



Fixed Point Theorems for Generalized θ - ϕ -Contractions in S -metric Spaces

Somkiat Chaipornjareansri¹

Department of Mathematics, Faculty of Science,
Lampang Rajabhat University, Thailand
e-mail : somkiat.chai@gmail.com

Abstract : We introduce the notion of generalized θ - ϕ contraction and establish some new fixed point theorems for this contraction in the setting of complete S -metric spaces. The results presented in the paper improve, extend, and unify some known results. Finally, we give an example to illustrate them.

Keywords : Fixed Point, S -metric spaces, generalized θ - ϕ contraction.

2000 Mathematics Subject Classification : 47H10; 54H25.

1 Introduction

Metric spaces are very important in various areas of mathematics such as analysis, topology, applied mathematics etc. So various generalizations of metric spaces have been studied and several fixed point results were obtained (for example, see [1–5]). Recently, Sedghi, Shobe and Aliouche have defined the concept of an S -metric space [6]. This notion is a generalization of a G -metric space [7] and a D^* -metric space [8].

In 2014, Jleli and Samet [4] introduced a new type of contraction called θ -contraction. Just recently, Zheng et al. [10] introduced the notion of θ - ϕ contraction in metric spaces which generalized θ -contraction and other contractions (see [5, 10] and the references therein).

Inspired by [1, 10], we introduce the notion of generalized θ - ϕ -contraction and establish some new fixed point theorems for this contraction in the setting of complete S -metric spaces. The results presented in the paper improve and extend the corresponding results of Sedghi, Shobe and Aliouche [6], Sedghi and Dung [9],

¹Copyright © 2019 by the Mathematical Association of Thailand.
All rights reserved.

Ćirićs [11], Kannan [12] and Browder [13]. Also, we give an example to illustrate them.

2 Preliminaries

We begin with the following definitions:

Definition 2.1 ([6], Definition 2.1). *Let X be a nonempty set. An S -metric on X is a function $S : X^3 \rightarrow [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$.*

- (S1) $S(x, y, z) \geq 0$;
- (S2) $S(x, y, z) = 0$ if and only if $x = y = z$;
- (S3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

Then the pair (X, S) is called an S -metric space.

The following is an intuitive geometric example for S -metric spaces.

Example 2.2 ([6], Example 2.4). *Let $X = \mathbb{R}^2$ and d be an ordinary metric on X . Put*

$$S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$$

for all $x, y, z \in \mathbb{R}^2$, that is, S is the perimeter of the triangle given by x, y, z . Then S is an S -metric on X .

Lemma 2.3 ([6], Lemma 2.5). *Let (X, S) be an S -metric space. Then $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.*

Remark 2.4. *Let (X, S) be an S -metric space. From Definition 2.1 we have,*

$$S(x, x, z) \leq 2S(x, x, y) + S(y, y, z)$$

then

$$\frac{1}{3}S(x, x, z) \leq \max \{S(x, x, y), S(y, y, z)\}$$

for all $x, y, z \in X$.

Definition 2.5 ([6]). *Let (X, S) be an S -metric space.*

- (i) *A sequence $\{x_n\} \subset X$ is said to converge to $x \in X$ if $S(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $S(x_n, x_n, x) < \varepsilon$. We write $x_n \rightarrow x$ for brevity.*
- (ii) *A sequence $\{x_n\} \subset X$ is called a Cauchy sequence if $S(x_n, x_n, x_m) \rightarrow 0$ as $n, m \rightarrow \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $S(x_n, x_n, x_m) < \varepsilon$.*

- (iii) The S -metric space (X, S) is said to be complete if every Cauchy sequence is a convergent sequence

Lemma 2.6 ([6], Lemma 2.12). Let (X, S) be an S -metric space. If $x_n \rightarrow x$ and $y_n \rightarrow y$, then $S(x_n, x_n, y_n) \rightarrow S(x, x, y)$.

According to [1, 10], denote by Θ the set of functions $\theta : (0, \infty) \rightarrow (1, \infty)$ satisfying the following conditions:

- (Θ 1) θ is nondecreasing.
 (Θ 2) For each sequence $\{t_n\} \subset (0, \infty)$,
 $\lim_{n \rightarrow \infty} \theta(t_n) = 1$ if and only if $\lim_{n \rightarrow \infty} t_n = 0^+$.
 (Θ 3) θ is continuous on $(0, \infty)$.

And by Φ the set of functions $\phi : [1, \infty) \rightarrow [1, \infty)$ satisfies the following conditions:

- (Φ 1) $\phi : [1, \infty) \rightarrow [1, \infty)$ is nondecreasing.
 (Φ 2) For each $t > 1$, $\lim_{n \rightarrow \infty} \phi^n(t) = 1$.
 (Φ 3) ϕ is continuous on $[1, \infty)$.

Lemma 2.7 ([10], Lemma 2.1). If $\phi \in \Phi$, then $\phi(1) = 1$ and $\phi(t) < t$ for each $t > 1$.

3 Main Results

Based on the functions $\theta \in \Theta$ and $\phi \in \Phi$, we give the following definition.

Definition 3.1. Let (X, S) be a S -metric space. A mapping $T : X \rightarrow X$ is said to be a **generalized θ - ϕ -contraction** if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that, for any $x, y, z \in X$,

$$S(Tx, Ty, Tz) \neq 0 \implies \theta(S(Tx, Ty, Tz)) \leq \phi[\theta(N(x, y, z))], \quad (3.1)$$

where

$$N(x, y, z) = \max \left\{ S(x, y, z), S(x, x, Tx), S(y, y, Ty), S(z, z, Tz), \frac{1}{3}S(x, x, Ty), \frac{1}{3}S(y, y, Tz), \frac{1}{3}S(z, z, Tx), \frac{1}{6}(S(x, x, Ty) + S(y, y, Tz) + S(z, z, Tx)) \right\}. \quad (3.2)$$

Theorem 3.2. Let (X, S) be a complete S -metric space and let $T : X \rightarrow X$ be a generalized θ - ϕ contraction. Then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Proof. Let $x_0 \in X$ be an arbitrary point. We define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$, for all $n \in \mathbb{N}$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $x^* = x_n$ is a fixed point for T . Next, we assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Then $S(x_n, x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. Applying inequality (3.1) with $x = x_n, y = x_n, z = x_{n+1}$, we obtain

$$\theta(S(Tx_n, Tx_n, Tx_{n+1})) \leq \phi[\theta(N(x_n, x_n, x_{n+1}))], \quad (3.3)$$

where

$$\begin{aligned} & N(x_n, x_n, x_{n+1}) \\ &= \max \left\{ S(x_n, x_n, x_{n+1}), S(x_n, x_n, x_{n+1}), S(x_n, x_n, x_{n+1}), S(x_{n+1}, x_{n+1}, x_{n+2}), \right. \\ & \quad \left. \frac{1}{3}S(x_n, x_n, x_{n+1}), \frac{1}{3}S(x_n, x_n, x_{n+2}), \frac{1}{3}S(x_{n+1}, x_{n+1}, x_{n+1}), \right. \\ & \quad \left. \frac{1}{6}(S(x_n, x_n, x_{n+1}) + S(x_n, x_n, x_{n+2}) + S(x_{n+1}, x_{n+1}, x_{n+1})) \right\} \\ &= \max \left\{ S(x_n, x_n, x_{n+1}), S(x_{n+1}, x_{n+1}, x_{n+2}), \frac{1}{3}S(x_n, x_n, x_{n+2}), \right. \\ & \quad \left. \frac{1}{3}S(x_{n+1}, x_{n+1}, x_{n+2}), \frac{1}{6}(S(x_n, x_n, x_{n+2}) + S(x_{n+1}, x_{n+1}, x_{n+2})) \right\} \\ &= \max \left\{ S(x_n, x_n, x_{n+1}), S(x_{n+1}, x_{n+1}, x_{n+2}), \frac{1}{3}S(x_n, x_n, x_{n+2}), \right. \\ & \quad \left. \frac{1}{6}(S(x_n, x_n, x_{n+2}) + S(x_{n+1}, x_{n+1}, x_{n+2})) \right\} \\ & \quad \left(\because \frac{1}{3}S(x_{n+1}, x_{n+1}, x_{n+2}) \leq S(x_{n+1}, x_{n+1}, x_{n+2}) \right) \\ &= \max \left\{ S(x_n, x_n, x_{n+1}), S(x_{n+1}, x_{n+1}, x_{n+2}), \right. \\ & \quad \left. \frac{1}{6}(S(x_n, x_n, x_{n+2}) + S(x_{n+1}, x_{n+1}, x_{n+2})) \right\} \quad (\text{see Remark 2.4}) \\ &= \max \left\{ S(x_n, x_n, x_{n+1}), S(x_{n+1}, x_{n+1}, x_{n+2}) \right\} \\ & \quad \left(\because \frac{1}{6}S(x_n, x_n, x_{n+2}) + \frac{1}{6}S(x_{n+1}, x_{n+1}, x_{n+2}) \leq \frac{M}{2} + \frac{M}{2}, \text{ where} \right. \\ & \quad \left. M = \max \left\{ S(x_n, x_n, x_{n+1}), S(x_{n+1}, x_{n+1}, x_{n+2}) \right\} \right). \end{aligned} \quad (3.4)$$

If $N(x_n, x_n, x_{n+1}) = S(x_{n+1}, x_{n+1}, x_{n+2})$, then it follows from (3.1) that

$$\begin{aligned} \theta(S(x_{n+1}, x_{n+1}, x_{n+2})) &= \theta(S(Tx_n, Tx_n, Tx_{n+1})) \\ &\leq \phi[\theta(N(x_n, x_n, x_{n+1}))] \\ &= \phi[\theta(S(x_{n+1}, x_{n+1}, x_{n+2}))] \\ &< \theta(S(x_{n+1}, x_{n+1}, x_{n+2})), \quad (\text{by Lemma 2.7}), \end{aligned} \quad (3.5)$$

which is a contradiction. Hence, for $\forall n \in \mathbb{N}$,

$$N(x_n, x_n, x_{n+1}) = S(x_n, x_n, x_{n+1}). \quad (3.6)$$

Thus, (3.1) becomes

$$\theta(S(Tx_n, Tx_n, Tx_{n+1})) \leq \phi[\theta(S(x_n, x_n, x_{n+1}))]. \quad (3.7)$$

Repeating this process, we get

$$\begin{aligned} \theta(S(x_n, x_n, x_{n+1})) &= \theta(S(Tx_{n-1}, Tx_{n-1}, Tx_n)) \\ &\leq \phi[\theta(S(x_{n-1}, x_{n-1}, x_n))] \\ &\leq \phi^2[\theta(S(x_{n-2}, x_{n-2}, x_{n-1}))] \\ &\leq \phi^3[\theta(S(x_{n-3}, x_{n-3}, x_{n-2}))] \\ &\leq \dots \leq \phi^n[\theta(S(x_0, x_0, x_1))]. \end{aligned} \quad (3.8)$$

By the definition of θ and $(\Phi 2)$, we have

$$\lim_{n \rightarrow \infty} \phi^n[\theta(S(x_0, x_0, x_1))] = 1. \quad (3.9)$$

By (S1) and $(\Theta 2)$, we obtain

$$\lim_{n \rightarrow \infty} S(x_n, x_n, x_{n+1}) = 0. \quad (3.10)$$

Now, we shall prove that x_n is a Cauchy sequence in X . Suppose, on the contrary, that, there exist a positive real number $\varepsilon_0 > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that, for all $k \in \mathbb{N}$,

$$k < n(k) < m(k), S(x_{n(k)}, x_{n(k)}, x_{m(k)-1}) < \frac{\varepsilon_0}{2} \quad \text{and} \quad \varepsilon_0 \leq S(x_{n(k)}, x_{n(k)}, x_{m(k)}). \quad (3.11)$$

Then

$$\begin{aligned} \varepsilon_0 &\leq S(x_{n(k)}, x_{n(k)}, x_{m(k)}) \\ &\leq 2S(x_{n(k)}, x_{n(k)}, x_{m(k)-1}) + S(x_{m(k)-1}, x_{m(k)-1}, x_{m(k)}) \\ &< \varepsilon_0 + S(x_{m(k)-1}, x_{m(k)-1}, x_{m(k)}). \end{aligned} \quad (3.12)$$

From (3.10) and (3.12) and also, for all given $p_1 = p_2 \in \mathbb{Z}$,

$$\lim_{k \rightarrow \infty} S(x_{n(k)}, x_{n(k)}, x_{m(k)}) = \lim_{k \rightarrow \infty} S(x_{n(k)+p_1}, x_{n(k)+p_1}, x_{m(k)+p_2}) = \varepsilon_0. \quad (3.13)$$

From (3.10), by Pick k large enough,

$$\begin{aligned}
& N(x_{n(k)}, x_{n(k)}, x_{m(k)}) \\
&= \max \left\{ S(x_{n(k)}, x_{n(k)}, x_{m(k)}), S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}), S(x_{m(k)}, x_{m(k)}, x_{m(k)+1}), \right. \\
&\quad \left. \frac{1}{3}S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}), \frac{1}{3}S(x_{n(k)}, x_{n(k)}, x_{m(k)+1}), \frac{1}{3}S(x_{m(k)}, x_{m(k)}, x_{n(k)+1}), \right. \\
&\quad \left. \frac{1}{6}(S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}) + S(x_{n(k)}, x_{n(k)}, x_{m(k)+1}) + S(x_{m(k)}, x_{m(k)}, x_{n(k)+1})) \right\} \\
&= \max \left\{ S(x_{n(k)}, x_{n(k)}, x_{m(k)}), \frac{1}{3}S(x_{m(k)}, x_{m(k)}, x_{n(k)+1}) \right\} \\
&\leq \max \left\{ S(x_{n(k)}, x_{n(k)}, x_{m(k)}), S(x_{m(k)}, x_{m(k)}, x_{n(k)+1}) \right\} \\
&\rightarrow \varepsilon_0 \quad (\text{as } k \rightarrow \infty) \quad (\text{By (3.13)}).
\end{aligned} \tag{3.14}$$

Using the contractivity condition (3.3),

$$\begin{aligned}
\theta(S(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)+1})) &= \theta(S(Tx_{n(k)}, Tx_{n(k)}, Tx_{m(k)})) \\
&\leq \phi[\theta(N(x_{n(k)}, x_{n(k)}, x_{m(k)}))].
\end{aligned} \tag{3.15}$$

Passing to limit as $k \rightarrow \infty$, then we get $\theta(\varepsilon_0) \leq \phi[\theta(\varepsilon_0)]$. By Lemma 2.7, $\phi[\theta(\varepsilon_0)] < \theta(\varepsilon_0)$, then $\theta(\varepsilon_0) \leq \phi[\theta(\varepsilon_0)] < \theta(\varepsilon_0)$, which is a contradiction. Thus, $\{x_n\}$ is a Cauchy sequence in X .

Taking into account the fact that (X, S) is complete, there exists $x^* \in X$ such that $\{x_n\}$ converges to x^* . In particular,

$$\lim_{n \rightarrow \infty} S(x_n, x_n, x^*) = 0 \tag{3.16}$$

Using the fact that S is continuous on each variable,

$$S(x^*, x^*, Tx^*) = \lim_{n \rightarrow \infty} S(x_{n+1}, x_{n+1}, Tx^*). \tag{3.17}$$

We claim that x^* is a fixed point of T . Suppose, on the contrary, if $x^* \neq Tx^*$, then by (3.16), (3.17),

$$\begin{aligned}
N(x_n, x_n, x^*) &= \max \left\{ S(x_n, x_n, x^*), S(x_n, x_n, x_{n+1}), S(x^*, x^*, Tx^*), \right. \\
&\quad \left. \frac{1}{3}S(x_n, x_n, x_{n+1}), \frac{1}{3}S(x_n, x_n, Tx^*), \frac{1}{3}S(x^*, x^*, x_{n+1}), \right. \\
&\quad \left. \frac{1}{6}(S(x_n, x_n, x_{n+1}) + S(x_n, x_n, Tx^*) + S(x^*, x^*, x_{n+1})) \right\} \\
&\rightarrow S(x^*, x^*, Tx^*) \quad (\text{as } n \rightarrow \infty).
\end{aligned} \tag{3.18}$$

Using the contractivity condition (3.3),

$$\theta(S(x_{n+1}, x_{n+1}, Tx^*)) = \theta(S(Tx_n, Tx_n, Tx^*)) \leq \phi[\theta(N(x_n, x_n, x^*))]. \tag{3.19}$$

Passing to limit as $n \rightarrow \infty$, then we have

$$\theta(S(x^*, x^*, Tx^*)) \leq \phi[\theta(S(x^*, x^*, Tx^*))]. \tag{3.20}$$

By Lemma 2.7, $\phi[\theta(S(x^*, x^*, Tx^*))] < \theta(S(x^*, x^*, Tx^*))$. Then

$$\theta(S(x^*, x^*, Tx^*)) \leq \phi[\theta(S(x^*, x^*, Tx^*))] < \theta(S(x^*, x^*, Tx^*)), \quad (3.21)$$

which is a contradiction. As a consequence, we conclude that $Tx^* = x^*$.

Now, we will prove that T has at most one fixed point. Suppose, on the contrary, that there exists another distinct fixed point y^* of T such that $Tx^* = x^* \neq Ty^* = y^*$. Therefore, $S(Tx^*, Tx^*, Ty^*) = S(x^*, x^*, y^*) > 0$, and $N(x^*, x^*, y^*) = S(x^*, x^*, y^*)$, and then by (3.3),

$$\begin{aligned} \theta(S(x^*, x^*, y^*)) &= \theta(S(Tx^*, Tx^*, Ty^*)) \\ &\leq \phi[\theta(N(x^*, x^*, y^*))] \\ &= \phi[\theta(S(x^*, x^*, y^*))]. \end{aligned} \quad (3.22)$$

By Lemma 2.7,

$$\theta(S(x^*, x^*, y^*)) \leq \phi[\theta(S(x^*, x^*, y^*))] < \theta(S(x^*, x^*, y^*)),$$

which is a contradiction. Therefore, the fixed point of T is unique. \square

Corollary 3.3. *Let (X, S) be a complete S -metric space and let $T : X \rightarrow X$ be a self-mapping. Assume that there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that, for any $x, y \in X$,*

$$S(Tx, Tx, Ty) \neq 0 \implies \theta(S(Tx, Tx, Ty)) \leq \phi[\theta(N(x, x, y))], \quad (3.23)$$

where

$$\begin{aligned} N(x, x, y) = \max \left\{ S(x, x, y), S(x, x, Tx), S(y, y, Ty), \frac{1}{3}S(x, x, Ty), \right. \\ \left. \frac{1}{3}S(y, y, Tx), \frac{1}{6}(S(x, x, Ty) + S(y, y, Ty) + S(y, y, Tx)) \right\}. \end{aligned} \quad (3.24)$$

Then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Theorem 3.4. *Let (X, S) be a complete S -metric space and let $T : X \rightarrow X$ be a self-mapping which satisfies the following condition, for all $x, y \in X$,*

$$\begin{aligned} S(Tx, Tx, Ty) \leq \max \left\{ aS(x, x, y), 2b(S(x, x, Tx) + 2S(y, y, Ty)), \right. \\ \left. b(S(x, x, Ty) + S(y, y, Ty) + S(y, y, Tx)) \right\}, \end{aligned} \quad (3.25)$$

where $0 \leq a < 1$ and $0 \leq b < 1/6$. Then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Proof. Let $\lambda = \max\{a, 6b\}$; then $0 \leq \lambda < 1$. And let $\theta(t) = e^t$, $\phi(t) = t^\lambda$; then $\theta \in \Theta$ and $\phi \in \Phi$. Since

$$\begin{aligned}
& \max\left\{aS(x, x, y), 2b(S(x, x, Tx) + 2S(y, y, Ty)), \right. \\
& \quad \left. b(S(x, x, Ty) + S(y, y, Ty) + S(y, y, Tx))\right\} \\
& \leq \lambda \max\left\{S(x, x, y), \frac{1}{3}(S(x, x, Tx) + 2S(y, y, Ty)), \right. \\
& \quad \left. \frac{1}{6}(S(x, x, Ty) + S(y, y, Ty) + S(y, y, Tx))\right\} \quad (3.26) \\
& \leq \lambda \max\left\{S(x, x, y), S(x, x, Tx), S(y, y, Ty), \right. \\
& \quad \left. \frac{1}{6}(S(x, x, Ty) + S(y, y, Ty) + S(y, y, Tx))\right\} \\
& \leq \lambda N(x, x, y).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\theta(S(Tx, Tx, Ty)) &= e^{S(Tx, Tx, Ty)} \leq e^{\lambda N(x, x, y)} \\
&= (e^{N(x, x, y)})^\lambda \quad (3.27) \\
&= \phi(\theta(N(x, x, y))).
\end{aligned}$$

From Corollary 3.3, we can see that T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$. \square

The following Theorem 3.5 is Ćirićs fixed point result [11].

Theorem 3.5. *Let (X, S) be a complete S -metric space and let $T : X \rightarrow X$ be a self-mapping which satisfies the following condition, for all $x, y, z \in X$,*

$$\begin{aligned}
S(Tx, Ty, Tz) &\leq k \max\left\{S(x, y, z), S(x, x, Tx), S(y, y, Ty), S(z, z, Tz), \right. \\
& \quad \left. S(x, x, Ty), S(y, y, Tz), S(z, z, Tx)\right\}, \quad (3.28)
\end{aligned}$$

where $0 \leq k < 1/3$. Then T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.

Proof. Let $\lambda = 3k$; then $0 \leq \lambda < 1$. And let $\theta(t) = e^t$, $\phi(t) = t^\lambda$; then $\theta \in \Theta$ and $\phi \in \Phi$. Since

$$\begin{aligned}
& k \max\left\{S(x, y, z), S(x, x, Tx), S(y, y, Ty), S(z, z, Tz), S(x, x, Ty), \right. \\
& \quad \left. S(y, y, Tz), S(z, z, Tx)\right\} \\
& = \lambda \max\left\{\frac{1}{3}S(x, y, z), \frac{1}{3}S(x, x, Tx), \frac{1}{3}S(y, y, Ty), \right. \\
& \quad \left. \frac{1}{3}S(z, z, Tz), \frac{1}{3}S(x, x, Ty), \frac{1}{3}S(y, y, Tz), \frac{1}{3}S(z, z, Tx)\right\} \quad (3.29) \\
& \leq \lambda N(x, y, z),
\end{aligned}$$

therefore,

$$\begin{aligned}\theta(S(Tx, Ty, Tz)) &= e^{S(Tx, Ty, Tz)} \leq e^{\lambda N(x, y, z)} \\ &= (e^{N(x, y, z)})^\lambda \\ &= \phi(\theta(N(x, y, z))).\end{aligned}\tag{3.30}$$

From Theorem 3.2, we can see that T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$. \square

Theorem 3.6. *Let (X, S) be a complete S -metric space and let $T : X \rightarrow X$ be a self-mapping. Assume that there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that, for any $x, y, z \in X$,*

$$S(Tx, Ty, Tz) \neq 0 \implies \theta(S(Tx, Ty, Tz)) \leq \phi[\theta(S(x, y, z))].\tag{3.31}$$

Then T has a unique fixed point $x^ \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.*

Theorem 3.7. *Let (X, S) be a complete S -metric space and let $T : X \rightarrow X$ be a self-mapping such that there exists $\lambda \in [0, 1)$ satisfying, for any $x, y, z \in X$,*

$$S(Tx, Ty, Tz) \leq \lambda S(x, y, z).\tag{3.32}$$

Then T has a unique fixed point $x^ \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.*

Proof. Let $\theta(t) = e^t$, $\phi(t) = t^\lambda$; then $\theta \in \Theta$ and $\phi \in \Phi$. $S(Tx, Ty, Tz) \leq \lambda S(x, y, z)$ is equivalent to $e^{S(Tx, Ty, Tz)} \leq e^{\lambda S(x, y, z)} = (e^{S(x, y, z)})^\lambda$; that is, $\theta(S(Tx, Ty, Tz)) \leq \phi(\theta(S(x, y, z)))$. From Theorem 3.6, we can see that T has a unique fixed point $x^* \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$. \square

Corollary 3.8. *Let (X, S) be a complete S -metric space and let $T : X \rightarrow X$ be a self-mapping. Assume that there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that, for any $x, y \in X$,*

$$S(Tx, Tx, Ty) \neq 0 \implies \theta(S(Tx, Tx, Ty)) \leq \phi[\theta(S(x, x, y))].\tag{3.33}$$

Then T has a unique fixed point $x^ \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.*

Corollary 3.9 ([6], Theorem 3.1). *Let (X, S) be a complete S -metric space and let $T : X \rightarrow X$ be a self-mapping such that there exists $\lambda \in [0, 1)$ satisfying, for any $x, y \in X$,*

$$S(Tx, Tx, Ty) \leq \lambda S(x, x, y).\tag{3.34}$$

Then T has a unique fixed point $x^ \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.*

Corollary 3.10. *Let (X, S) be a complete S -metric space and let $T : X \rightarrow X$ be a self-mapping. Assume that there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that, for any $x, y, z \in X$,*

$$S(Tx, Ty, Tz) \neq 0 \implies \theta(S(Tx, Ty, Tz)) \leq \phi \left[\theta \left(\frac{S(x, x, Tx) + S(y, y, Ty) + S(z, z, Tz)}{3} \right) \right]. \quad (3.35)$$

Then T has a unique fixed point $x^ \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.*

Corollary 3.11. *Let (X, S) be a complete S -metric space and let $T : X \rightarrow X$ be a self-mapping there exists $\lambda \in [0, 1)$ satisfying, for any $x, y \in X$,*

$$S(Tx, Ty, Tz) \leq \lambda \left(\frac{S(x, x, Tx) + S(y, y, Ty) + S(z, z, Tz)}{3} \right). \quad (3.36)$$

Then T has a unique fixed point $x^ \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.*

Corollary 3.12. *Let (X, S) be a complete S -metric space and let $T : X \rightarrow X$ be a self-mapping. Assume that there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that, for any $x, y, z \in X$, $S(Tx, Tx, Ty) \neq 0$, we have*

$$\theta(S(Tx, Tx, Ty)) \leq \phi \left[\theta \left(\max \left\{ \frac{S(x, x, Tx) + S(y, y, Ty)}{2}, \frac{S(y, y, Ty) + S(z, z, Tz)}{2}, \frac{S(x, x, Tx) + S(z, z, Tz)}{2} \right\} \right) \right]. \quad (3.37)$$

Then T has a unique fixed point $x^ \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.*

Corollary 3.13 ([9], Corollary 2.21). *Let (X, S) be a complete S -metric space and let $T : X \rightarrow X$ be a self-mapping which satisfies the following condition, for all $x, y, z \in X$,*

$$S(Tx, Ty, Tz) \leq k \max \left\{ S(x, x, Tx) + S(y, y, Ty), S(y, y, Ty) + S(z, z, Tz), S(x, x, Tx) + S(z, z, Tz) \right\} \quad (3.38)$$

where $0 \leq k < 1/3$. Then T has a unique fixed point $x^ \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.*

Corollary 3.14. *Let (X, S) be a complete S -metric space and let $T : X \rightarrow X$ be a self-mapping. Assume that there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that, for any $x, y \in X$,*

$$S(Tx, Tx, Ty) \neq 0 \implies \theta(S(Tx, Tx, Ty)) \leq \phi \left[\theta \left(\frac{S(x, x, Tx) + S(y, y, Ty)}{2} \right) \right]. \quad (3.39)$$

Then T has a unique fixed point $x^ \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.*

Corollary 3.15. *Let (X, S) be a complete S -metric space and let $T : X \rightarrow X$ be a self-mapping such that there exists $\lambda \in [0, 1)$ satisfying, for any $x, y \in X$,*

$$S(Tx, Tx, Ty) \leq \lambda \left(\frac{S(x, x, Tx) + S(y, y, Ty)}{2} \right). \quad (3.40)$$

Then T has a unique fixed point $x^ \in X$ such that the sequence $\{T^n x\}$ converges to x^* for every $x \in X$.*

The following theorem is Browders theorem [13] ,

Theorem 3.16. *Let (X, S) be a complete S -metric space and let $T : X \rightarrow X$ be a mapping such that, for all $x, y, z \in X$,*

$$S(Tx, Ty, Tz) \leq \varphi(S(x, y, z)), \quad (3.41)$$

where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is an increasing continuous function such that $\lim_{n \rightarrow \infty} \varphi^n(t) = 0$ for $t > 0$. Then T has a unique fixed point $x^ \in X$ and for every $x \in X$ the sequence $\{T^n x\}$ converges to x^* .*

Proof. Let $\theta(t) = e^t$ for all $t \in [0, \infty)$, and $\phi(t) = e^{\varphi(\ln t)}$ for all $t \in [1, \infty)$. Obviously, $\theta \in \Theta$, $\phi \in \Phi$. By the definition of ϕ , we have $\phi(e^t) = e^{\varphi(t)}$.

$$\begin{aligned} \theta(S(Tx, Ty, Tz)) &= e^{S(Tx, Ty, Tz)} \leq e^{\varphi(S(x, y, z))} \\ &= \phi \left[e^{S(x, y, z)} \right] \\ &= \phi[\theta(S(x, y, z))]. \end{aligned} \quad (3.42)$$

Therefore, from Theorem 3.6, T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}$ converges to x^* . \square

Remark 3.17. *According to fixed point theory of metric spaces, we divide contractions into different type in the setting of S -metrics. Then Theorem 3.7 and Corollary 3.9 belong to Banach type, Theorem 3.5 is Ćirić type, Corollaries 3.10-3.15 are Kannan type [12] and Theorem 3.7 is Browder type [13]. To some extent, our results unify them.*

4 Example

Example 4.1. *Let $X = \mathbb{Z}$ with the usual S -metric given in Example 2.2. Let us define the function $T : X \rightarrow X$ as*

$$Tx = \begin{cases} 0 & \text{if } x = 0 \\ -(n-1) & \text{if } x = n \\ n-1 & \text{if } x = -n \end{cases} \quad (4.1)$$

for all $x \in X$ and $n \in \mathbb{N}$. At first, we observe that Theorem 3.7 and Theorem 3.16 cannot be applied since for all $x = y = n > z = m > 2$, $S(Tx, Ty, Tz) = S(Tn, Tn, Tm) = 2(n - m) = S(n, n, m)$. And Theorem 3.4 cannot be applied too. In fact, let $x = n > 2, y = 0$; then $S(Tx, Tx, Ty) = S(Tn, Tn, T0) = 2(n - 1)$, while

$$\begin{aligned} & \max \left\{ aS(x, x, y), 2b(S(x, x, Tx) + 2S(y, y, Ty)), b(S(x, x, Ty) + S(y, y, Ty) + S(y, y, Tx)) \right\} \\ &= \max \left\{ aS(n, n, 0), 2b(S(n, n, -(n - 1))), b(S(n, n, 0) + S(0, 0, -(n - 1))) \right\} \\ &= \max \left\{ 2an, 4b(2n - 1) \right\} \\ &\leq \max \left\{ 2an, \frac{2}{3}(2n - 1) \right\}, \end{aligned} \tag{4.2}$$

$$S(Tx, Tx, Ty) \leq \max \left\{ aS(x, x, y), 2b(S(x, x, Tx) + 2S(y, y, Ty)), b(S(x, x, Ty) + S(y, y, Ty) + S(y, y, Tx)) \right\},$$

is equivalent to $2(n - 1) \leq \max \left\{ 2an, \frac{2}{3}(2n - 1) \right\}$.

Since $2(n - 1) \leq \max \left\{ 2an, \frac{2}{3}(2n - 1) \right\}$ for all $n > 2$, we have $a = 1$, which yields a contradiction since $a < 1$. By the same way, we can see that Theorem 3.5 cannot be applied. Now, let the function $\theta : (0, \infty) \rightarrow (1, \infty)$ be defined by

$$\theta(t) = 7^t \tag{4.3}$$

And define $\phi : [1, \infty) \rightarrow [1, \infty)$ by

$$\phi(t) = \begin{cases} 1, & \text{if } 1 \leq t \leq 2; \\ t - 1, & \text{if } t > 2. \end{cases} \tag{4.4}$$

Obviously, $\theta \in \Theta, \phi \in \Phi$. In what follows, we prove that T is some $\theta - \phi$ Kannan-type contraction; that is, T satisfies the condition of Corollary 3.14.

We consider three cases.

Case 1 ($x = n \geq 1, y = 0$ or $x = -n(n \geq 1), y = 0$). In this case, we have

$$\begin{aligned} S(Tx, Tx, Ty) &= 2(n - 1), \\ S(x, x, Tx) &= 2(2n - 1), \\ S(y, y, Ty) &= 0, \\ \theta(S(Tx, Tx, Ty)) &= \theta(2(n - 1)) = 7^{2(n-1)}, \\ \phi \left(\theta \left(\frac{S(x, x, Tx) + S(y, y, Ty)}{2} \right) \right) & \tag{4.5} \\ &= \phi \left(\theta \left(\frac{S(x, x, Tx)}{2} \right) \right) = \phi(\theta(2n - 1)) \\ &= \phi(7^{2n-1}) = 7^{2n-1} - 1 = 7 \cdot 7^{2(n-1)} - 1 \\ &\geq 7^{2(n-1)} = \theta(S(Tx, Ty, Ty)). \end{aligned}$$

Case 2 ($x = n > y = m \geq 1$ or $x = -n < y = -m \leq -1$). In this case, we have

$$\begin{aligned}
 S(Tx, Tx, Ty) &= 2(n - m), \\
 S(x, x, Tx) &= 2(2n - 1), \\
 S(y, y, Ty) &= 2(2m - 1), \\
 \phi\left(\theta\left(\frac{S(x, x, Tx) + S(y, y, Ty)}{2}\right)\right) & \\
 &= \phi(\theta(2(n + m - 1))) = \phi(7^{2(n+m-1)}) \\
 &= 7^{4m-2} \cdot 7^{2(n-m)} - 1 \\
 &\geq 7^{2(n-m)} = \theta(S(Tx, Tx, Ty)).
 \end{aligned} \tag{4.6}$$

Case 3 ($x = n, y = -m, n > m \geq 1$ or $x = -n, y = m, n > m \geq 1$). In this case, we have

$$\begin{aligned}
 S(Tx, Tx, Ty) &= 2(n + m - 2), \\
 S(x, x, Tx) &= 2(2n - 1), \\
 S(y, y, Ty) &= 2(2m - 1), \\
 \phi\left(\theta\left(\frac{S(x, x, Tx) + S(y, y, Ty)}{2}\right)\right) & \\
 &= \phi(\theta(2(n + m - 1))) = \phi(7^{2(n+m-1)}) \\
 &= 49 \cdot 7^{2(n+m-2)} - 1 \\
 &\geq 7^{2(n+m-2)} = \theta(S(Tx, Tx, Ty)).
 \end{aligned} \tag{4.7}$$

Therefore, we have for all $x, y \in X$,

$$\begin{aligned}
 &\theta(S(Tx, Tx, Ty)) \\
 &\leq \phi\left[\theta\left(\frac{S(x, x, Tx) + S(y, y, Ty)}{2}\right)\right].
 \end{aligned} \tag{4.8}$$

Thus, T is a θ - ϕ Kannan-type contraction. So all the hypotheses of Corollary 3.14 are satisfied, thus T has a fixed point. In this example $x = 0$ is the fixed point.

References

- [1] Jleli M, Samet B. A new generalization of the Banach contraction principle. Journal of Inequalities and Applications. 2014; 1(38): 8 pages.
- [2] Sedghi S, Shobe N, Dösenović. Fixed point results in S -metric spaces, Non-linear Func. Anal. Appl. 2015; 20(1): 55–67.
- [3] Chaipornjareansri S. Fixed Point Theorems for F_w -Contractions in Complete S -Metric Spaces. Thai J. Math. 2016; 14: 98–109.

- [4] Sedghi S, Gholidahneh A, Dösenović, Esfahani J, Radenović S. Common fixed point of four maps in S_b -metric spaces. *J. Linear Topol. Algebra.* 2016; 5(2): 93–104.
- [5] N. Y. Ozgur N Y, Tas N. Some fixed point theorems on S -metric spaces. *Mat. Vesnik.* 2017; 69(1): 39–52.
- [6] Sedghi S, Shobe N, Aliouche A. A generalization of fixed point theorems in S -metric spaces. *Mat. Vesnik.* 2012; 64(3): 258–266.
- [7] Mustafa Z, Sims B. A new approach to generalized metric spaces. *Journal of Nonlinear and Convex Analysis.* 2006; 7(2): 289–297.
- [8] Sedghi S, Shobe N, Zhou H. A common fixed point theorem in D^* -metric spaces. *Fixed Point Theory Appl.* 2007; 1–13.
- [9] Sedghi S, Dung N V. A generalization of fixed point theorems in S -metric spaces. *Mat. Vesnik.* 2014; 66(1): 258–266.
- [10] Zheng D, Cai Z, Wang P. New fixed point theorems for $\theta - \phi$ contraction in complete metric spaces. *Journal of Nonlinear Sciences and Applications.* 2017; 10(5):2662–2670.
- [11] Ćirić L B. A generalization of Banachs contraction principle. *Proc. Amer. Math. Soc.* 1974; 45: 267–273.
- [12] Kannan R. Some results on fixed points, II. *Amer. Math. Monthly.* 1969; 76:, 405–408.
- [13] Browder F E. On the convergence of successive approximations for nonlinear functional equations. *Indagationes Mathematicae (Proceedings).* 1968; 71: 27–35.

(Received 4 July 2018)

(Accepted 22 November 2018)