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The S-Intermixed Iterative Method for Equilibrium Problems

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Abstract : Inspired by the work of Yao [12], the S-intermixed iteration for equilibrium problems is proposed. Under some control conditions, a strong convergence theorem for approximating a common solution of two finite familes of equilibrium problems is proved. Finally, a numerical example for a main theorem is given to support the result.

Keywords : equilibrium problem, fixed point, intermixed iteration.
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1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be bifunction. *The equilibrium problem* for F is to determine its equilibrium point, i.e., the set

$$EP(F) = \{x \in C : F(x, y) \geq 0, \forall y \in C\}. \tag{1.1}$$

Equilibrium problems were introduced by [3] in 1994 where such problems have had a significant impact and influence in the development of several branches of pure and applied sciences. Various problems in physics, optimization, and economics are related to seeking some elements of $EP(F)$, see [3, 7]. Many authors have been investigated iterative algorithms for the equilibrium problems, see, for example, [7, 11].

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In 2013, Suwannaut and Kangtunyakarn [11] introduced *the combination of equilibrium problem* which is to find $u \in C$ such that

$$\sum_{i=1}^{N} a_i F_i(x,y) \geq 0, \forall y \in C, \tag{1.2}$$

where $F_i : C \times C \to \mathbb{R}$ be bifunctions and $a_i \in (0,1)$ with $\sum_{i=1}^{N} a_i = 1$, for every $i = 1, 2, \ldots, N$. The set of solution (1.2) is denoted by

$$EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP(F_i).$$

If $F_i = F, \forall i = 1, 2, \ldots, N$, then the combination of equilibrium problem (1.2) reduces to the equilibrium problem (1.1).

The fixed point problem for the mapping $T : C \to C$ is to find $x \in C$ such that $x = Tx$. We denote the fixed point set of a mapping T by $Fix(T)$.

Definition 1.1. *Let $T : C \to C$ be a mapping. Then*

(i) a mapping T is called contractive if there exists $\alpha \in (0,1)$ such that

$$\|Tx - Ty\| \leq \alpha \|x - y\|, \forall x, y \in C;$$

(ii) a mapping T is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C;$$

(iii) T is said to be κ-strictly pseudo-contractive *if there exists a constant $\kappa \in [0,1)$ such that*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(I-T)x - (I-T)y\|^2, \forall x, y \in C.$$

Note that the class of κ-strictly pseudo-contractions strictly includes the class of nonexpansive mappings, that is, a nonexpansive mapping is a 0-strictly pseudo-contractive mapping.

For the last decades, many researcher have studied fixed point theorems associated with various types of nonlinear mappings, see, for instance, [8, 9, 10].

Over the past decades, many others have constructed various types of iterative methods to approximate fixed points. The first one is the Mann iteration introduced by Mann [1] in 1953 and is defined as follows:

$$\begin{cases} x_0 \in H \text{ arbitrary chosen}, \\ x_{n+1} = (1-\alpha_n) x_n + \alpha_n T x_n, \forall n \geq 0, \end{cases} \tag{1.3}$$

where C is a nonempty closed convex subset of a normed space, $T : C \to C$ is a mapping and the sequence $\{\alpha_n\}$ is in the interval (0,1). But this algorithm has only weak convergence. Thus, many mathematicians have been trying to modify Mann's iteration (1.3) and construct new iterative method to obtain the strong convergence theorem.

By modification of Mann's iteration (1.3), the next iteration process is referred to as Ishikawa's iteration process [2] which is defined recursively as follows:

$$\begin{cases} x_0 \in H \text{ arbitrary chosen}, \\ y_n = \beta_n x_n + (1-\beta_n) T x_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) T y_n, \forall n \geq 0, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$. He also obtain the strong convergence theorem for the iterative method (1.4) converging to a fixed point of mapping T. Observe that if $\beta_n = 1$, then the Ishikawa's iteration (1.4) reduces to the Mann's iteration (1.3).

In 2000, Moudafi [4] introduced the viscosity approximation method for nonexpansive mapping S as follows:
Let C be a closed convex subset of a real Hilbert space H and let $S : C \to C$ be a nonexpansive mapping such that $Fix(S)$ is nonempty. Let $f : C \to C$ be a contraction, that is, there exists $\alpha \in (0,1)$ such that $\|fx - fy\| \leq \alpha \|x - y\|, \forall x, y \in C$, and let $\{x_n\}$ be a sequence defined by

$$\begin{cases} x_1 \in C \text{ arbitrary chosen}, \\ x_{n+1} = \frac{1}{1+\epsilon_n} S x_n + \frac{\epsilon_n}{1+\epsilon_n} f(x_n), \forall n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $\{\varepsilon_n\} \subset (0,1)$ satisfies certain conditions. Then the sequence $\{x_n\}$ converges strongly to $z \in Fix(S)$, where $z = P_{Fix(S)} f(z)$ and $P_{Fix(S)}$ is the metric projection of H onto $Fix(S)$.

Recently, in 2015, Yao *et al.* [12] proposed the intermixed algorithm for two strict pseudocontractions S and T as follows:

Algorithm 1.2. *For arbitrarily given $x_0 \in C, y_0 \in C$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by*

$$\begin{aligned} x_{n+1} &= (1-\beta_n) x_n + \beta_n P_C [\alpha_n f(y_n) + (1-k-\alpha_n) x_n + kTx_n], n \geq 0, \\ y_{n+1} &= (1-\beta_n) y_n + \beta_n P_C [\alpha_n g(x_n) + (1-k-\alpha_n) y_n + kSy_n], n \geq 0, \end{aligned} \tag{1.6}$$

where $T : C \to C$ is a λ-strictly pseudo-contraction, $f : C \to H$ is a ρ_1-contraction and $g : C \to H$ is a ρ_2-contraction, $k \in (0, 1-\lambda)$ is a constant and $\{\alpha_n\}$, $\{\beta_n\}$ are two real number sequences in $(0,1)$.

Furthermore, under some control conditions, they proved that the iterative sequences $\{x_n\}$ and $\{y_n\}$ defined by (1.6) converges independently to $P_{Fix(T)} f(y^*)$ and $P_{Fix(S)} g(x^*)$, respectively, where $x^* \in Fix(T)$ and $y^* \in Fix(S)$.

Motivated by Yao *et al.* [12], we introduce the new iterative method called *the S-intermixed iteration* for two finite families of nonlinear mappings as in the following algorithm:

Algorithm 1.3. *Starting with $x_1, y_1, z_1 \in C$, let the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be defined by*

$$x_{n+1} = (1-\beta_n)\,x_n + \beta_n\,(\alpha_n f_1\,(y_n) + (1-\alpha_n)\,Sx_n)\,,$$
$$y_{n+1} = (1-\beta_n)\,y_n + \beta_n\,(\alpha_n f_2\,(x_n) + (1-\alpha_n)\,Ty_n)\,, n \geq 1,$$

where $S, T : C \to C$, is a nonlinear mapping with $Fix(S) \cap Fix\,(T) \neq \emptyset$, $f_i : C \to C$ is a contractive mapping with coefficients $\alpha_i; i = 1, 2$ and $\{\beta_n\}$,$\{\alpha_n\}$ are real sequences in $(0, 1)$, $\forall n \geq 1$.

Inspired by the previous research, we introduce the S-intermixed iteration for equilibrium problems without considering the constant k. Under appropriate conditions, we prove a strong convergence theorem for finding a common solution of two finite families of equilibrium problems. Finally, we give a numerical example for the main theorem in a space of real numbers.

2 Preliminaries

In this section, some well-known definitions and Lemmas are recalled. Let H be a real Hilbert space and C be a nonempty closed convex subset of H. We denote weak convergence and strong convergence by notations " $\rightharpoonup$" and " $\to$", respectively. For every $x \in H$, there is a unique nearest point $P_C x$ in C such that

$$\|x - P_C x\| \leq \|x - y\|, \forall y \in C.$$

Such an operator P_C is called the metric projection of H onto C.

Lemma 2.1 ([5]). *For a given $z \in H$ and $u \in C$,*

$$u = P_C z \Leftrightarrow \langle u - z, v - u\rangle \geq 0, \forall v \in C.$$

Furthermore, P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle\,, \forall x, y \in H.$$

Lemma 2.2 ([6]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n)s_n + \delta_n, \forall n \geq 0,$$

where α_n is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F and C satisfy the following conditions:
(A1) $F(x,x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;
(A3) For each $x, y, z \in C$,

$$\lim_{t \to 0^+} F(tz + (1-t)x, y) \leq F(x,y);$$

(A4) For each $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous.

Lemma 2.3 ([11]). *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be bifunctions satisfying $(A1) - (A4)$ with $\bigcap_{i=1}^{N} EP(F_i) \neq \emptyset$. Then,*

$$EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP(F_i),$$

where $a_i \in (0,1)$ for every $i = 1, 2, \ldots, N$ and $\sum_{i=1}^{N} a_i = 1$.

Lemma 2.4 ([3]). *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C.$$

Lemma 2.5 ([7]). *Assume that $F : C \times C \to \mathbb{R}$ satisfies $(A1) - (A4)$. For $r > 0$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}$$

for all $x \in H$. Then, the following hold:

(i) T_r is single-valued;

(ii) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle;$$

(iii) $Fix(T_r) = EP(F)$;

(iv) $EP(F)$ is closed and convex.

Remark 2.6 ([11]). *Since $\sum_{i=1}^{N} a_i F_i$ satisfies (A1)-(A4), by Lemma 2.3 and Lemma 2.5, we obtain*

$$Fix(T_r) = EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP(F_i),$$

where $a_i \in (0,1)$, for each $i = 1, 2, \ldots, N$, and $\sum_{i=1}^{N} a_i = 1$.

3 Strong convergence theorem

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, \dots, N$, let $F_i, G_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$. Let $f, g : C \to C$ be a contractive mapping with coefficients α_1 and α_2, respectively, with $\alpha = \max_{i \in \{1,2\}} \alpha_i$. Assume that $\Omega_1 := \bigcap_{i=1}^{N} EP(F_i) \neq \emptyset$ and $\Omega_2 := \bigcap_{i=1}^{N} EP(G_i) \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ be generated by $x_1, y_1 \in C$ and*

$$\begin{cases} \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ \sum_{i=1}^{N} b_i G_i(v_n, y) + \frac{1}{s_n} \langle y - v_n, v_n - y_n \rangle \geq 0, \forall y \in C, \\ x_{n+1} = (1 - \beta_n) x_n + \beta_n (\alpha_n f(y_n) + (1 - \alpha_n) u_n), \\ y_{n+1} = (1 - \beta_n) y_n + \beta_n (\alpha_n g(x_n) + (1 - \alpha_n) v_n), \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{r_n\}, \{s_n\} \subseteq (0, 1)$ and $0 \leq a_i, b_i \leq 1$ for every $i = 1, 2, \dots, N$, satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \tau \leq \beta_n \leq \upsilon < 1$, *for some* $\tau, \upsilon > 0$;

(iii) $0 < \epsilon \leq r_n \leq \eta < \infty$, *for some* $\epsilon, \eta > 0$;

(iv) $0 < \delta \leq s_n \leq \mu < \infty$, *for some* $\delta, \mu > 0$;

(v) $\sum_{i=1}^{N} a_i = 1$ *and* $\sum_{i=1}^{N} b_i = 1$;

(vi) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$.

Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $\tilde{x} = P_{\Omega_1} f(\tilde{y})$ and $\tilde{y} = P_{\Omega_2} g(\tilde{x})$, respectively.

Proof. Since $\sum_{i=1}^{N} a_i F_i$ and $\sum_{i=1}^{N} b_i G_i$ satisfy (A1)-(A4) and (3.1), by Lemma 2.5 and Remark 2.6, we have $u_n = T_{r_n}^1 x_n$, $v_n = T_{s_n}^2 y_n$, $Fix(T_{r_n}^1) = \bigcap_{i=1}^{N} EP(F_i)$ and $Fix(T_{s_n}^2) = \bigcap_{i=1}^{N} EP(G_i)$.

Step 1 We show that $\{x_n\}$ and $\{y_n\}$ are bounded.

Let $x^* \in \Omega_1$ and $y^* \in \Omega_2$. Then we derive

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|(1-\beta_n)(x_n - x^*) + \beta_n(\alpha_n(f(y_n) - x^*) + (1-\alpha_n)(u_n - x^*))\| \\
&\le (1-\beta_n)\|x_n - x^*\| + \beta_n\Big[\alpha_n\|f(y_n) - x^*\| + (1-\alpha_n)\|u_n - x^*\|\Big] \\
&\le (1-\beta_n)\|x_n - x^*\| + \beta_n\Big[\alpha_n\|f(y_n) - f(y^*)\| + \alpha_n\|f(y^*) - x^*\| \\
&\quad + (1-\alpha_n)\left\|T^1_{r_n}x_n - x^*\right\|\Big] \\
&\le (1-\beta_n)\|x_n - x^*\| + \beta_n\Big[\alpha_n\alpha_1\|y_n - y^*\| + \alpha_n\|f(y^*) - x^*\| \\
&\quad + (1-\alpha_n)\|x_n - x^*\|\Big] \\
&= (1-\alpha_n\beta_n)\|x_n - x^*\| + \beta_n\alpha_n\alpha\|y_n - y^*\| + \beta_n\alpha_n\|f(y^*) - x^*\|.
\end{aligned} \tag{3.1}$$

Using the same argument as (3.1), we also obtain

$$\|y_{n+1} - y^*\| \le (1-\alpha_n\beta_n)\|y_n - y^*\| + \beta_n\alpha_n\alpha\|x_n - x^*\| + \beta_n\alpha_n\|g(x^*) - y^*\|. \tag{3.2}$$

Combining (3.1) and (3.2), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| \\
&\le (1 - \alpha_n\beta_n(1-\alpha))(\|x_n - x^*\| + \|y_n - y^*\|) \\
&\quad + \alpha_n\beta_n(\|f(y^*) - x^*\| + \|g(x^*) - y^*\|) \\
&\le \max\left\{\|x_1 - x^*\| + \|y_1 - y^*\|, \frac{\|f(y^*) - x^*\| + \|g(x^*) - y^*\|}{1-\alpha}\right\}.
\end{aligned}$$

By induction, we get

$$\begin{aligned}
&\|x_n - x^*\| + \|y_n - y^*\| \\
&\le \max\left\{\|x_1 - x^*\| + \|y_1 - y^*\|, \frac{\|f(y^*) - x^*\| + \|g(x^*) - y^*\|}{1-\alpha}\right\}.
\end{aligned}$$

This implies that $\{x_n\}$ and $\{y_n\}$ are bounded. So are $\{u_n\}$ and $\{v_n\}$.

Step 2. Derive that $\|x_{n+1} - x_n\| \to 0$ and $\|y_{n+1} - y_n\| \to 0$ as $n \to \infty$.
Using the same method as in [11], we get

$$\|u_n - u_{n-1}\| \le \|x_n - x_{n-1}\| + \frac{1}{\epsilon}|r_n - r_{n-1}|\,\|u_n - x_n\|. \tag{3.3}$$

Take $p_n = \alpha_n f(y_n) + (1-\alpha_n)u_n$ and $q_n = \alpha_n g(x_n) + (1-\alpha_n)v_n$. Then, by (3.3), we obtain

$$
\begin{aligned}
&\|p_n - p_{n-1}\| \\
\leq&\alpha_n\|f(y_n) - f(y_{n-1})\| + |\alpha_n - \alpha_{n-1}|\,\|f(y_{n-1})\| + (1-\alpha_n)\|u_n - u_{n-1}\| \\
&+ |\alpha_n - \alpha_{n-1}|\,\|u_{n-1}\| \\
\leq&\alpha_n\alpha\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\,\|f(y_{n-1})\| + (1-\alpha_n)\|x_n - x_{n-1}\| \\
&+ \frac{1}{\epsilon}\,|r_n - r_{n-1}|\,\|u_n - x_n\| + |\alpha_n - \alpha_{n-1}|\,\|u_{n-1}\|. \qquad (3.4)
\end{aligned}
$$

From (3.4), we have

$$
\begin{aligned}
\|x_{n+1} - x_n\| \leq&(1-\beta_n)\,\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\,\|x_{n-1}\| + \beta_n\,\|p_n - p_{n-1}\| \\
&+ |\beta_n - \beta_{n-1}|\,\|p_{n-1}\| \\
\leq&(1-\alpha_n\beta_n)\,\|x_n - x_{n-1}\| + \alpha_n\beta_n\alpha\,\|y_n - y_{n-1}\| \\
&+ |\alpha_n - \alpha_{n-1}|\,(\|f(y_{n-1})\| + \|u_{n-1}\|) + \frac{1}{\epsilon}\,|r_n - r_{n-1}|\,\|u_n - x_n\| \\
&+ |\beta_n - \beta_{n-1}|\,(\|x_{n-1}\| + \|p_{n-1}\|)\,. \qquad (3.5)
\end{aligned}
$$

Applying the same proof as (3.5), we get

$$
\begin{aligned}
\|y_{n+1} - y_n\| \leq&(1-\alpha_n\beta_n)\,\|y_n - y_{n-1}\| + \alpha_n\beta_n\alpha\,\|x_n - x_{n-1}\| \\
&+ |\alpha_n - \alpha_{n-1}|\,(\|g(x_{n-1})\| + \|v_{n-1}\|) + \frac{1}{\delta}\,|s_n - s_{n-1}|\,\|v_n - y_n\| \\
&+ |\beta_n - \beta_{n-1}|\,(\|y_{n-1}\| + \|q_{n-1}\|)\,. \qquad (3.6)
\end{aligned}
$$

By (3.5) and (3.6), we derive

$$
\begin{aligned}
&\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| \\
\leq&\,(1-\alpha_n\beta_n(1-\alpha))\,(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|) + \frac{1}{\epsilon}\,|r_n - r_{n-1}|\,\|u_n - x_n\| \\
&+ \frac{1}{\delta}\,|s_n - s_{n-1}|\,\|v_n - y_n\| + |\alpha_n - \alpha_{n-1}|\,\big(\,\|f(y_{n-1})\| + \|g(x_{n-1})\| + \|u_{n-1}\| \\
&+ \|v_{n-1}\|\,\big) + |\beta_n - \beta_{n-1}|\,(\|x_{n-1}\| + \|y_{n-1}\| + \|p_{n-1}\| + \|q_{n-1}\|)\,.
\end{aligned}
$$

By Lemma 2.2 and the conditions (i), (ii), (vi), we obtain

$$\|x_{n+1} - x_n\| \to 0 \text{ as } n \to \infty \qquad (3.7)$$

and

$$\|y_{n+1} - y_n\| \to 0 \text{ as } \to 0 \text{ as } n \to \infty. \qquad (3.8)$$

Step 3. Prove that $\lim_{n\to\infty}\|u_n - x_n\| = 0$ and $\lim_{n\to\infty}\|v_n - y_n\| = 0$.
Observe that

$$x_{n+1} - x_n = \beta_n\left[\alpha_n\left(f(y_n) - x_n\right) + (1-\alpha_n)\left(u_n - x_n\right)\right] \qquad (3.9)$$

and

$$y_{n+1} - y_n = \beta_n \left[\alpha_n \left(g(x_n) - y_n\right) + (1-\alpha_n)\left(v_n - y_n\right)\right]. \tag{3.10}$$

It follows by (3.9) that

$$\beta_n (1-\alpha_n) \|u_n - x_n\| \le \alpha_n \beta_n \|f(y_n) - x_n\| + \|x_{n+1} - x_n\|.$$

From the condition (i), (ii) and (3.7), we have

$$\|u_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{3.11}$$

Shortly, from (3.10), we also obtain

$$\|v_n - y_n\| \to 0 \text{ as } n \to \infty. \tag{3.12}$$

Step 4 Claim that $\limsup_{n\to\infty} \langle f(\tilde{y}) - \tilde{x}, x_n - \tilde{x}\rangle \le 0$, where $\tilde{x} = P_{\Omega_1} f(\tilde{y})$ and $\limsup_{n\to\infty} \langle g(\tilde{x}) - \tilde{y}, y_n - \tilde{y}\rangle \le 0$, where $\tilde{y} = P_{\Omega_2} g(\tilde{x})$.
Without of generality, we can assume that $x_{n_k} \rightharpoonup \omega_1$ as $k \to \infty$. From (3.11), it follows that $u_{n_k} \rightharpoonup \omega_1$ as $k \to \infty$. Continuiing the same method as in Step 4 of [11], we get

$$\omega_1 \in \Omega_1. \tag{3.13}$$

By (3.13) and $x_{n_k} \rightharpoonup \omega_1$ as $k \to \infty$, we derive that

$$\limsup_{n\to\infty} \langle f(\tilde{y}) - \tilde{x}, x_n - \tilde{x}\rangle = \lim_{k\to\infty} \langle f(\tilde{y}) - \tilde{x}, x_{n_k} - \tilde{x}\rangle = \langle f(\tilde{y}) - \tilde{x}, \omega_1 - \tilde{x}\rangle \le 0. \tag{3.14}$$

Similarly, we can assume that $y_{n_k} \rightharpoonup \omega_2$ as $k \to \infty$ and we have that $v_{n_k} \rightharpoonup \omega_2$ as $k \to \infty$. This implies that $\omega_2 \in \Omega_2$. Thus, we also obtain

$$\limsup_{n\to\infty} \langle g(\tilde{x}) - \tilde{y}, y_n - \tilde{y}\rangle = \lim_{k\to\infty} \langle g(\tilde{x}) - \tilde{y}, y_{n_k} - \tilde{y}\rangle = \langle g(\tilde{x}) - \tilde{y}, \omega_2 - \tilde{y}\rangle \le 0. \tag{3.15}$$

Step 5 Show that $\{x_n\}$ and $\{y_n\}$ converge strongly to $\tilde{x} = P_{\Omega_1} f(\tilde{y})$ and $\tilde{y} = P_{\Omega_2} g(\tilde{x})$, respectively.
Hence, we derive

$$\begin{aligned}
&\|x_{n+1} - \tilde{x}\|^2 \\
=& \|(1-\beta_n)(x_n - \tilde{x}) + \beta_n (\alpha_n (f(y_n) - \tilde{x}) + (1-\alpha_n)(u_n - \tilde{x}))\|^2 \\
\le& \|(1-\beta_n)(x_n - \tilde{x}) + \beta_n (1-\alpha_n)(u_n - \tilde{x})\|^2 + 2\alpha_n\beta_n \langle f(y_n) - \tilde{x}), x_{n+1} - \tilde{x}\rangle \\
\le& [(1-\beta_n)\|x_n - \tilde{x}\| + \beta_n(1-\alpha_n)\|u_n - \tilde{x}\|]^2 + 2\alpha_n\beta_n \|f(y_n) - f(\tilde{y})\| \|x_{n+1} - \tilde{x}\| \\
&+ 2\alpha_n\beta_n \langle f(\tilde{y}) - \tilde{x}), x_{n+1} - \tilde{x}\rangle \\
\le& (1-\alpha_n\beta_n)^2 \|x_n - \tilde{x}\|^2 + 2\alpha_n\beta_n\alpha \|y_n - \tilde{y}\| \|x_{n+1} - \tilde{x}\| \\
&+ 2\alpha_n\beta_n \langle f(\tilde{y}) - \tilde{x}), x_{n+1} - \tilde{x}\rangle \\
\le& (1-\alpha_n\beta_n)^2 \|x_n - \tilde{x}\|^2 + \alpha_n\beta_n\alpha \left[\|y_n - \tilde{y}\|^2 + \|x_{n+1} - \tilde{x}\|^2\right] \\
&+ 2\alpha_n\beta_n \langle f(\tilde{y}) - \tilde{x}), x_{n+1} - \tilde{x}\rangle.
\end{aligned}$$

This yields that

$$\begin{aligned}\|x_{n+1}-\tilde{x}\|^2 \leq & \frac{(1-\alpha_n\beta_n)^2}{1-\alpha_n\beta_n\alpha}\|x_n-\tilde{x}\|^2+\frac{\alpha_n\beta_n\alpha}{1-\alpha_n\beta_n\alpha}\|y_n-\tilde{y}\|^2 \\ & +\frac{2\alpha_n\beta_n}{1-\alpha_n\beta_n\alpha}\langle f(\tilde{y})-\tilde{x}, x_{n+1}-\tilde{x}\rangle. \end{aligned} \tag{3.16}$$

Applying the similar argument as (3.16), we also get

$$\begin{aligned}\|y_{n+1}-\tilde{y}\|^2 \leq & \frac{(1-\alpha_n\beta_n)^2}{1-\alpha_n\beta_n\alpha}\|y_n-\tilde{y}\|^2+\frac{\alpha_n\beta_n\alpha}{1-\alpha_n\beta_n\alpha}\|x_n-\tilde{x}\|^2 \\ & +\frac{2\alpha_n\beta_n}{1-\alpha_n\beta_n\alpha}\langle g(\tilde{x})-\tilde{y}, y_{n+1}-\tilde{y}\rangle. \end{aligned} \tag{3.17}$$

Combining (3.16) and (3.17), we obtain

$$\begin{aligned} & \|x_{n+1}-\tilde{x}\|^2+\|y_{n+1}-\tilde{y}\|^2 \\ \leq & \frac{(1-\alpha_n\beta_n)^2+\alpha_n\beta_n\alpha}{1-\alpha_n\beta_n\alpha}\left(\|x_n-\tilde{x}\|^2+\|y_n-\tilde{y}\|^2\right) \\ & +\frac{2\alpha_n\beta_n}{1-\alpha_n\beta_n\alpha}\left(\langle f(\tilde{y})-\tilde{x}, x_{n+1}-\tilde{x}\rangle+\langle g(\tilde{x})-\tilde{y}, y_{n+1}-\tilde{y}\rangle\right) \\ = & \left(1-\frac{2\alpha_n\beta_n(1-\alpha)}{1-\alpha_n\beta_n\alpha}\right)\left(\|x_n-\tilde{x}\|^2+\|y_n-\tilde{y}\|^2\right) \\ & +\frac{2\alpha_n\beta_n(1-\alpha)}{1-\alpha_n\beta_n\alpha}\left(\frac{\alpha_n\beta_n}{2(1-\alpha)}\left(\|x_n-\tilde{x}\|^2+\|y_n-\tilde{y}\|^2\right)\right. \\ & \left.+\frac{1}{1-\alpha}\left(\langle f(\tilde{y})-\tilde{x}, x_{n+1}-\tilde{x}\rangle+\langle g(\tilde{x})-\tilde{y}, y_{n+1}-\tilde{y}\rangle\right)\right). \end{aligned}$$

By Lemma 2.2 and the conditions (i),(ii), we can conclude that $\{x_n\}$ and $\{y_n\}$ converge strongly to $\tilde{x}=P_{\Omega_1}f(\tilde{y})$ and $\tilde{y}=P_{\Omega_2}g(\tilde{x})$, respectively. Furthermore, from (3.11) and (3.12), hence we get $\{u_n\}$ and $\{v_n\}$ converge strongly to $\tilde{x}=P_{\Omega_1}f(\tilde{y})$ and $\tilde{y}=P_{\Omega_2}g(\tilde{x})$, respectively.. This completes the proof.

□

The following Corollary is a direct consequence of Theorem 3.1.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F,G : C\times C\to\mathbb{R}$ be a bifunction satisfying $(A1)-(A4)$. Let $f,g : C\to C$ be a contractive mapping with coefficients α_1 and α_2, respectively, with $\alpha=\max_{i\in\{1,2\}}\alpha_i$. Assume that $EP(F), EP(G)\neq\emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ be generated by $x_1, y_1\in C$ and*

$$\begin{cases} F(u_n,y)+\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle\geq 0, \forall y\in C, \\ G(v_n,y)+\frac{1}{s_n}\langle y-v_n, v_n-y_n\rangle\geq 0, \forall y\in C, \\ x_{n+1}=(1-\beta_n)x_n+\beta_n(\alpha_n f(y_n)+(1-\alpha_n)u_n), \\ y_{n+1}=(1-\beta_n)y_n+\beta_n(\alpha_n g(x_n)+(1-\alpha_n)v_n), \forall n\geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{r_n\}, \{s_n\} \subseteq (0,1)$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(ii) $0 < \tau \leq \beta_n \leq \upsilon < 1$*, for some* $\tau, \upsilon > 0$*;*

(iii) $0 < \epsilon \leq r_n \leq \eta < \infty$*, for some* $\epsilon, \eta > 0$*;*

(iv) $0 < \delta \leq s_n \leq \mu < \infty$*, for some* $\delta, \mu > 0$*;*

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$*,* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$*,* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$*,* $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$*.*

Then the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $\tilde{x} = P_{EP(F)} f(\tilde{y})$ *and* $\tilde{y} = P_{EP(G)} g(\tilde{x})$*, respectively.*

Proof. Put $F \equiv F_i$ and $G \equiv G_i$, for every $i = 1, 2, \ldots, N$. Then, from Theorem 3.1, the result of this corollary can be obtained. □

4 A Numerical Example

In this section, we give a numerical example to support our main theorem.

Example 4.1. *Let* $\mathbb{R}$ *be the set of real numbers. For every* $i = 1, 2, \ldots, N$*, let* $F_i, G_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be defined by*

$$F_i(x,y) = i(y-x)(y+5x+6),$$
$$G_i(x,y) = i(y-x)(y+5x-6), \text{ for all } x, y \in \mathbb{R}.$$

Moreover, let $f, g : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$fx = \frac{x}{2}$$
$$gx = \frac{x}{6}, \text{ for all } x \in \mathbb{R}.$$

Put $a_i = \frac{3}{5^i} + \frac{1}{N5^N}$ *and* $b_i = \frac{2}{9^i} + \frac{1}{N9^N}$*, for every* $i = 1, 2, \ldots, N$*. Let* $\alpha_n = \frac{1}{100n}$*,* $\beta_n = \frac{3n}{5n+3}$*,* $r_n = \frac{3n+7}{4n+9}$ *and* $s_n = \frac{7n+1}{9n+11}$ *for every* $n \in \mathbb{N}$*. Then, the sequences* $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* -1 *and* 1*, respectively.*

Solution. *Since* $a_i = \frac{3}{5^i} + \frac{1}{N5^N}$*, we obtain*

$$\begin{aligned}\sum_{i=1}^{N} a_i F_i(x,y) &= \sum_{i=1}^{N} \left(\frac{3}{5^i} + \frac{1}{N5^N}\right) i(y-x)(y-2x+1) \\ &= \xi(y-x)(y+5x+6), \end{aligned} \tag{4.1}$$

where $\xi = \sum_{i=1}^{N} \left(\frac{3}{5^i} + \frac{1}{N5^N}\right) i$*. It is clear to check that* $\sum_{i=1}^{N} a_i F_i$ *satisfies all conditions (A1)-(A4) and* $-1 \in EP(\sum_{i=1}^{N} a_i F_i) = \bigcap_{i=1}^{N} EP(F_i)$*. Using (4.1), we also obtain that*

$$\sum_{i=1}^{N} b_i G_i(x,y) = \varepsilon(y-x)(y+5x-6),$$

where $\varepsilon = \sum_{i=1}^{N} \frac{2}{9^i} + \frac{1}{N9^N} i$. *Thus we also get* $1 \in EP(\sum_{i=1}^{N} b_i G_i) = \bigcap_{i=1}^{N} EP(G_i)$. *Observe that*

$$\begin{aligned}
0 &\leq \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \\
&= \xi(y - u_n)(y + 5u_n + 6) + \frac{1}{r_n} (y - u_n)(u_n - x_n) \\
&\Leftrightarrow \\
0 &\leq r_n \xi(y - u_n)(y + 5u_n + 6) + (y - u_n)(u_n - x_n) \\
&= \xi r_n y^2 + 6\xi r_n y + u_n y + 4\xi r_n u_n y - x_n y - 6\xi r_n u_n - u_n^2 - 5\xi r_n u_n^2 + u_n x_n.
\end{aligned} \tag{4.2}$$

Let $G(y) = \xi r_n y^2 + 6\xi r_n y + u_n y + 4\xi r_n u_n y - x_n y - 6\xi r_n u_n - u_n^2 - 5\xi r_n u_n^2 + u_n x_n$. *Then* $G(y)$ *is a quadratic function of* y *with coefficients* $a = \xi r_n$, $b = 6\xi r_n + u_n + 4\xi r_n u_n - x_n$, *and* $c = -6\xi r_n u_n - u_n^2 - 5\xi r_n u_n^2 + u_n x_n$. *Determine the discriminant* Δ *of* G *as follows:*

$$\begin{aligned}
\Delta =& b^2 - 4ac \\
=& (6\xi r_n + u_n + 4\xi r_n u_n - x_n)^2 - 4(xir_n)\left(-6\xi r_n u_n - u_n^2 - 5\xi r_n u_n^2 + u_n x_n\right) \\
=& \ 36\xi^2 r_n^2 + 12\xi r_n u_n + 72xi^2 r_n^2 u_n + 12\xi r_n u_n^2 + 36\xi^2 r_n^2 u_n^2 - 12\xi r_n x_n - 2u_n x_n \\
& - 12\xi r_n u_n x_n + x_n^2 \\
=& (6\xi r_n + u_n + 6\xi r_n u_n - x_n)^2 .
\end{aligned}$$

From (4.2), we have $G(y) \geq 0$, *for every* $y \in \mathbb{R}$. *If* $G(y)$ *has most one solution in* $\mathbb{R}$, *thus we have* $\Delta \leq 0$. *This implies that*

$$u_n = \frac{x_n - 6\xi r_n}{1 + 6\xi r_n}. \tag{4.3}$$

Similar to (4.3), we also obtain

$$v_n = \frac{y_n + 6\varepsilon s_n}{1 + 6\varepsilon s_n}. \tag{4.4}$$

Clearly, all sequences and parameters are satisfied all conditions of Theorem 3.1. Hence, by Theorem 3.1, we can conclude that the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* -1 *and* 1 *respectively.*

Table 1 and Figure 1 show the numerical results of sequences $\{u_n\}$, $\{x_n\}$, $\{v_n\}$ *and* $\{y_n\}$ *with* $x_1 = 1$, $y_1 = -1$, $N = 20$ *and* $n = 30$.

n	u_n	x_n	v_n	y_n
1	-0.624549	1.000000	0.384615	-1.000000
2	-0.737571	0.391261	0.621027	-0.481587
3	-0.835139	-0.128592	0.771185	0.026030
4	-0.901507	-0.480452	0.865187	0.397286
5	-0.942797	-0.698698	0.922050	0.640175
⋮	⋮	⋮	⋮	⋮
15	-0.999587	-0.997836	0.999583	0.997881
⋮	⋮	⋮	⋮	⋮
26	-0.999851	-0.999219	0.999883	0.999391
27	-0.999857	-0.999251	0.999888	0.999416
28	-0.999863	-0.999281	0.999893	0.999440
29	-0.999868	-0.999308	0.999897	0.999461
30	-0.999872	-0.999333	0.999901	0.999481

Table 1: The values of $\{u_n\}$, $\{x_n\}$, $\{v_n\}$ and $\{y_n\}$ with $x_1 = 1$, $y_1 = -1$, $N = 20$ and $n = 30$.

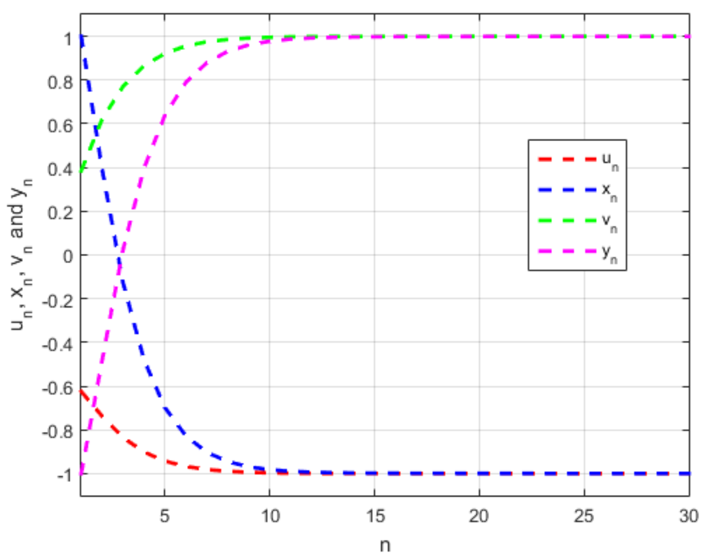

Figure 1: An independent convergence of $\{u_n\}$, $\{x_n\}$, $\{v_n\}$ and $\{y_n\}$ with $x_1 = 1$, $y_1 = -1$, $N = 20$ and $n = 30$.

Remark 4.2. *From the previous example, we can conclude that*

(i) Table 1 and Figure 1 show that the sequences $\{u_n\}$, $\{x_n\}$ converge to $-1 \in \Omega_1$ and $\{v_n\}$,$\{y_n\}$ converge to $1 \in \Omega_2$, independently.

(ii) The convergence of $\{u_n\}$, $\{x_n\}$, $\{v_n\}$ and $\{y_n\}$ can be guaranteed by Theorem 3.1.

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