector spaces. To improve the results in [10], Li and Fang [19] proposed a relaxed assumption which related to monotonicity properties, and employed it to study the lower semicontinuity of the weak vector solutions and global vector solutions to parametric generalized Ky Fan inequalities. However, the monotonicity assumption on the objective mapping may cause the fact that solution set is a singleton. By this observation, Zhang et al. [28] introduced Hölder-related assumptions to establish the lower semicontinuity of the efficient and weak solution mappings for parametric vector equilibrium problems without using the assumptions related to monotonicity. However, this assumption is unnatural and hard to apply in practical situations because it require to know the information concerning solution set of the reference problem. In order to overcome these drawbacks, the linear/nonlinear scalarization method has been employed by many authors; see, for instance, [12, 19, 21, 27, 23, 22] and the references therein. Nevertheless, this approach is effective only for handling weak vector equilibrium problems, not for strong ones.

Motivated and inspired by above observations, we investigate the stability properties, such as upper semicontinuity, lower semicontinuity and continuity of the solution mappings to parametric strong vector equilibrium problems under some new conditions with neither monotonicity properties nor any information of the solution mappings. The linear scalarization method is not, of course, employed in our study. The obtained results in this paper are new and improve the existing ones in the literature, for instance, [5, 11]. Furthermore, we also discuss dual strong vector equilibrium problems. As far as we know, there is no work with contribution to the stability of solution mappings to dual strong vector equilibrium problems.

The rest of this paper is organized as follows. Section 2 is devoted to some preliminary facts, and statements of parametric strong vector equilibrium problem and its dual problem. The main results on upper semicontinuity, lower semicontinuity and continuity of the solution mappings of these problems are respectively

discussed in Section 3. Some examples to illustrate that the obtained results improve the existing ones in the recent literature are also given. The last section, Section 4, contains some particular cases as illustrative examples, where we derive consequences of the main results.

2 Preliminaries

Throughout this paper, unless otherwise specified, let X, Y and Z be Hausdorff topological vector spaces, $C \subset Y$ be a pointed closed convex cone with nonempty topological interior and Λ be a nonempty subset of Z. We consider the parametric strong vector equilibrium problem under perturbations in terms of perturbing the constraint set K and the objective mapping f by a parameter λ varying on Λ.

(VEP_λ) finding $\bar{x} \in K(\lambda)$ such that $f(\lambda, \bar{x}, y) \in C, \forall y \in K(\lambda)$,

where $f : \Lambda \times X \times X \to Y$ is a vector-valued trifunction and $K : \Lambda \rightrightarrows X$ is a constraint mapping. The dual problem of the strong vector equilibrium problem is under question of

(DVEP_λ) finding $\bar{x} \in K(\lambda)$ such that $f(\lambda, y, \bar{x}) \in -C, \forall y \in K(\lambda)$.

In what follows, instead of writing $\{(\text{VEP}_\lambda) \mid \lambda \in \Lambda\}$ and $\{(\text{DVEP}_\lambda) \mid \lambda \in \Lambda\}$ for the families of such problems, one simply writes (VEP) and (DVEP), respectively. For each $\lambda \in \Lambda$, the solution sets of (VEP) and (DVEP) corresponding to λ are respectively denoted by $S(\lambda)$ and $T(\lambda)$. In this paper, we focus on the continuity of solution mappings S and T, hence we always assume that $S(\lambda)$ and $T(\lambda)$ are nonempty for each λ in a neighborhood of the reference point. The existence results for such problems can be found in [5, 4, 18].

We first recall some notions and well-known results needed in the sequel.

Definition 2.1. Let X, Y be two topological vector spaces and F be a set-valued mapping from X to Y.

(a) F is said to be *lower semicontinuous* (l.s.c.) at x_0 if for any open subset U of Y with $F(x_0) \cap U \neq \emptyset$, there exits a neighborhood N of x_0 such that for all $x \in N, F(x) \cap U \neq \emptyset$.

(b) F is said to be *upper semicontinuous* (u.s.c.) at x_0 if for any open superset U of $F(x_0)$, there exists a neighborhood N of x_0 such that $F(x) \subset U$ for all $x \in N$.

(c) F is said to be *continuous* at x_0 if it is both u.s.c. and l.s.c. at x_0.

In the sequel, we say that a mapping satisfies a certain property on a subset $A \subset X$ if so does it at every point of A. When $A = X$, we omit "on X" in the statement.

Lemma 2.2. (See e.g. [13, Proposition 2.5.6 and Proposition 2.5.9]) *Let* $F : X \rightrightarrows Y$ *be a set-valued mapping. The following statements hold true.*

(a) *F is l.s.c. at $\bar{x}$ if and only if, for every sequence $\{x_n\}$ in X with $x_n \to \bar{x}$ and $\bar{y} \in F(\bar{x})$, there exists a sequence $\{y_n\}$ of $F(x_n)$ such that $y_n \to \bar{y}$.*

(b) *If $F(\bar{x})$ is compact, then F is u.s.c. at $\bar{x}$ if and only if for any sequence $\{x_n\}$ in X with $x_n \to \bar{x}$ and $y_n \in F(x_n)$, there is a subsequence $\{y_{n_k}\}$ that converges to some $\bar{y} \in F(\bar{x})$.*

We next recall the concepts and properties of cone upper and lower semicontinuity for vector-valued mappings, which are generalizations of ordinary upper and lower semicontinuity on real-valued functions.

Definition 2.3. (See [24]) Let X, Y be topological vector spaces and C be a pointed solid convex cone in Y. A mapping $g : X \to Y$ is said to be:

(a) *C-lower semicontinuous* (C-lsc) at $\bar{x}$ if, for any neighborhood V of the origin in Y, there exists a neighborhood U of $\bar{x}$ such that

$$g(x) \in g(\bar{x}) + V + C, \quad \forall x \in U;$$

(b) *C-upper semicontinuous* (C-usc) at $\bar{x}$ if, $-g$ is C-lower semicontinuous at $\bar{x}$;

(c) *C-continuous* at $\bar{x}$ if it is both C-usc and C-lsc at $\bar{x}$.

The abbreviation lsc (usc) for cone lower semicontinuity (upper semicontinuity) of vector-valued mappings is utilized to stress the difference between lower semicontinuity (upper semicontinuity) of a vector-valued mapping and lower semicontinuity (upper semicontinuity) of a set-valued mapping.

Proposition 2.4. (See [24]) *For X, Y, C and g defined as in Definition 2.3, the following assertions are equivalent:*

(i) *g is C-lsc;*

(ii) *For each $\bar{x} \in X$ and $c \in \text{int}C$, there is an neighborhood U of $\bar{x}$ such that $g(x) \in g(\bar{x}) - c + \text{int}C$, for all $x \in U$;*

(iii) *For each $\bar{x} \in X$ and $a \in Y$, $g^{-1}(a + \text{int}C)$ is open.*

Similarly, we have an analogous result for the C-upper semicontinuity property of a vector-valued mapping.

Proposition 2.5. *For X, Y, C and g defined as in Definition 2.3, the following assertions are equivalent:*

(i) *g is C-usc;*

(ii) *For each $\bar{x} \in X$ and $c \in \text{int}C$, there is an neighborhood U of $\bar{x}$ such that $g(x) \in g(\bar{x}) + c - \text{int}C$, for all $x \in U$;*

(iii) *For each $\bar{x} \in X$ and $a \in Y$, $g^{-1}(a - \text{int}C)$ is open.*

Proposition 2.6. (See [24]) *Let X, Y, C be as in Definition 2.3, f and g be two mapping from X into Y. The following assertions hold true.*

(i) *αf is C-usc (resp., C-lsc) for each $\alpha > 0$, if so is f.*

(ii) *$f + g$ is C-usc (resp., C-lsc), if so are f and g.*

Definition 2.7. (See [20, Definition 6.1]) Let X and Y be vector spaces, C be a solid convex pointed cone in Y, and K be a nonempty convex subset of X. A vector-valued mapping $g : K \to Y$ is said to be:

(a) *C-convex* on K if for any $x_1, x_2 \in K$ and $t \in [0, 1]$,

$$g(tx_1 + (1 - t)x_2) \in tg(x_1) + (1 - t)g(x_2) - C;$$

(b) *C-strictly convex* on K if for any $x_1, x_2 \in K$, $x_1 \neq x_2$ and $t \in (0, 1)$,

$$g(tx_1 + (1 - t)x_2) \in tg(x_1) + (1 - t)g(x_2) - \text{int } C;$$

(c) *C-quasiconvex* on K if for $y \in Y, x_1, x_2 \in K$, $t \in [0, 1]$,

$$g(x_1) \in y - C, g(x_2) \in y - C \text{ imply } g(tx_1 + (1 - t)x_2) \in y - C;$$

(d) *strictly C-quasiconvex* on K if for $y \in Y, x_1, x_2 \in K, x_1 \neq x_2, t \in (0, 1)$,

$$g(x_1) \in y - C, g(x_2) \in y - C \text{ imply } g(tx_1 + (1 - t)x_2) \in y - \text{int } C.$$

The mapping g is said to be *C-concave* (respectively, *strictly C-concave, C-quasiconcave, strictly C-quasiconcave*) if $-g$ is C-convex (respectively, strictly C-convex, C-quasiconvex, strictly C-quasiconvex).

In a particular case where $Y = \mathbb{R}, C = \mathbb{R}^+$, we obtain the ordinary definition of (quasi)convex and strictly (quasi) convex functions.

3   The main results

In this section, we discuss the upper semicontinuity and lower semicontinuity of the solution mappings for (VEP) and (DVEP) under relaxed conditions.

Theorem 3.1. *Assume that K is continuous and compact-valued at $\bar{\lambda}$. Then, the following assertions hold:*

(a) *S is u.s.c. and compact-valued at $\bar{\lambda}$, if f is C-usc.*

(b) *T is u.s.c. and compact-valued at $\bar{\lambda}$, if f is C-lsc.*

Proof. We present only the proof for (b); the other one is treated similarly. We first show that T is u.s.c. at $\bar{\lambda}$. Suppose to the contrary that there exist an open set U containing $T(\bar{\lambda})$ along with a sequence $\{\lambda_n\}$ tending to $\bar{\lambda}$ such that there are $x_n \in T(\lambda_n) \setminus U$ for all n. Because K is u.s.c and compact-valued at $\bar{\lambda}$, one can assume that $\{x_n\}$ converges to some point $\bar{x}$ in $K(\bar{\lambda})$. If $\bar{x} \notin T(\bar{\lambda})$, there is $\bar{y} \in K(\bar{\lambda})$ such that $f(\bar{\lambda}, \bar{y}, \bar{x}) \notin -C$, and hence, there exists a neighborhood B of the origin in Y such that

$$f(\bar{\lambda}, \bar{y}, \bar{x}) + B \subset Y \setminus (-C). \tag{3.1}$$

The lower semicontinuity of K at $\bar{\lambda}$ in turn shows the existence of a sequence of points $y_n \in K(\lambda_n)$ converging to $\bar{y}$. Since $x_n \in T(\lambda_n)$ for each n,

$$f(\lambda_n, y_n, x_n) \in -C. \tag{3.2}$$

Taking into account the C-lower semicontinuity of f at $(\bar{\lambda}, \bar{y}, \bar{x})$, we have

$$f(\lambda_n, y_n, x_n) \in f(\bar{\lambda}, \bar{y}, \bar{x}) + B + C, \text{ for } n \text{ sufficiently large.}$$

Combining this with inclusion (3.1), we obtain

$$f(\lambda_n, y_n, x_n) \subset Y \setminus (-C) + C \subset Y \setminus (-C),$$

which contradicts (3.2). Thus, $\bar{x} \in T(\bar{\lambda})$, which is another contradiction as $x_n \notin U$, for all n. Therefore, T is u.s.c. at $\bar{\lambda}$. The proof of the compact-valuedness of T at $\bar{\lambda}$ is similar. $\square$

We next establish sufficient conditions for the lower semicontinuity of solution mapping S to (VEP).

Theorem 3.2. *Suppose that the following conditions hold:*

(i) *K is continuous and compact-convex-valued at $\bar{\lambda}$;*

(ii) *f is C-lsc on $\{\bar{\lambda}\} \times K(\bar{\lambda}) \times K(\bar{\lambda})$;*

(iii) *$f(\bar{\lambda}, \cdot, y)$ is strictly C-quasiconcave on $K(\bar{\lambda})$ for all $y \in K(\lambda)$;*

(iv) *$(\text{VEP}_{\bar{\lambda}})$ has at least two solutions.*

Then, S is l.s.c. at $\bar{\lambda}$.

Proof. Suppose to the contrary that there exist $x_0 \in S(\bar{\lambda})$ and a neighborhood W_0 of the origin in X such that for any neighborhood U of $\bar{\lambda}$, there is $\lambda \in U$ satisfying

$$S(\lambda) \cap (x_0 + W_0) = \emptyset.$$

Thus, there exists a sequence $\{\lambda_n\}$ converging to $\bar{\lambda}$ such that

$$S(\lambda_n) \cap (x_0 + W_0) = \emptyset, \forall n \in \mathbb{N}. \tag{3.3}$$

Since $S(\bar{\lambda})$ is not a singleton, we can choose $\bar{x} \in S(\bar{\lambda})$ with $\bar{x} \neq x_0$. Thus, for any $y \in K(\bar{\lambda})$, we have

$$f(\bar{\lambda}, x_0, y) \in C \text{ and } f(\bar{\lambda}, \bar{x}, y) \in C. \tag{3.4}$$

By the strict C-quasiconcavity of $f(\bar{\lambda}, \cdot, y)$ on $K(\bar{\lambda})$, for all $t \in (0, 1)$,

$$f(\bar{\lambda}, t\bar{x} + (1 - t)x_0, y) \in \text{int } C. \tag{3.5}$$

Because of the convexity of $K(\bar{\lambda})$, $x(t) := t\bar{x} + (1 - t)x_0$ belongs to $K(\bar{\lambda})$. It is worth noting that for the chosen W_0, there are a neighborhood W_1 of the origin in X and $t_0 \in (0, 1)$ such that $W_1 + W_1 \subset W_0$ and $x(t_0) \in x_0 + W_1$. Consequently,

$$x(t_0) + W_1 \subset x_0 + W_1 + W_1 \subset x_0 + W_0.$$

Since $x(t_0) \in K(\bar{\lambda})$ and K is l.s.c. at $\bar{\lambda}$, there are $\hat{x}_n(t_0) \in K(\lambda_n)$ such that $\hat{x}_n(t_0)$ tend to $x(t_0)$. Hence, $\hat{x}_n(t_0) \in x(t_0) + W_1 \subset x_0 + W_0$, for n sufficiently large. Combining this with (3.3), one has $\hat{x}_n(t_0) \notin S(\lambda_n)$, for all n. Thus, there exists $\hat{y}_n \in K(\lambda_n)$ such that

$$f(\lambda_n, \hat{x}_n(t_0), \hat{y}_n) \in Y \setminus C. \tag{3.6}$$

Since K is u.s.c. and compact-valued at $\bar{\lambda}$, one can assume that $\{\hat{y}_n\}$ converges to some point $\hat{y} \in K(\bar{\lambda})$ (taking a subsequence if necessary). For each neighborhood B of the origin in Y, there exists a balanced neighborhood B_1 of the origin in Y (i.e., $-B_1 = B_1$) satisfying $B_1 \subset B$. The C-lower semicontinuity of f implies that

$$f(\lambda_n, \hat{x}_n(t_0), \hat{y}_n) \in f(\bar{\lambda}, x(t_0), \hat{y}) + B_1 + C.$$

This together with the balance of B_1 leads to

$$f(\bar{\lambda}, x(t_0), \hat{y}) \in f(\lambda_n, \hat{x}_n(t_0), \hat{y}_n) + B_1 - C,$$

and hence, by (3.6),

$$f(\bar{\lambda}, x(t_0), \hat{y}) \in Y \setminus C + B_1 - C \subset B + Y \setminus C \subset B + \text{cl}(Y \setminus C).$$

Since B is arbitrarily chosen and $\text{cl}(Y \setminus C)$ is closed,

$$f(\bar{\lambda}, x(t_0), \hat{y}) \in \text{cl}(Y \setminus C),$$

which contradicts (3.5). This brings the proof to its end. $\qquad \square$

Remark 3.3. The lower semicontinuity of the solution mapping S cannot be guaranteed by only imposed continuity assumption on f, even in particular case where f is a scalar function and K is a constant mapping. Therefore, one has to impose additionally some stronger conditions, for instance, S is a singleton or conditions (iii) and (iv) is satisfied. We give here an example to illustrate this statement.

Example 3.4. Let $X = \mathbb{R}^2, Y = Z = \mathbb{R}, \Lambda = [0, 1], \bar{\lambda} = 0$ and $C = \mathbb{R}_+$. Let $K : \Lambda \rightrightarrows X$ and $f : \Lambda \times X \times X \to \mathbb{R}$ be defined, respectively, by

$$K(\lambda) = \{(x_1, x_2) \in \mathbb{R}_+^2 \mid x_1 + x_2 = 1\} =: A, \forall \lambda \in \Lambda,$$

and

$$f(\lambda, x, y) = (1 - \lambda)(x_1 - y_1) + x_2 - y_2.$$

It is obvious that f is continuous and K is constant. However, $f(\bar{\lambda}, \cdot, y)$ is not strictly C-quasiconcave on K. Indeed, for any $y \in K(\bar{\lambda})$, let $\hat{x} = (0.25, 0.75)$ and $\bar{x} = (0, 1)$ belong to $K(\bar{\lambda})$ and $t \in (0, 1)$. Then, we have $f(\bar{\lambda}, \hat{x}, y) = f(\bar{\lambda}, \bar{x}, y) = 0$ but $f(\bar{\lambda}, t\hat{x} + (1 - t)\bar{x}, y) = 0 \notin \operatorname{int} \mathbb{R}_+$. By direct computations one has

$$S(\lambda) = \begin{cases} A, & \text{if } \lambda = 0, \\ \{(0, 1)\}, & \text{if } \lambda \in (0, 1]. \end{cases}$$

Therefore, S is not l.s.c. at 0.

Remark 3.5. In [5, Chapter 9], the Authors established the lower semicontinuity of the solution mapping for (VEP) under C-pseudocontinuity assumption imposed on the objective mapping f. We recall that a vector-valued mapping $g : X \to Y$ is said to be *C-pseudocontinuous* at $x \in X$, if for each $k \in C \setminus \{0\}$, there exists a neighborhood $U \subset X$ of x such that $g(u) \in g(x) - k + C$ for all $u \in U$. Theorem 3.2 strictly improves Theorem 9.49 in [5]. Namely, the C-pseudocontinuity assumption on the objective mapping f is now relaxed to the C-lower semicontinuity. The following example shows a simple case where Theorem 3.2 is applicable while Theorem 9.49 in [5] is not.

Example 3.6. Let $X = Z = \mathbb{R}, Y = \mathbb{R}^2, \Lambda = [1, 2]$, and $C = \mathbb{R}_+^2$. Let $K : \Lambda \rightrightarrows X$ and $f : \Lambda \times X \times X \to Y$ be defined, respectively, as $K(\lambda) = [-\lambda, \lambda]$, and $f(\lambda, x, y) = (x + 1, y + 1)$ for any $\lambda \in \Lambda$. This is evident that all assumptions of Theorem 3.2 are satisfied. Direct calculation gives $S(\lambda) = [-1, 1]$, which is l.s.c.. However, Theorem 9.49 in [5] is not applicable in this case because f is not C-pseudocontinuous. Indeed, for each $\lambda \in \Lambda$, we show that $f(\lambda, \cdot, \cdot)$ is not C-pseudocontinuous at $(1, 0)$. Suppose to the contrary that for all $(k, l) \in C \setminus \{0\}$, there exists a neighborhood U of $(1, 0)$ such that for any $(u, v) \in U$ we have

$$(u + 1, v + 1) \in (2, 1) - (k, l) + C. \tag{3.7}$$

Without loss of generality, the neighborhood U can be taken as the form $(1 - \varepsilon, 1 + \varepsilon) \times (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Now let $(k, l) = (1, 0)$. The inclusion in (3.7) yields that $(u, v) \in C$ for all $(u, v) \in U$. This is impossible when we take $v < 0$.

Passing to (DVEP), we state a similar result concerning the lower semicontinuity of solution mapping. The proof is analogous to that for Theorem 3.2, therefore we omit it with some suitable modifications.

Theorem 3.7. *Suppose that the following conditions hold:*

(i) *K is continuous and compact-convex-valued at $\bar{\lambda}$;*

(ii) *f is C-usc on $\{\bar{\lambda}\} \times K(\bar{\lambda}) \times K(\bar{\lambda})$;*

(iii) *$f(\bar{\lambda}, y, \cdot)$ is strictly C-quasiconvex on $K(\bar{\lambda})$ for all $y \in K(\lambda)$;*

(iv) *$(\text{DVEP}_{\bar{\lambda}})$ has at least two solutions.*

Then, T is l.s.c. at $\bar{\lambda}$.

By combining results from the lower semicontinuity and upper semicontinuity cases of solution mappings to (VEP) and (DVEP), we obtain the following results concerning the continuity of the solution mappings to such problems.

Theorem 3.8. *Suppose that the following conditions hold:*

(i) *K is continuous and compact-convex-valued at $\bar{\lambda}$;*

(ii) *f is C-continuous on $\{\bar{\lambda}\} \times K(\bar{\lambda}) \times K(\bar{\lambda})$;*

(iii) *$f(\bar{\lambda}, y, \cdot)$ is strictly C-quasiconvex on $K(\bar{\lambda})$ for each $y \in K(\lambda)$.*

Then, T is continuous at $\bar{\lambda}$.

Proof. According to Theorem 3.1(b), T is u.s.c. at $\bar{\lambda}$. Now two situations must be considered. If $(\text{DVEP}_{\bar{\lambda}})$ has a unique solution, then T is continuous at $\bar{\lambda}$ because of its upper semicontinuity. If $(\text{DVEP}_{\bar{\lambda}})$ has at least two solutions, then by Theorem 3.7, T is l.s.c. at $\bar{\lambda}$. Therefore, T is continuous at $\bar{\lambda}$. The proof is complete. □

By the same argument as that in the proof of Theorem 3.8, we obtain the following result.

Theorem 3.9. *Suppose that the following conditions hold:*

(i) *K is continuous and compact-convex-valued at $\bar{\lambda}$;*

(ii) *f is C-continuous on $\{\bar{\lambda}\} \times K(\bar{\lambda}) \times K(\bar{\lambda})$;*

(iii) *$f(\bar{\lambda}, \cdot, y)$ is strictly C-quasiconcave on $K(\bar{\lambda})$ for each $y \in K(\lambda)$.*

Then, S is continuous at $\bar{\lambda}$.

4   Applications

Because vector equilibrium problem setting encompasses several problems in optimization, such as including vector optimization problem, vector Nash equilibrium problem, vector variational inequality, etc, we can apply the obtained results in Section 3 to such problems. In this section, we only provide some discussions on vector optimization problem and vector variational inequality as illustrative examples.

4.1 Vector optimization

Let X, Y, Z, C, Λ and K be as in Section 2. Let $\varphi : \Lambda \times X \to Y$ be a vector-valued mapping. For each $\lambda \in \Lambda$, we consider the following vector optimization problem:

$$(\text{VOP}_\lambda) \qquad\qquad C\text{-}\min \varphi(x) \text{ such that } x \in K(\lambda).$$

A point $\bar{x}$ is called *absolute solution* of (VOP_λ) if $\varphi(\lambda, y) - \varphi(\lambda, x) \in C$ for all $y \in K(\lambda)$ (see, e.g., [20, 17]). By $\Omega(\lambda)$ we denote the set of absolute solutions to (VOP_λ). Similarly as for (VEP), the family of vector optimization problems is simply written by (VOP). Based on the results obtained in Section 3, we can give sufficient conditions for the continuity of Ω.

Corollary 4.1. *Suppose that the following conditions hold:*

(i) *K is continuous and compact-convex-valued at $\bar{\lambda}$;*

(ii) *φ is C-continuous on $\{\bar{\lambda}\} \times K(\bar{\lambda})$;*

(iii) *$\varphi(\bar{\lambda}, \cdot)$ is strictly C-quasiconvex on $K(\bar{\lambda})$.*

Then, Ω is continuous at $\bar{\lambda}$.

Proof. Consider the mapping $f : \Lambda \times X \times X \to Y$ be defined by $f(\lambda, x, y) = \varphi(\lambda, y) - \varphi(\lambda, x)$. Then, the conclusion immediately follows from Theorem 3.9. $\square$

4.2 Vector variational inequalities

Let X, Y, Z, C, Λ and K be as in Section 2. Let $\mu : \Lambda \times X \to L(X, Y)$ be an operator, where $L(X, y)$ stands for the set of all linear and continuous operators from X into Y. For a given $\lambda \in \Lambda$, the strong vector variational inequality of Stampacchia type consists in

$$(\text{SVVI}_\lambda) \qquad \text{finding } \bar{x} \in K(\lambda) \text{ such that } \langle \mu(\lambda, \bar{x}), y - \bar{x} \rangle \in C, \forall y \in K(\lambda),$$

and Minty type is of

$$(\text{MVVI}_\lambda) \qquad \text{finding } \bar{x} \in K(\lambda) \text{ such that } \langle \mu(\lambda, y), \bar{x} - y \rangle \in -C, \forall y \in K(\lambda).$$

It is worth noting that, if we take $f : \Lambda \times X \times X \to Y$ defined as $f(\lambda, x, y) = \langle \mu(\lambda, x), y - x \rangle$ (respectively, $f(\lambda, x, y) = \langle \mu(\lambda, y), x - y \rangle$), then the strong vector variational inequality of Stampacchia (respectively, Minty) type becomes the vector equilibrium problem. For $\lambda \in \Lambda$, the solution set of (SVVI) and (MVVI) are denoted by $\Psi(\lambda)$ and $\Upsilon(\lambda)$, respectively. In conclusion, the following results hold true.

Corollary 4.2. *Suppose that the following conditions hold:*

(i) *K is continuous and compact-convex-valued at $\bar{\lambda}$;*

(ii) *The mapping $(\lambda, x, y) \mapsto \langle \mu(\lambda, x), y - x \rangle$ is C-continuous on $\{\bar{\lambda}\} \times K(\bar{\lambda}) \times K(\bar{\lambda})$;*

(iii) *For all $\lambda \in \Lambda$ and $y \in K(\bar{\lambda})$, the mapping $x \mapsto \langle \mu(\lambda, x), y - x \rangle$ is strictly C-quasiconcave on $K(\bar{\lambda})$.*

Then, Ψ is continuous at $\bar{\lambda}$.

Proof. Consider the mapping $f : \Lambda \times X \times X \to Y$ defined as

$$f(\lambda, x, y) = \langle \mu(\lambda, x), y - x \rangle.$$

It is obvious that we can apply Theorem 3.9, and hence the conclusion follows. □

Corollary 4.3. *Suppose that the following conditions hold:*

(i) *K is continuous and compact-convex-valued at $\bar{\lambda}$;*

(ii) *The mapping $(\lambda, x, y) \mapsto \langle \mu(\lambda, x), y - x \rangle$ is C-continuous on $\{\bar{\lambda}\} \times K(\bar{\lambda}) \times K(\bar{\lambda})$;*

(iii) *For all $\lambda \in \Lambda$ and $x \in K(\bar{\lambda})$, the mapping $y \mapsto \langle \mu(\lambda, y), x - y \rangle$ is strictly C-quasiconvex on $K(\bar{\lambda})$.*

Then, Υ is continuous at $\bar{\lambda}$.

Proof. Consider the mapping $f : \Lambda \times X \times X \to Y$ defined as

$$f(\lambda, x, y) = \langle \mu(\lambda, y), x - y \rangle.$$

Then, the conclusion follows from Theorem 3.8. □

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