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# Some Generalizations of Weak Cyclic Compatible Contractions

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**Abstract :** In this paper, we establish the existence theorems for weak cyclic compatible contractions and cyclic compatible $M_k$-contractions. Our results extend and improve existing known results in [15] as well as other results in the literature. We provide examples to illustrate and support our main results.

**Keywords :** weak cyclic compatible contraction; cyclic idle contraction; cyclic compatible $M_k$-contraction; cyclic idle $M_k$-contraction
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## 1 Introduction

Let $U : X \to X$ and $V : X \to X$ be any two mappings. $U$ and $V$ are said to have a coincidence point at $x \in X$ if $Ux = Vx$ and then $Vx$ is called a point of coincidence. Further, a point $x \in X$ is called a fixed point of $U$ if $Ux = x$.

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In 1922, S. Banach [3] formulated the concept of contraction mapping theorem known as the Banach contraction principle. This principle is one of the most important results in fixed theory and is the most powerful and useful tools in non-linear analysis in general. Overtime, this principle was extended and improved in many ways and various fixed point theorems were obtained. A number of famous mathematicians have established contractive type mappings which are the standard generalizations of the well-known Banach contraction on a complete metric space $(X, d)$. One of the more notable generalizations of this basic principle was given by Kirk et al.[13] in 2003. Following the work of Kirk et al.[13], several authors stated many fixed point results for cyclic mappings satisfying various contractive conditions. For more details, the readers may refer to [1, 6, 12, 17] and references therein.

We collect the following notion of cyclic representation.

**Definition 1.1** (see [13]). *Let $A$ and $B$ be non-empty subsets of a metric space $(X, d)$ and $U : A \cup B \to A \cup B$ be a mapping. Then $U$ is called a cyclic mapping if $U(A) \subset B$ and $U(B) \subset A$.*

Throughout this paper, we assume that $\mathbb{R}^+ = [0, \infty)$, $\mathbb{N}$= the set of positive intergers.

Let $U$ and $V$ be self mappings on a complete metric space $(X, d)$, then

1. $U, V$ are said to be weakly compatible if and only if

$$Ux = Vx \text{ implies } UVx = VUx, (\text{see [8]}).$$

2. $U$ is said to be a $\phi$-weak contraction if and only if

$$d(Ux, Uy) \le d(x, y) - \phi(d(x, y)),$$

where $\phi :\ [0, +\infty) \to [0, +\infty)$ with $\phi(0) = 0$, (see [19]).

3. $U$ is a Meir-Keeler contraction(abbreviated as $M_k$-contraction) if and only if $\epsilon \le d(x, y) < \epsilon + \delta(\epsilon) \Rightarrow d(Ux, Uy) < \epsilon$, for all $\epsilon > 0$, and $\delta(\epsilon) > 0$, (see [16]).

Let $A$ and $B$ be non-empty closed subsets of a metric space $(X, d)$, and let $U, V : A \cup B \to A \cup B$ be cyclic maps, then

1. $U$ is called a cyclic orbital contraction if and only if

$$d(U^{2n}x, Uy) \le \gamma d(U^{2n-1}x, y),$$

for all $x \in A, y \in B$, $\gamma \in (0, 1)$, (see [9]).

2. $U$ is called a cyclic orbital stronger $M_k$ $\psi$-contraction if and only if

$$d(U^{2n}x, Uy) \le \psi(d(U^{2n-1}x, y))d(U^{2n-1}x, y),$$

where $\psi : \mathbb{R}^+ \to [0, 1)$ is a strong $M_k$ type mapping, $n \in \mathbb{N}$, $y \in A, y \in B$ (see[5]).

3. $U, V$ are called cyclic compatible contractions if and only if

$$d_\alpha(U^{2n}x, Uy) \leq \gamma d_\alpha(U^{2n-1}x, Vy),$$

for all $n \in \mathbb{N}$ and $x \in A, y \in B, \gamma \in (0,1)$, (see [15]).

The background concept of above topics are very deep and strong. Some interesting results in this direction can be found in [8, 9, 14, 15, 16, 19]. A number of authors also obtained more other interesting related results in this area, see [2, 4, 5, 7, 10, 11, 18, 20, 21], for examples. We now present the following essential definition.

**Definition 1.2** (see [14])**.** *Let $X$ be a non-empty set and $d_\alpha : X \times X \to \mathbb{R}^+$ be a family of mappings. Then $(X, d_\alpha)$ is predominantly known as generating space of b-quasi-metric family(abbreviated as $G_{bq}$-family), if it satisfies the following conditions, for any $x, y, z \in X$ and $s \geq 1$ :*

1. $d_\alpha(x, y) = 0$ *if and only if* $x = y$;
2. $d_\alpha(x, y) = d_\alpha(y, x)$;
3. $d_\alpha(x, z) \leq s[d_\beta(x, y) + d_\beta(y, z)]$, *for any* $\alpha \in (0, 1]$ *and some* $\beta \in (0, \alpha]$;
4. $d_\alpha(x, y)$ *is left continuous in* $\alpha$ *and non-increasing.*

The topological concepts of $G_{bq}$-*convergence*, $G_{bq}$-*limit point*, $G_{bq}$-Cauchy sequence and $G_{bq}$-completeness can be found in [14].

In this paper, motivated and inspired by the works of Kumari and Panti in [15], we establish coincidence point and fixed point theorems for generalized weak cyclic compatible contractions via $\phi$-weak contractions and $M_k$-contractions.

## 2 Main Results

We introduce the concept of generalized weak cyclic compatible contractions as follows.

**Definition 2.1.** *Let $A$ and $B$ be non-empty subsets of a $G_{bq}$-family $(X, d_\alpha)$. Suppose $U, V : A \cup B \to A \cup B$ are cyclic maps such that $U(X) \subset V(X)$. We say that $U, V$ are* weak cyclic compatible contractions*, if for some $x \in A$,*

$$d_\alpha(U^{2n}x, Uy) \leq d_\alpha(U^{2n-1}x, Vy) - \phi(d_\alpha(U^{2n-1}x, Vy)); \quad (2.1)$$

where $\phi : [0, +\infty) \to [0, +\infty)$ is an increasing function with $\phi(0) = 0, \phi(t) > 0$, for all $t \in (0, \infty)$ and $x \in A,\ y \in B$.

**Theorem 2.2.** *Let $A$ and $B$ be non-empty subsets of a complete $G_{bq}$-family $(X, d_\alpha)$. Suppose $U, V : A \cup B \to A \cup B$ are cyclic maps. If $U, V$ are weak cyclic compatible contractions, then $U$ and $V$ have a point of coincidence and a unique common fixed point in $A \cap B$.*

*Proof.* Let $x_0 \in A$ be fixed. Since $U(X) \subset V(X)$, we may choose $x_1 \in X$ such that

$$Ux_0 = Vx_1.$$

So we can construct a sequence $\{x_n\}$ in $X$ by $Ux_n = Vx_{n+1}$ for $n \in \mathbb{N} \cup \{0\}$. Now consider,

$$\begin{aligned} d_\alpha(U^{2n}x, U^{2n+1}x) &\leq d_\alpha(U^{2n-1}x, V^{2n+1}x) - \phi(d_\alpha(U^{2n-1}x, V^{2n+1}x)) \\ &\leq d_\alpha(U^{2n-1}x, V^{2n+1}x) \\ &= d_\alpha(U^{2n-1}x, U^{2n}x). \end{aligned} \tag{2.2}$$

Similarly,

$$d_\alpha(U^{2n+1}x, U^{2n+2}x) \leq d_\alpha(U^{2n}x, U^{2n+1}x).$$

In general, we have

$$d_\alpha(U^n x, U^{n+1}x) \leq d_\alpha(U^{n-1}x, U^n x), \ \forall n \in \mathbb{N}.$$

Thus the sequence $\{d_\alpha(U^n x, U^{n+1}x)\}$ is decreasing, so it is convergent. This implies that there exists $\kappa \geq 0$ such that

$$\lim_{n\to\infty} d_\alpha(U^n x, U^{n+1}x) = \kappa.$$

From (2.2), we have,

$$d_\alpha(U^{2n}x, U^{2n+1}x) \leq d_\alpha(U^{2n-1}x, U^{2n}x) - \phi(d_\alpha(U^{2n-1}x, U^{2n}x)). \tag{2.3}$$

Taking the limit as $n \to \infty$, we get,

$$\kappa \leq \kappa - \lim_{n\to\infty} \phi(d_\alpha(U^{2n-1}x, U^{2n}x)) \leq \kappa.$$

Therefore

$$\lim_{n\to\infty} \phi(d_\alpha(U^{2n-1}x, U^{2n}x)) = 0. \tag{2.4}$$

Suppose $\kappa > 0$, since $\kappa = inf\{d_\alpha(U^n x, U^{n+1}x) : n \in \mathbb{N}\}$, thus $0 < \kappa \leq d_\alpha(U^n x, U^{n+1}x)$ for $n \in \mathbb{N}$. Since $\phi$ is increasing, we have

$$0 < \phi(\kappa) \leq \phi(d_\alpha(U^n x, U^{n+1}x)),$$

which is a contradiction to (2.4). Hence $\kappa = 0$. This implies

$$\lim_{n\to\infty} d_\alpha(U^n x, U^{n+1}x) = 0.$$

Now, for $n, m \in \mathbb{N},\ m > n$, and by Definition 1.2(3), we have

$$\begin{aligned} d_\alpha(U^n x, U^m x) &\leq s[d_\beta(U^n x, U^{n+1}x) + d_\beta(U^{n+1}x, U^m x)] \\ &\leq s d_\beta(U^n x, U^{n+1}x) + s^2 d_\beta(U^{n+1}x, U^{n+2}x) \\ &\quad + s^3 d_\beta(U^{n+2}x, U^{n+3}x) + ... \end{aligned} \tag{2.5}$$

Letting $n, m \to \infty$, we get

$$\lim_{n,m\to\infty} d_\alpha(U^n x, U^m x) = 0.$$

Thus $\{U^n x\}$ is a cauchy sequence. Since $(X, d_\alpha)$ is a complete $G_{bq}$-family, there exists two sequences $\{U^{2n}x\}$ in $A$ and $\{U^{2n-1}x\}$ in $B$ such that $\lim\limits_{n\to\infty} U^{2n}x \to u$ and $\lim\limits_{n\to\infty} U^{2n-1}x \to u$, which yields $\lim\limits_{n\to\infty} V^{2n+1}x \to u$ and $\lim\limits_{n\to\infty} V^{2n}x \to u$. Since $A$ and $B$ are closed in $X$, we have $u \in A \cap B$. Now we shall prove that $Uz = u$. Since $V(X)$ is closed in $X$, there exists $z$ in $X$ such that $Vz = u$. From

$$\begin{aligned} d_\alpha(U^{2n}x, Uz) &\le d_\alpha(U^{2n-1}x, Vz) - \phi(d_\alpha(U^{2n-1}x, Vz)) \\ &\le d_\alpha(U^{2n-1}x, Vz) \end{aligned} \tag{2.6}$$

By taking the limit as $n \to \infty$, $d_\alpha(u, Uz) = 0 \Rightarrow Uz = u$. Therefore $Vz = Uz = u$. Hence $u$ is a coincidence point of $U$ and $V$. From weak compatibility, we get

$$Uu = Vu. \tag{2.7}$$

Now we prove $Vu = u = Uu$. Let us assume $u \neq Vu$, then

$$\begin{aligned} d_\alpha(u, Vu) &\le \lim_{n\to\infty} d_\alpha(U^{2n-1}x, Vu) - \phi(d_\alpha(U^{2n-1}x, Vu)) \\ &< \lim_{n\to\infty} d_\alpha(U^{2n-1}x, Vu) \\ &= d_\alpha(u, Vu), \end{aligned} \tag{2.8}$$

which is a contradiction. Therefore

$$u = Vu \tag{2.9}$$

From (2.7) and (2.9), we get $Uu = Vu = u$. Therefore $u$ is a common fixed point of $U$ and $V$. To prove uniqueness, suppose $v$ is another fixed point of $U$ and $V$. Thus

$$\begin{aligned} d_\alpha(u, v) &\le \lim_{n\to\infty} d_\alpha(U^{2n}x, Uv) \\ &\le \lim_{n\to\infty} d_\alpha(U^{2n-1}x, Vv) - \phi(d_\alpha(U^{2n-1}x, Vv)) \\ &< \lim_{n\to\infty} d_\alpha(U^{2n-1}x, Vv) \\ &= d_\alpha(u, v), \end{aligned} \tag{2.10}$$

which is a contradiction. Thus $u = v$. This completes our proof. □

**Remark 2.3.** *We can obtain special cases of Theorem 2.2, if we*

1. *replace a $G_{bq}$-family $(X, d_\alpha)$ by a $G_q$-family, according to Definition 2.1 in [2], by putting $s = 1$;*
2. *replace a $G_{bq}$-family $(X, d_\alpha)$ by a b-metric space, and taking $d$ instead of $d_\alpha$;*

3. *replace a $G_{bq}$-family $(X, d_\alpha)$ by a complete metric space, by taking $d$ instead of $d_\alpha$ and letting $s = 1$.*

**Example 2.4.** *Let $A = B = X = [0,1]$. Let $d : X \times X \to \mathbb{R}^+$ defined by $d(x,y) = (x-y)^2$. This is a b-metric with $s = 2$ ($d$ is not a usual metric). Define*

$$Vx = \begin{cases} 0, & \text{if } 0 \leq x < \frac{1}{2} \\ \frac{1}{5}, & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

*and*

$$Ux = \frac{1}{5}, \text{ if } 0 \leq x \leq 1.$$

*Clearly $U(X) \subset V(X)$. Define $\phi(t) = t - \frac{t}{t+1}$.*

*For any $x \in [0,1]$, we get*

$$Ux = U^2x = ...U^nx = \frac{1}{5}, \ \forall n.$$

*For any $y \in [0,1]$, we get*

$$Vy = \begin{cases} 0, & \text{if } 0 \leq y < \frac{1}{2} \\ \frac{1}{5}, & \text{if } \frac{1}{2} \leq y \leq 1 \end{cases}$$

**Case(i):** $0 \leq y < \frac{1}{2}$, *we have*

$$Vy = 0,$$

$$d(U^{2n}x, Uy) = d(\frac{1}{5}, \frac{1}{5}) = 0,$$

*and*

$$\begin{aligned} d(U^{2n-1}x, Vy) - \phi(d(U^{2n-1}x, Vy)) &= d(\frac{1}{5}, 0) - \phi(d(\frac{1}{5}, 0)) \\ &= \frac{1}{25} - \phi(\frac{1}{25}) \\ &> 0 = d(U^{2n}x, Uy). \end{aligned}$$

**Case(ii):**$\frac{1}{2} \leq y \leq 1$, *we have*

$$Vy = \frac{1}{5},$$

$$d(U^{2n}x, Uy) = d(\frac{1}{5}, \frac{1}{5}) = 0,$$

*and*

$$\begin{aligned} d(U^{2n-1}x, Vy) - \phi(d(U^{2n-1}x, Vy)) &= d(\frac{1}{5}, \frac{1}{5}) - \phi(d(\frac{1}{5}, \frac{1}{5})) \\ &= 0 = d(U^{2n}x, Uy). \end{aligned}$$

*Therefore*

$$d(U^{2n}x, Uy) \leq d(U^{2n-1}x, Vy) - \phi(d(U^{2n-1}x, Vy)).$$

*Thus $U$, $V$ are weak cyclic compatible contractions. All the conditions of Theorem 2.2 hold true and $U$, $V$ have a unique common fixed point $u = \frac{1}{5}$.*

**Example 2.5.** *Let $A = B = X = [0, 1]$. Let $d : X \times X \to \mathbb{R}$ defined by $d(x, y) = |x - y|$. So $(X, d)$ is a complete metric space. Define*

$$Vx = \frac{x}{1+x}, \text{ whenever } x \in [0, 1]$$

*and*

$$Ux = 0, \text{ whenever } x \in [0, 1].$$

*We can see that $U(X) \subset V(X)$. Define $\phi(t) = t - \frac{t}{t+1}$. For any $x \in [0, 1]$, we have*

$$Ux = U^2x = ... = U^nx = 0, \forall n.$$

*For $y \in [0, 1]$, we have*

$$Vy = \frac{y}{1+y}$$

*and*

$$d(U^{2n}x, Uy) = d(0, 0) = 0.$$

*Consider,*

$$\begin{aligned} d(U^{2n-1}x, Vy) - \phi(d(U^{2n-1}x, Vy)) &= d(0, \frac{y}{1+y}) - \phi(d(0, \frac{y}{1+y})) \\ &= \frac{y}{1+y} - \phi(\frac{y}{1+y}) \\ &\geq 0 = d(U^{2n}x, Uy) \end{aligned}$$

*Therefore*

$$d(U^{2n}x, Uy) \leq d(U^{2n-1}x, Vy) - \phi(d(U^{2n-1}x, Vy)).$$

*Thus $U$, $V$ are weak cyclic compatible contractions. All the conditions of Theorem 2.2 hold true and $U$, $V$ has a unique common fixed point. Here $u = 0$ is the unique common fixed point of $U$ and $V$.*

If we take $S = T = U$ in the definition of Kumari and Panthi in [15], and take $U = V$ and $V = I$ in the Definition 2.1 above, we can obtain the pertinent definition.

**Definition 2.6.** *Let $U : A \cup B \to A \cup B$ be a cyclic mapping, then*

1. *$U$ is called a* cyclic idle contraction *if and only if*

$$d_\alpha(U^{2n}x, Uy) \leq \gamma d_\alpha(U^{2n-1}x, Uy), \gamma \in (0, 1).$$

*2. $U$ is called a* weak cyclic idle contraction *if and only if*

$$d_\alpha(U^{2n}x, Uy) \leq d_\alpha(U^{2n-1}x, Uy) - \phi(d_\alpha(U^{2n-1}x, Uy))$$

*3. $U$ is called* weak cyclic orbital contraction *if and only if*

$$d_\alpha(U^{2n}x, Uy) \leq d_\alpha(U^{2n-1}x, y) - \phi(d_\alpha(U^{2n-1}x, y))$$

*where $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$.*

We now obtain the following results.

**Theorem 2.7.** *Let $A$ and $B$ be non-empty subsets of a complete $G_{bq}$-family $(X, d_\alpha)$. Suppose $U : A \cup B \to A \cup B$ is a cyclic map. If $U$ is a weak cyclic idle contraction, then $U$ has a unique fixed point in $A \cap B$.*

**Theorem 2.8.** *Let $A$ and $B$ be non-empty subsets of a complete $G_{bq}$-family $(X, d_\alpha)$. Suppose $U : A \cup B \to A \cup B$ is a cyclic map. If $U$ is a weak cyclic orbital contraction, then $U$ has a unique fixed point in $A \cap B$.*

Next, we introduce another definition involving *cyclic compatible contractions* and *cyclic orbital stronger $M_k$ $\psi$-contractions.*

**Definition 2.9.** *Let $A$and $B$ be non-empty subsets of a $G_{bq}$-family $(X, d_\alpha)$. Suppose $U, V : A \cup B \to A \cup B$ are cyclic maps. Then $U, V$ are called* cyclic compatible $M_k$-contractions, *if there exists a $M_k$ type mapping $\psi : \mathbb{R}^+ \to [0, 1)$ such that*

$$d_\alpha(U^{2n}x, Uy) \leq \psi(d_\alpha(U^{2n-1}x, Vy))d_\alpha(U^{2n-1}x, y) \tag{2.11}$$

*for $x \in A, y \in B$.*

**Theorem 2.10.** *Let $A$ and $B$ be non-empty closed subsets of a $G_{bq}$-family $(X, d_\alpha)$. Let $U, V : A \cup B \to A \cup B$ be cyclic maps. Suppose $U, V$ are cyclic compatible $M_k$-contractions, then $U$ and $V$ have a coincidence point and a unique common fixed point in $A \cap B$.*

*Proof.* Let $x_0 \in A$ be fixed. Since $U(X) \subset V(X)$, we may choose $x_1 \in X$ such that

$$Ux_0 = Vx_1.$$

Thus we can define a sequence $\{x_n\}$ in $X$ by $Ux_n = Vx_{n+1}$, $n \in \mathbb{N} \cup \{0\}$. Now consider

$$\begin{aligned} d_\alpha(U^{2n}x, U^{2n+1}x) &\leq \psi(d_\alpha(U^{2n-1}x, V^{2n+1}x))d_\alpha(U^{2n-1}x, V^{2n+1}x) \\ &\leq d_\alpha(U^{2n-1}x, V^{2n+1}x) \\ &= d_\alpha(U^{2n-1}x, U^{2n}x). \end{aligned}$$

Similarly, we have

$$\begin{aligned} d_\alpha(U^{2n+1}x, U^{2n+2}x) &\leq \psi(d_\alpha(U^{2n}x, V^{2n+2}x))d_\alpha(U^{2n}x, V^{2n+2}x) \\ &\leq d_\alpha(U^{2n}x, V^{2n+2}x) \\ &= d_\alpha(U^{2n}x, U^{2n+1}x). \end{aligned}$$

In general, we have

$$d_\alpha(U^n x, U^{n+1}x) \leq d_\alpha(U^{n-1}x, U^n x), n \in \mathbb{N}.$$

Therefore, the sequence $\{d_\alpha(U^n x, U^{n+1}x)\}$ is non-increasing and hence it is convergent. Let $\lim_{n\to\infty} d_\alpha(U^n x, U^{n+1}x) = \eta$. There exists $k_0 \in \mathbb{N}$ and $\delta > 0$ such that for all $n \geq k_0$, $\eta \leq d_\alpha(U^n x, U^{n+1}x) < n + \delta$. By hypothesis, there exists $\gamma_\eta \in [0, 1)$ such that

$$\psi(d_\alpha(U^{k_0+n}x, U^{k_0+n+1}x)) < \gamma_\eta.$$

Thus, by definition of *cyclic compatible* $M_k$ *contraction,* we have

$$\begin{aligned} d_\alpha(U^{k_0+n}x, U^{k_0+n+1}x) &\leq \psi(d_\alpha(U^{k_0+n-1}x, V^{k_0+n+1}x))d_\alpha(U^{k_0+n-1}x, V^{k_0+n+1}x) \\ &\leq \gamma_\eta d_\alpha(U^{k_0+n-1}x, U^{k_0+n}x) \end{aligned}$$

and it follows that, for each $n \in \mathbb{N}$,

$$\begin{aligned} d_\alpha(U^{k_0+n}x, U^{k_0+n+1}x) &\leq \gamma_\eta d_\alpha(U^{k_0+n-1}x, U^{k_0+n}x) \\ &\vdots \\ &\leq \gamma_\eta^n d_\alpha(U^{k_0}x, U^{k_0+n}x). \end{aligned}$$

Hence

$$\lim_{n\to\infty} d_\alpha(U^{k_0+n}x, U^{k_0+n+1}x) = 0,$$

since $\gamma_n \in [0, 1)$.

Now Consider, for $n, m \in \mathbb{N},\ m > n$,

$$\begin{aligned} d_\alpha(U^{k_0+n}x, U^{k_0+m}x) &\leq s[d_\beta(U^{k_0+n}x, U^{k_0+n+1}x) + d_\beta(U^{k_0+n+1}x, U^{k_0+m}x)] \\ &= sd_\beta(U^{k_0+n}x, U^{k_0+n+1}x) + sd_\beta(U^{k_0+n+1}x, U^{k_0+m}x) \\ &\leq sd_\beta(U^{k_0+n}x, U^{k_0+n+1}x) + s^2 d_\beta(U^{k_0+n+1}x, U^{k_0+n+2}x) \\ &\qquad + s^3 d_\beta(U^{k_0+n+2}x, U^{k_0+n+3}x) + ... \\ &\leq s\gamma_\eta^n d_\alpha(U^{k_0}x, U^{k_0+1}x) + s^2\gamma_\eta^{n+1} d_\alpha(U^{k_0}x, U^{k_0+1}x) \\ &\qquad + s^3\gamma_\eta^{n+2} d_\alpha(U^{k_0}x, S^{k_0+1}x) + ... \\ &= s\gamma_\eta^n[1 + s\gamma_\eta + (s\gamma_\eta)^2 + ...]d_\alpha(U^{k_0}x, U^{k_0+1}x) \\ &< \frac{s\gamma_\eta^n}{1 - s\gamma_\eta} d_\alpha(U^{k_0}x, U^{k_0+1}x) \end{aligned}$$

By taking the limits as $n, m \to \infty$, and since $0 < \gamma_\eta < \frac{1}{s} < 1$, we get

$$\lim_{n,m\to\infty} d_\alpha(U^{k_0+n}x, U^{k_0+m}x) = 0.$$

Thus $\{U^n x\}$ is a Cauchy sequence. Since $(X, d_\alpha)$ is a $G_{bq}$-family, there exists sequences $\{U^{2n}x\}$ in $A$ and $\{U^{2n-1}x\}$ in $B$ such that $\lim\limits_{n\to\infty} U^{2n}x \to u$ and $\lim\limits_{n,m\to\infty} U^{2n-1}x \to u$. Thus $\lim\limits_{n\to\infty} V^{2n+1}x \to u$ and $\lim\limits_{n,m\to\infty} V^{2n}x \to u$. Since $A$ and $B$ are closed in $X$, $u \in A \cap B$. Now we will prove that $Uz = u$. Since $V(X)$ is closed in $X$, there exists $z$ in $X$ such that $Vz = u$.
Consider,

$$\begin{aligned} d_\alpha(U^{2n}x, Uz) &\le \psi(d_\alpha(U^{2n-1}x, Vz))d_\alpha(U^{2n-1}x, Vz) \\ &< d_\alpha(U^{2n-1}x, Vz) \end{aligned}$$

By taking the limit as $n \to \infty$, $d_\alpha(u, Uz) = 0$. This implies $Uz = u$. Thus $Vz = Uz = u$. Hence $u$ is a coincidence point of $U$ and $V$. From weak compatibility, we get

$$Uu = Vu. \tag{2.12}$$

Now we will prove that $Vu = u = Uu$. Assume $u \neq Vu$, then

$$\begin{aligned} d_\alpha(u, Vu) &= \lim_{n\to\infty} d_\alpha(U^{2n}x, Uu) \\ &\le \lim_{n\to\infty} \psi(d_\alpha(U^{2n-1}x, Vu))d_\alpha(U^{2n-1}x, Vu) \\ &< \lim_{n\to\infty} d_\alpha(U^{2n-1}x, Vu) \\ &= d_\alpha(u, Vu), \end{aligned}$$

which is a contradiction. Hence

$$u = Vu. \tag{2.13}$$

From (2.12) and (2.13), we get $Uu = Vu = u$. Thus $u$ is a common fixed point of $U$ and $V$. To prove uniqueness, suppose $v$ be another fixed point of $U$ and $V$.
Then,

$$\begin{aligned} d_\alpha(u, v) &= \lim_{n\to\infty} d_\alpha(U^{2n}x, Uv) \\ &\le \lim_{n\to\infty} \psi(d_\alpha(U^{2n-1}x, Vv))d_\alpha(U^{2n-1}x, Vv) \\ &< \lim_{n\to\infty} d_\alpha(U^{2n-1}x, Vv) \\ &= d_\alpha(u, v), \end{aligned}$$

which is a contradiction. Hence $u = v$. This completes our proof. □

**Remark 2.11.** *We can obtain special cases of Theorem 2.10, if we*

1. *replace a $G_{bq}$-family $(X, d_\alpha)$ by a $G_q$-family, according to Definition 2.1 in [2], by putting $s = 1$;*

2. *replace a $G_{bq}$-family $(X, d_\alpha)$ by a b-metric space, and taking $d$ instead of $d_\alpha$;*
3. *replace a $G_{bq}$-family $(X, d_\alpha)$ by a complete metric space, by taking $d$ instead of $d_\alpha$ and letting $s = 1$.*

**Example 2.12.** *Let $A = B = X = [0,1]$. Let $d : X \times X \to \mathbb{R}^+$ defined by $d(x,y) = (x-y)^2$. This is a b-metric with $s = 2$ ($\neq$ a usual metric, since $d(0,1) \nleq d(0,\frac{1}{2}) + d(\frac{1}{2},1)$). Define $U, V : A \cup B \to A \cup B$ as follows:*

$$Vx = \begin{cases} 0, & \text{if } x = 0 \\ x+2, & \text{if } 0 < x \leq \frac{1}{2} \\ x-3, & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

*and*

$$Ux = 0, \text{if } 0 \leq x \leq 1.$$

*Clearly $U(X) \subset V(X)$, and define $\psi(t) = \frac{t}{t+1}$ if $0 \leq t \leq 1$.*
*For $y = 0$, we get*

$$Vy = 0.$$

*If $y \in (0, \frac{1}{2}]$, we get*

$$Vy = y + 2.$$

*If $y \in (\frac{1}{2}, 1]$, we get*

$$Vy = y - 3.$$

**Case(i):** *If $y = 0$, we have*

$$\begin{aligned} \psi(d(U^{2n-1}x, Vy))d(U^{2n-1}x, Vy) &= \psi(d(0,0))d(0,0) \\ &= 0 = d(U^{2n}x, Uy) \end{aligned}$$

**Case(ii):** *$y \in (0, \frac{1}{2}]$, we have*

$$\begin{aligned} \psi(d(U^{2n-1}x, Vy))d(U^{2n-1}x, Vy) &= \psi(d(0, y+2))d(0, y+2) \\ &= \psi((y+2)^2)(y+2)^2 \\ &= \frac{(y+2)^2}{(y+2)^2+1}(y+2)^2 > 0 = d(U^{2n}x, Uy) \end{aligned}$$

**Case(iii):** *$y \in (\frac{1}{2}, 1]$, we have*

$$\begin{aligned} \psi(d(U^{2n-1}x, Vy)).d(U^{2n-1}x, Vy) &= \psi(d(0, y-3)).d(0, y-3) \\ &= \frac{(3-y)^2}{(3-y)^2+1}(3-y)^2 > 0 = d(U^{2n}x, Uy) \end{aligned}$$

*Therefore*

$$d(U^{2n}x, Uy) \leq \psi(d(U^{2n-1}x, Vy))d(U^{2n-1}x, Vy).$$

*Thus $U$, $V$ are are cyclic compatible $M_k$-contractions. All the conditions of Theorem 2.10 hold true and $U$, $V$ have a unique common fixed point. Here $u = 0$ is the unique common fixed point of $U$ and $V$.*

**Example 2.13.** *Let $A = B = X = [0,1]$. Define $d(x,y) = |x-y|$, so $(X,d)$ is a complete metric space, where $d$ is a mapping $X \times X$ to $\mathbb{R}^+$.*
*Define*

$$Vx = \begin{cases} 0, & \text{if } 0 \leq x < \frac{1}{2} \\ \frac{1}{2}, & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

*and*

$$Ux = \frac{1}{2}, \text{if } 0 \leq x \leq 1.$$

*Clearly $U(X) \subset V(X)$. Define $\psi(t) = \frac{t}{t+1}$, $0 \leq t \leq 1$. For any $x \in [0,1]$, we have $Ux = U^2x = U^3x = ...U^nx = \frac{1}{2}, \forall n$.*
*For $y \in [0, \frac{1}{2})$, we have*

$$Vy = 0.$$

*For $y \in [\frac{1}{2}, 1]$, we have*

$$Vy = \frac{1}{2}.$$

**Case(i):**$y \in [0, \frac{1}{2})$, *we have*

$$d(U^{2n}x, Uy) = d(\frac{1}{2}, \frac{1}{2}) = 0$$

*and*

$$\begin{aligned} \psi(d(U^{2n-1}x, Vy))d(U^{2n-1}x, Vy) &= \psi(d(\frac{1}{2}, 0)).d(\frac{1}{2}, 0) \\ &= \psi(\frac{1}{2}).\frac{1}{2} \\ &= \frac{1}{6}. \end{aligned}$$

**Case(ii)** : $y \in [\frac{1}{2}, 1]$, *we have*

$$d(U^{2n}x, Uy) = d(\frac{1}{2}, \frac{1}{2}) = 0$$

*and*

$$\begin{aligned} \psi(d(U^{2n-1}x, Vy)).d(U^{2n-1}x, Vy) &= \psi(d(\frac{1}{2}, \frac{1}{2})).d(\frac{1}{2}, \frac{1}{2}) \\ &= 0. \end{aligned}$$

*From both cases, we conclude that*

$$d(U^{2n}x, Uy) \leq \psi(d(U^{2n-1}x, Vy))d(U^{2n-1}x, Vy).$$

*Thus $U$, $V$ are are cyclic compatible $M_k$-contractions. All the conditions of Theorem 2.10 hold true and $U$, $V$ have a unique common fixed point. Here $u = \frac{1}{2}$ is the unique common fixed point of $U$ and $V$.*

Now we introduce the following definition.

**Definition 2.14.** *Let $A$ and $B$ be non empty subsets of a $G_{bq}$-family $(X, d_\alpha)$. Suppose that $V : A \cup B \to A \cup B$ is a cyclic mapping then,*

*(1) $V$ is called* cyclic idle $M_k$-contraction *if and only if*

$$d_\alpha(V^{2n}x, Vy) \leq \psi(d_\alpha(V^{2n-1}x, Vy))d_\alpha(V^{2n-1}x, Vy).$$

*(2) $U$ is called* cyclic orbital $M_k$-contraction *if and only if*

$$d_\alpha(U^{2n}x, Uy) \leq \psi(d_\alpha(U^{2n-1}x, y))d_\alpha(U^{2n-1}x, y).$$

*where $\psi : \mathbb{R}^+ \to [0, 1)$ is a $M_k$ type mapping, for $n \in \mathbb{N}$ and $x \in A,\ y \in B$.*

We obtain the following new results.

**Theorem 2.15.** *Let $A$ and $B$ be non-empty closed subsets of a $G_{bq}$-family $(X, d_\alpha)$. Let $V : A \cup B \to A \cup B$ be a cyclic map. Suppose $V$ are cyclic idle $M_k$-contraction, then $V$ has a unique fixed point in $A \cap B$.*

*Proof.* Take $U = V$ in Theorem 2.10. □

**Theorem 2.16.** *Let $A$and $B$ be non-empty closed subsets of a $G_{bq}$-family $(X, d_\alpha)$. Let $U : A \cup B \to A \cup B$ be a cyclic map. Suppose $U, V$ are cyclic orbital $M_k$-contraction, then $U$ has a unique fixed point in $A \cap B$.*

*Proof.* Take $V = I$ in Theorem 2.10. □

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## References

[1] R. P. Agarwal, M. A. Alghamdi and N. Shahzad, Fixed point theory for cyclic generalized contractions in partial metric spaces. *Fixed Point Theory Appl.* 2012(2012):40 , 1-11.

[2] A. Aliouche, A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type, *J. Math. Anal. Appl.*, **322**, No. 2(2006), 796-802.

[3] S. Banach, Sur les opérations dans les ensembles abstraits et leur applications aux equations intégrales. *Fundam. Math.* **3**, (1922),133-181.

[4] S.S. Chang, YJ. Cho, B.S. Lee, J.S. Jung, SM. Kang, Coincidence point theorems and minimization theorems in fuzzy metric spaces. *Fuzzy Sets Syst.* **88**(1997), 119-127.

[5] A. Erduran, I. Altun, Fixed Point Theorems for Weakly Contractive Mapping on $G$-Metric Spaces and a Homotopy Result, *Int. J. Nonlinear Anal. Appl.* **3**,No.1(2012),1-8.

[6] Bui T. N. Han and Nguyen T. Hieu, A fixed point theorem for generalized cyclic contractive mappings in $B$-metric spaces. *Facta. Uni. Ser. Math. Inform.* **31**, No 2 (2016), 399-415.

[7] J. Jachymski, Equivalent conditions and the Meir-Keeler type theorems, *J. Math. Anal. Appl.* **194** (1995) 293-303.

[8] G, Jungck, Common fixed points for noncontinuous non-self maps on non-metric spaces, *Far East J. Math. Sci.* **4** (1996), 199-215 .

[9] S. Karpagam, S. Agrawal, Best proximity points theorems for cyclic Meir-Keeler contraction maps, *Nonlinear Anal.* 74 (2011), 1040 -1046.

[10] E. Karapinar, Fixed point theory for cyclic weak $\phi$-contraction, *Appl. Math. Lett.*,**24**(2011), 822-825.

[11] E. Karapinar, S. Ramaguera, T. Kenan, Fixed points for cyclic orbital generalized contractions on complete metric spaces, *Central European Journal of Mathematics,* **11**, Issue 3(2013), 552-560.

[12] E. Karapinar and K. Sadarangani, Fixed point theory for cyclic ($\varphi$-$\psi$)-contractions. *Fixed Point Theory Appl.* 2011 (2011):69, 1-8.

[13] W. A. Kirk, P. S. Srinivasan and P. Veeramani, Fixed points for mappings satisfying cyclical contractive conditions. *Fixed Point Theory* **4** (2003), No. 1, 79-89.

[14] P.S. Kumari, D. Panthi, Cyclic contractions and fixed point theorems on various generating spaces, *Fixed Point Theory Appl.* 2015 (2015):153.

[15] P.S. Kumari, D. Panthi, Cyclic compatible contraction and related fixed point theorems, *Fixed Point Theory Appl.* 2016 (2016):28.

[16] A. Meir, E. Keeler, A theorem on contraction mappings, *J. Math. Anal. Appl.* **28** (1969), 326-329.

[17] H. K. Nashine, W. Sintunavarat and P. Kumam, Cyclic generalized contractions and fixed point results with applications to an integral equation. *Fixed Point Theory Appl.* 2012(2012):217, 1-13.

[18] S. Park, B.E. Rhoades, $M_k$ type contractive conditions, *Math. Japon.* **26** (1981), 13-20.

[19] B.E. Rhoades, Some theorems on weakly contractive maps, *Nonlinear Anal.*, **47** (2001), 2683-2693.

[20] S.L. Singh, R. Chugh,R. Kamal and A. Kumar, A new common fixed point theorem for Suzuki-$M_k$ contractions, *Filomat,* **28**:2(2014), 257-262.

[21] B. Zlatanov, Best proximity points for p-summing cyclic orbital Meir Keeler contractions, *Nonlinear Analysis: Modelling and Control,* **20**, No. 4(2015), 528-544.

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