Thai Journal of Mathematics : 34-45
Special Issue: Annual Meeting in Mathematics 2018

http://thaijmath.in.cmu.ac.th
Online ISSN 1686-0209

Domination Game Played on a Graph Constructed from 1-Sum of Paths

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Abstract : The domination game consists of two players, Dominator and Staller, who construct a dominating set in a given graph G by alternately choosing a vertex from G, with the restriction that in each turn at least one new vertex must be dominated. Dominator wants to minimize the size of the dominating set, while Staller wants to maximize it. In the game, both play optimally. The game domination number $\gamma_g(G)$ is the number of vertices chosen in the game which Dominator starts, and $\gamma_g'(G)$ is the number of vertices chosen in the game which Staller starts. In this paper these two numbers are analyzed when the game is played on a graph constructed from paths on n vertices, P_n, and on two vertices, P_2, by gluing them together at a vertex. This type of operation is called 1-sum. The motivation behind our research is to study the game domination number of a tree that can be constructed from 1-sum of paths.

Keywords : domination game; dominating set; game domination number; 1-sum; paths.

2010 Mathematics Subject Classification : 05C57; 05C69; 91A43.

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[0]This research project was supported by the Development and Promotion of Science and Technology Talent Project Scholarship (DPST), the Thailand Research Fund (MRG6080201), and Faculty of Science, Mahidol University

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1 Introduction

In this paper, the domination game is played on a finite simple graph G. The domination game was first introduced by Breŝar, Klavẑar and Rall in 2010 [1]. It is basically different from the domination number of a graph G (the minimum size of its dominating set), $\gamma(G)$, although $\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1$, see [1]. The game domination numbers, γ_g and γ_g', of some simple graphs such as paths and cycles are determined in [2, 3]. For a tree T, a connected graph with no cycles, the problem of determining its game domination numbers are non-trivial and the lower bound of $\gamma_g(T)$ is given in terms of the number of vertices and maximum degree of T [4]. To explain the relationship between $\gamma_g(G)$ and $\gamma_g'(G)$ of a graph G, they use imagination strategy, which compares the moves in a real game with an imaginary game both played on G. It is showed in [7] that these two numbers can differ only by 1, $|\gamma_g(G) - \gamma_g'(G)| \leq 1$. We call a pair (k, l) is *realized* by G if $\gamma_g(G) = k$ and $\gamma_g'(G) = l$. Some possible realizable pairs are studied in [1, 4]. All possible realizable pairs are given in [5]. For example, for every k, $(k, k+1)$ can be realized by a tree [4], and for all $k \geq 2$, $(2k, 2k-1)$ can be realized by a class of 2-connected graphs[5]. One way to study the game domination numbers of a graph is by considering graph operations such as deletion of a vertex or of an edge. As proved in [6], for a graph G and an edge e in G, the game domination numbers of G and G deleted e can deffer only by 2, $|\gamma_g(G) - \gamma_g(G-e)| \leq 2$ and $|\gamma_g'(G) - \gamma_g'(G-e)| \leq 2$. The same result holds for deleting a vertex in G.

We can think of a tree as joining paths together at vertices. The operation of combining two graphs by identifying a vertex of one graph with a vertex of another is called the *1-sum*. Then a tree can be constructed from 1-sum of paths. In our paper, we consider the game domination numbers of a tree constructed from 1-sum of a path on n vertices, P_n, and a path on two vertices, P_2. To state our main result we need to define a few graphs. Let $x_1, x_2, ..., x_n$ be vertices of P_n, and let v_1, v_2 be vertices of P_2. We define a graph Q_{n+1}, $n \geq 4$, to be a 1-sum of $P_{n\geq 4}$ and P_2 at x_2 and v_1, see Figure 1.

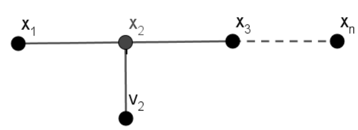

Figure 1: Graph Q_{n+1}

In a graph G, vertices u and v in G are *neighbours* if uv is an edge in G. Let $N[u]$ be the set consisting of u and all its neighbours. Note that a vertex in a graph is called *dominated* if it is chosen or it is a neighbour of the vertex chosen. Let S be a subset of the vertex set of G, $V(G)$. Then a *partially dominated graph* $G|S$ is a subgraph of G where the vertices of S are already dominated. So these vertices do not need to be dominated in the course of the game. The *residual graph* corresponding to $G|S$ is a graph obtained from G by deleting all edges between dominated vertices and all vertices u that cannot be chosen any more, $N[u] \subseteq S$. Our main results are as follows.

Theorem 1.1. $\gamma(Q_{n+1}) \leq 1 + \gamma_g'(Q_{n+1}|N[x_2]) < 1 + \gamma_g'(Q_{n+1}|N[x_3])$.

Theorem 1.2. $\gamma_g'(Q_{n+1}) \geq 1 + \gamma_g(Q_{n+1}|N[x_3])$.

Theorem 1.3. *In a Staller-start game played on Q_{n+1}, for $n \equiv 3 \pmod 4$, if the Staller first move is v_2, then Dominator cannot choose x_4.*

For the rest of the paper, we start with introducing our tools used in our proofs. Then we analyze domination games played on Q_{n+1}. Finally, we consider a Dominator-start game played on 1-sum of P_n and P_2.

2 Basic Lemmas

In this section, we introduce our main tools, which are the continuation principal, properties of realization, and formulas involving the game domination numbers of a path P_n.

Theorem 2.1 (Continuation Principle)**.** *[7] Let G be a graph and $A, B \subseteq V(G)$. If $B \subseteq A$, then $\gamma_g(G|A) \leq \gamma_g(G|B)$ and $\gamma_g'(G|A) \leq \gamma_g'(G|B)$.*

The next theorem shows the relation between the game domination numbers.

Theorem 2.2. *[7] For any graph G, $|\gamma_g(G) - \gamma_g'(G)| \leq 1$.*

Suppose that $\gamma_g(G) = k$ and $\gamma_g'(G) = m$. Theorem 2.2 implies that the realization of G is $(k, k), (k, k+1), (k, k-1)$, where $m = \{k-1, k, k+1\}$. We call *equal* for the case (k, k), *plus* for the case $(k, k+1)$, and *minus* for the case $(k, k-1)$. If G is a family of forests, then the realization is (k, k) or $(k, k+1)$.

Theorem 2.3. *[1, 7] Forests are no-minus graphs.*

If the disjoint union of no-minus graphs has at least one equal graph (component), then the following holds.

Theorem 2.4. *[8] Let $G_1|S_1$ and $G_2|S_2$ be partially dominated no-minus graphs. If $G_1|S_1$ realizes (k, k) and $G_2|S_2$ realizes (l, m) (where $m \in l, l+1$), then the disjoint union $(G_1 \cup G_2)|(S_1 \cup S_2)$ realizes $(k+l, k+m)$.*

In the case that both components of a no-minus graph are plus, the following statement holds.

Theorem 2.5. *[8] Let $G_1|S_1$ and $G_2|S_2$ be partially dominated no-minus graphs such that $G_1|S_1$ realizes $(k, k+1)$ and $G_2|S_2$ realizes $(l, l+1)$. Then*

$$k + l \leq \gamma_g((G_1 \cup G_2)|(S_1 \cup S_2)) \leq k + l + 1,$$

$$k + l + 1 \leq \gamma_g'((G_1 \cup G_2)|(S_1 \cup S_2)) \leq k + l + 2.$$

Let P''_n denote the partially dominated path of order $n+2$, which its endpoints are dominated, see Figure 2, and let P'_n denote the partially dominated path of order $n+1$, which only one of its endpoint is dominated, see Figure 2. In both cases, n vertices are not dominated. The following is an important lemma involving the proof of the game domination numbers of a path.

Figure 2: Partially dominated paths of P''_n (left) and P'_n (right)

Lemma 2.6. *[3] For every $n \geq 0$, we have*

$$\gamma_g(P''_n) = \begin{cases} \lceil \frac{n}{2} \rceil - 1; & n \equiv 3 \pmod 4, \\ \lceil \frac{n}{2} \rceil; & \textit{otherwise}, \end{cases}$$

$$\gamma'_g(P''_n) = \begin{cases} \lceil \frac{n}{2} \rceil + 1; & n \equiv 2 \pmod 4, \\ \lceil \frac{n}{2} \rceil; & \textit{otherwise}. \end{cases}$$

Moreover, for every $i, j \geq 0$ such that $i + j = n$, $i_r = (i \bmod 4)$ and $j_r = (j \bmod 4)$, we also have

$$\gamma_g(P''_i \cup P''_j) = \begin{cases} \gamma_g(P''_i) + \gamma_g(P''_j); & (i_r, j_r) \in \{0,1\} \times \{0,1,2,3\} \cup \\ & \qquad \{0,1,2,3\} \times \{0,1\}, \\ \gamma_g(P''_i) + \gamma_g(P''_j) + 1; & (i_r, j_r) \in \{2,3\} \times \{2,3\}, \end{cases}$$

$$\gamma'_g(P''_i \cup P''_j) = \begin{cases} \gamma_g(P''_i) + \gamma_g(P''_j); & (i_r, j_r) \in \{0,1\} \times \{0,1\}, \\ \gamma_g(P''_i) + \gamma_g(P''_j) + 1; & (i_r, j_r) \in \{0,1\} \times \{2,3\} \cup \\ & \qquad \{2,3\} \times \{0,1\} \cup \{(2,2)\}, \\ \gamma_g(P''_i) + \gamma_g(P''_j) + 2; & (i_r, j_r) \in \{(2,3), (3,2), (3,3)\}. \end{cases}$$

This lemma shows the optimal first move of both players playing on a partially dominated graph P''_n. Dominator always chooses a vertex distance two from the dominated endpoint, but Staller always choose dominated endpoint. Hence, both players play the same way in P'_n. The following statement holds.

Lemma 2.7. *[3] For every $n, m \geq 0$, we have*

$$\gamma_g(P'_n \cup P'_m) = \gamma_g(P''_n \cup P'_m) = \gamma_g(P''_n \cup P''_m) \text{ and}$$
$$\gamma'_g(P'_n \cup P'_m) = \gamma'_g(P''_n \cup P'_m) = \gamma'_g(P''_n \cup P''_m)$$

We can apply Lemmas 2.6 and 2.7 to determine the game domination number of paths.

Theorem 2.8. *[3] For every $n \geq 0$, we have*

$$\gamma_g(P_n) = \begin{cases} \lceil \frac{n}{2} \rceil - 1; & n \equiv 3 \pmod 4, \\ \lceil \frac{n}{2} \rceil; & \textit{otherwise}, \end{cases}$$
$$\gamma_g'(P_n) = \left\lceil \frac{n}{2} \right\rceil.$$

3 A Dominator-Start Game Played on Q_{n+1}

In this section, we analyze $\gamma_g(Q_{n+1})$. First, we study the game when the Dominator first move is vertex x_3.

Lemma 3.1. *Suppose the Dominator first move is x_3. Then*

$$\gamma_g'(Q_{n+1}|N[x_3]) \geq \begin{cases} \lceil \frac{n}{2} \rceil - 1; & n \equiv 3 \pmod 4, \\ \lceil \frac{n}{2} \rceil; & \textit{otherwise}. \end{cases}$$

Proof. After the Dominator first move at x_3, the residual graph is a disjoint union between graph $P_{x_1x_2v_2}$ and $P'_{n-4\geq 0}$, where $P_{x_1x_2v_2}$ is a path in P_n with the vertex set $\{x_1, x_2, v_2\}$. Notice that $\gamma_g'(Q_{n+1}|N[x_3]) = \gamma_g'(P_{x_1x_2v_2} \cup P'_{n-4})$. We calculate the game domination number directly ,and obtain that $\gamma_g(P_{x_1x_2v_2}) = 1$ and $\gamma_g'(P_{x_1x_2v_2}) = 2$. So $P_{x_1x_2v_2}$ is a plus graph. We now consider $\gamma_g'(P_{x_1x_2v_2} \cup P'_{n-4})$. If P'_{n-4} is a plus graph where $n-4 \equiv 2, 3 \pmod 4$, then the residual graph is a disjoint union between plus graphs $P_{x_1x_2v_2}$ and P'_{n-4}. By Theorem 2.5, we have

$$\begin{aligned} \gamma_g'(P_{x_1x_2v_2} \cup P'_{n-4}) &\geq \gamma_g(P_{x_1x_2v_2}) + \gamma_g(P'_{n-4}) + 1 \\ &\geq 2 + \gamma_g(P'_{n-4}). \end{aligned}$$

If P'_{n-4} is an equal graph where $n-4 \equiv 0, 1 \pmod 4$, then the residual graph is a disjoint union between plus and equal graphs. By Theorem 2.4, we have

$$\begin{aligned} \gamma_g'(P_{x_1x_2v_2} \cup P'_{n-4}) &= \gamma_g'(P_{x_1x_2v_2}) + \gamma_g'(P'_{n-4}) \\ &= \gamma_g(P_{x_1x_2v_2}) + 1 + \gamma_g(P'_{n-4}) \\ &= 2 + \gamma_g(P'_{n-4}). \end{aligned}$$

We can easily check by hand for the case $n = 4$. Suppose that $n \geq 5$, we consider four cases according to the value of n mod 4. We apply Lemmas 2.6 and 2.7 to obtain the solution for all $k \geq 1$ as follows.

$$\begin{aligned} \gamma_g'(P_{x_1x_2v_2} \cup P'_{4(k-1)}) &= 2 + \gamma_g(P'_{4(k-1)}) \\ &= 2 + \gamma_g(P''_{4(k-1)}) \\ &= 2 + 2k - 2 = 2k, \end{aligned}$$

$$\begin{aligned}\gamma_g'(P_{x_1x_2v_2}\cup P'_{4(k-1)+1}) &= 2+\gamma_g(P'_{4(k-1)+1})\\ &= 2+\gamma_g(P''_{4(k-1)+1})\\ &= 2+2k-2+1 = 2k+1,\end{aligned}$$

$$\begin{aligned}\gamma_g'(P_{x_1x_2v_2}\cup P'_{4(k-1)+2}) &\geq 2+\gamma_g(P'_{4(k-1)+2})\\ &\geq 2+2(k-1)+1 = 2k+1,\end{aligned}$$

$$\begin{aligned}\gamma_g'(P_{x_1x_2v_2}\cup P'_{4(k-1)+3}) &\geq 2+\gamma_g(P'_{4(k-1)+3})\\ &\geq 2+2(k-1)+2-1 = 2k+1.\end{aligned}$$

□

Notice that $\gamma_g(Q_{n+1}) = 1 + \min_{x\in Q_{n+1}} \{\gamma_g'(Q_{n+1}|N[x])\}$. We obtain this equality when x is the Dominator first move in an optimal strategy. Since it does not guarantee that the Dominator first move at x_3 is an optimal strategy, we obtain the following corollary.

Corollary 3.1. $\gamma(Q_{n+1}) \leq 1+\gamma_g'(Q_{n+1}|N[x_3])$.

Next, we consider the game domination number on graph Q_{n+1} after the Dominator first move choosing vertex x_2.

Lemma 3.2. *If the Dominator first move is x_2, then*

$$\gamma_g'(Q_{n+1}|N[x_2]) = \begin{cases}\lceil \frac{n}{2}\rceil - 2; & n\equiv 3 \pmod 4,\\ \lceil \frac{n}{2}\rceil - 1; & \textit{otherwise.}\end{cases}$$

Proof. Suppose that the Dominator first move is x_2. Then vertices x_1, x_2, x_3, v_2 are all dominated, and the residual graph is P'_{n-3}. We consider four cases according to the value of n *mod* 4. Let $k\geq 1$. By Lemmas 2.6 and 2.7, we obtain that

$$\begin{aligned}\gamma_g'(P'_{4(k-1)+1}) &= \gamma_g'(P''_{4(k-1)+1})\\ &= \gamma_g(P''_{4(k-1)+1})\\ &= 2(k-1)+1 = 2k-1,\end{aligned}$$

$$\begin{aligned}\gamma_g'(P'_{4(k-1)+2}) &= \gamma_g'(P''_{4(k-1)+2})\\ &= \gamma_g(P''_{4(k-1)+2})+1 = 2k,\end{aligned}$$

$$\begin{aligned}\gamma_g'(P'_{4(k-1)+3}) &= \gamma_g'(P''_{4(k-1)+3})\\ &= \gamma_g(P''_{4(k-1)+3})+1 = 2k,\end{aligned}$$

$$\begin{aligned}\gamma'_g(P'_{4k}) &= \gamma'_g(P''_{4k}) \\ &= \gamma_g(P''_{4k}) = 2k.\end{aligned}$$

□

Proof of Theorem 1.1. We compare $\gamma'_g(Q_{n+1}|N[x_2])$ and $\gamma'_g(Q_{n+1}|N[x_3])$. Since $\gamma'_g(Q_{n+1}|N[x_2]) < \gamma'_g(Q_{n+1}|N[x_3])$, we obtain the result. □

We now consider some vertices which are not the Dominator first move in an optimal strategy.

Lemma 3.3. *In an optimal strategy of the Dominator-start game played on Q_{n+1}, the Dominator first move cannot be x_1, x_3, x_n and v_2.*

Proof. We know that $N[x_1]$ and $N[v_2]$ are subsets of $N[x_2]$, and $\{x_n\}$ is a subset of $N[x_{n-1}]$. By the continuation principle and Theorem 1.1, the result follows. □

From our analysis, we propose the following conjecture. In an optimal strategy of the Dominator-start game played on Q_{n+1}, the Dominator first move is x_2. Then

$$\begin{aligned}\gamma_g(Q_{n+1}) &= 1 + \gamma'_g(Q_{n+1}|N[x_2]) \\ &= \begin{cases} \lceil \frac{n}{2} \rceil - 1; & n \equiv 3 \pmod 4, \\ \lceil \frac{n}{2} \rceil; & \textit{otherwise.} \end{cases}\end{aligned}$$

4 A Staller-Start Game Played on Q_{n+1}

In this part, we consider the Staller-start game domination number on graph Q_{n+1}.

Lemma 4.1. *If the Staller first move is x_3, then*

$$\gamma_g(Q_{n+1}|N[x_3]) = \begin{cases} \lceil \frac{n}{2} \rceil - 1; & n \equiv 0, 1 \pmod 4, \\ \lceil \frac{n}{2} \rceil - 1 \textit{ or } \lceil \frac{n}{2} \rceil; & n \equiv 2 \pmod 4, \\ \lceil \frac{n}{2} \rceil - 1 \textit{ or } \lceil \frac{n}{2} \rceil - 2; & n \equiv 3 \pmod 4. \end{cases}$$

Proof. Suppose that the Staller first move is x_3. Then the residual graph is a disjoint union between $P_{x_1x_2v_2}$ and $P'_{n-4\geq 0}$, where $P_{x_1x_2v_2}$ is a path in P_n with the vertex set $\{x_1, x_2, v_2\}$. Notice that $\gamma_g(Q_{n+1}|N[x_3]) = \gamma_g(P_{x_1x_2v_2} \cup P'_{n-4})$. We can find the game domination number directly from the graph; $\gamma_g(P_{x_1x_2v_2}) = 1$ and $\gamma'_g(P_{x_1x_2v_2}) = 2$. So $P_{x_1x_2v_2}$ is a plus graph. Nest we find $\gamma_g(P_{x_1x_2v_2} \cup P'_{n-4})$. If P'_{n-4} is a plus graph, where $n-4 \equiv 2, 3 \pmod 4$, then the residual graph is a disjoint union between plus graphs $P_{x_1x_2v_2}$ and P'_{n-4}. By Theorem 2.5, we have that

$$\gamma_g(P_{x_1x_2v_2}) + \gamma_g(P'_{n-4}) \leq \gamma_g(P_{x_1x_2v_2} \cup P'_{n-4}) \leq \gamma_g(P_{x_1x_2v_2}) + \gamma_g(P'_{n-4}) + 1$$
$$1 + \gamma_g(P'_{n-4}) \leq \gamma_g(P_{x_1x_2v_2} \cup P'_{n-4}) \leq 2 + \gamma_g(P'_{n-4}).$$

If P'_{n-4} is an equal graph, where $n-4 \equiv 0, 1 \pmod 4$, then the residual graph is a disjoint union between plus graph and equal graph. By Theorem 2.4, we have that

$$\begin{aligned}\gamma_g(P_{x_1x_2v_2} \cup P'_{n-4}) &= \gamma_g(P_{x_1x_2v_2}) + \gamma_g(P'_{n-4}) \\ &= 1 + \gamma_g(P'_{n-4}).\end{aligned}$$

It can be easily checked for $n = 4$. Assume that $n \geq 5$. There are four cases according to the value of $n \mod 4$. Then we apply Lemmas 2.6 and 2.7 to obtain the solution for all $k \geq 1$.

$$\begin{aligned}\gamma_g(P_{x_1x_2v_2} \cup P'_{4(k-1)}) &= 1 + \gamma_g(P'_{4(k-1)}) \\ &= 1 + \gamma_g(P''_{4(k-1)}) \\ &= 1 + 2k - 2 = 2k - 1,\end{aligned}$$

$$\begin{aligned}\gamma_g(P_{x_1x_2v_2} \cup P'_{4(k-1)+1}) &= 1 + \gamma_g(P'_{4(k-1)+1}) \\ &= 1 + \gamma_g(P'_{4(k-1)+1}) \\ &= 1 + 2k - 2 + 1 = 2k,\end{aligned}$$

$$\begin{aligned}1 + \gamma_g(P'_{4(k-1)+2}) \leq \gamma_g(P_{x_1x_2v_2} \cup P'_{4(k-1)+2}) &\leq 2 + \gamma_g(P'_{4(k-1)+2}) \\ 1 + 2k - 2 + 1 \leq \gamma_g(P_{x_1x_2v_2} \cup P'_{4(k-1)+2}) &\leq 2 + 2k - 2 + 1 \\ 2k \leq \gamma_g(P_{x_1x_2v_2} \cup P'_{4(k-1)+2}) &\leq 2k + 1,\end{aligned}$$

$$\begin{aligned}1 + \gamma_g(P'_{4(k-1)+3}) \leq \gamma_g(P_{x_1x_2v_2} \cup P'_{4(k-1)+3}) &\leq 2 + \gamma_g(P'_{4(k-1)+3}) \\ 1 + 2k - 2 + 1 \leq \gamma_g(P_{x_1x_2v_2} \cup P'_{4(k-1)+3}) &\leq 2 + 2k - 2 + 1 \\ 2k \leq \gamma_g(P_{x_1x_2v_2} \cup P'_{4(k-1)+3}) &\leq 2k + 1.\end{aligned}$$

□

Proof of Theorem 1.2. We know that

$$\gamma_g(Q_{n+1}) = 1 + \max_{x \in V(Q_{n+1})} \{\gamma'_g(Q_{n+1}|N[x])\}.$$

We can obtain this equality when x is the Staller first move in an optimal strategy. Since it does not guarantee that the Staller first move at x_3 is an optimal strategy, we obtain the result. □

We next consider the Staller-start game domination number when Staller chooses v_2 and Dominator chooses x_2.

Lemma 4.2. *If the Staller first move is v_2 and the next move by Dominator is x_2, then*

$$\gamma_g'(Q_{n+1}|N[v_2,x_2]) = \begin{cases} \lceil \frac{n}{2} \rceil - 2; & n \equiv 3 \pmod 4, \\ \lceil \frac{n}{2} \rceil - 1; & \textit{otherwise}. \end{cases}$$

Proof. Suppose that the first move of Staller is v_2, then x_2 is dominated. If Dominator chooses x_2, then $\gamma_g(Q_{n+1}|N[v_2]) = 1 + \gamma_g'(Q_{n+1}|N[v_2,x_2])$. The corresponding residual graph is P_r', where $r \geq 0$ and $r = n-3$. We have that $\gamma_g'(P_{n-3}') = \gamma_g'(Q_{n+1}|N[v_2,x_2])$. There are four cases according to the value of n *mod* 4. For $k \geq 1$, by Lemmas 2.6 and 2.7, we have that

$$\begin{aligned}\gamma_g'(P_{4(k-1)+1}') &= \gamma_g'(P_{4(k-1)+1}'') \\ &= \gamma_g(P_{4(k-1)+1}'') \\ &= 2(k-1)+1 = 2k-1,\end{aligned}$$

$$\begin{aligned}\gamma_g'(P_{4(k-1)+2}') &= \gamma_g'(P_{4(k-1)+2}'') \\ &= \gamma_g(P_{4(k-1)+2}'') + 1 = 2k,\end{aligned}$$

$$\begin{aligned}\gamma_g'(P_{4(k-1)+3}') &= \gamma_g'(P_{4(k-1)+3}'') \\ &= \gamma_g(P_{4(k-1)+3}'') + 1 = 2k,\end{aligned}$$

$$\begin{aligned}\gamma_g'(P_{4k}') &= \gamma_g'(P_{4k}'') \\ &= \gamma_g(P_{4k}'') = 2k.\end{aligned}$$

□

We assume that the Dominator first move is x_4 in the Staller-start game.

Lemma 4.3. *In the Staller-start game, if the Staller first move is v_2 and the Dominator first move is x_4, then*

$$\gamma_g'(Q_{n+1}|N[v_2,x_4]) = \begin{cases} \lceil \frac{n}{2} \rceil - 2; & n \equiv 1 \pmod 4, \\ \lceil \frac{n}{2} \rceil - 1; & \textit{otherwise}. \end{cases}$$

Proof. Suppose that the Staller first move is v_2 and the Dominator first move is x_4. Then $\gamma_g(Q_{n+1}|N[x_4]) = 1 + \gamma_g'(Q_{n+1}|N[v_2,x_4])$. The corresponding residual graph is $P_1' \cup P_r'$, where $r \geq 0$ and $r+1 = n-4$. We have that $\gamma_g'(P_1' \cup P_{n-3}') = \gamma_g'(Q_{n+1}|N[v_2,x_4])$. There are four cases according to the value of n *mod* 4. For $k \geq 1$, by Lemmas 2.6 and 2.7, we have that

$$\begin{aligned}\gamma_g'(P_1' \cup P_{4(k-2)+3}') &= \gamma_g(P_1'') + \gamma_g(P_{4(k-2)+3}'') + 1 \\ &= 2 + \gamma_g(P_{4(k-2)+3}'') \\ &= 2 + 2(k-2) + 2 - 1 \\ &= 2 + 2k - 4 + 1 = 2k-1,\end{aligned}$$

$$\begin{aligned}\gamma_g'(P_1' \cup P_{4(k-1)}') &= \gamma_g(P_1'') + \gamma_g(P_{4(k-1)}'') \\ &= 1 + \gamma_g(P_{4(k-1)}'') \\ &= 1 + 2k - 2 = 2k - 1,\end{aligned}$$

$$\begin{aligned}\gamma_g'(P_1' \cup P_{4(k-1)+1}') &= \gamma_g(P_1'') + \gamma_g(P_{4(k-1)+1}'') \\ &= 1 + \gamma_g(P_{4(k-1)+1}'') \\ &= 1 + 2k - 2 + 1 = 2k,\end{aligned}$$

$$\begin{aligned}\gamma_g'(P_1' \cup P_{4(k-1)+2}') &= \gamma_g(P_1'') + \gamma_g(P_{4(k-1)+2}'') + 1 \\ &= 2 + \gamma_g(P_{4(k-1)+2}'') \\ &= 2 + 2k - 2 + 1 = 2k + 1.\end{aligned}$$

□

Proof of Theorem 1.3. Note that for $u \in Q_{n+1}$,

$$\gamma_g(Q_{n+1}|N[u]) = 1 + \min_{v \in V(Q_{n+1})-u} \{\gamma_g'(Q_{n+1}|N[u,v])\}.$$

From Lemmas 4.2 and 4.3, for $n \equiv 3 \pmod 4$,

$$\gamma_g'(Q_{n-1}|N[v_2, x_2]) = \left\lceil \frac{n}{2} \right\rceil - 2 < \left\lceil \frac{n}{2} \right\rceil - 1 = \gamma_g'(Q_{n-1}|N[v_2, x_4]).$$

So for Dominator, choosing x_2 is better than choosing x_4. □

From our analysis, we propose the following conjecture. In an optimal strategy of the Staller-start game, if the Staller first move is v_2 and the Dominator first move is x_{n-1}, then

$$\gamma_g'(Q_{n+1}) = 1 + \begin{cases} \lceil \frac{n}{2} \rceil; & n \equiv 1, 3 \pmod 4, \\ \lceil \frac{n}{2} \rceil + 1; & \textit{otherwise.} \end{cases}$$

5 A Dominator-Start Game Played on 1-sum of P_n and P_2

In this section, we analyze the game domination number on a graph T_{n+1}, which is a graph constructed from 1-sum of P_n and P_2 at x_k, for some $k = 2, ..., n-1$, and v_1, see figure 3. Then we find the upper bound of $\gamma_g(T_{n+1})$ by assuming that the Dominator first move is x_k. By applying Lemmas 2.6 and 2.7, we obtain the following lemma.

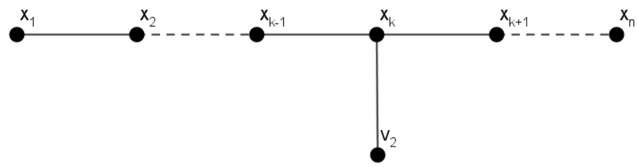

Figure 3: Graph T_{n+1}

Lemma 5.1. *If $k \equiv 0$ (mod 4), then*

$$\gamma_g(T_{n+1}) \leq \begin{cases} \lceil \frac{n}{2} \rceil; & n \equiv 0 \text{ (mod 4)}, \\ \lceil \frac{n}{2} \rceil + 1; & n \equiv 1 \text{ (mod 4)}, \\ \lceil \frac{n}{2} \rceil + 2; & n \equiv 2, 3 \text{ (mod 4)}. \end{cases}$$

If $k \equiv 1$ (mod 4), then

$$\gamma_g(T_{n+1}) \leq \begin{cases} \lceil \frac{n}{2} \rceil; & n \equiv 0 \text{ (mod 4)}, \\ \lceil \frac{n}{2} \rceil + 1; & n \equiv 1, 2 \text{ (mod 4)}, \\ \lceil \frac{n}{2} \rceil + 2; & n \equiv 3 \text{ (mod 4)}. \end{cases}$$

If $k \equiv 2$ (mod 4), then $\gamma_g(T_{n+1}) \leq \lceil \frac{n}{2} \rceil + 1$.
If $k \equiv 3$ (mod 4), then

$$\gamma_g(T_{n+1}) \leq \begin{cases} \lceil \frac{n}{2} \rceil + 1; & n \equiv 0, 1 \text{ (mod 4)}, \\ \lceil \frac{n}{2} \rceil + 2; & n \equiv 2, 3 \text{ (mod 4)}. \end{cases}$$

Proof. We can easily check by hand for the case $n = 4$. Assume that $n \geq 5$. Suppose that Dominator chooses x_k in the first move, then the residual graph is $P'_r \cup P'_s$, where $r + s = n - 3$. We now consider the following cases of the residual graph according to the value of n mod 4.

If $n \equiv 0$ (mod 4) or $n = 4j$, where $j > 0$, there are two cases: 1) $P'_{4l} \cup P'_{4m+1}$ where $l + m + 1 = j$ and $l, m \geq 0$; and 2) $P'_{4l+2} \cup P'_{4m+3}$ where $l + m + 2 = j$ and $l, m \geq 0$.

If $n \equiv 1$ (mod 4) or $n = 4j + 1$, where $j > 0$, there are three cases: 1) $P'_{4l+3} \cup P'_{4m+3}$ where $l+m+2 = j$ and $l, m \geq 0$; 2) $P'_{4l} \cup P'_{4m+2}$ where $l+m+1 = j$ and $l, m \geq 0$; and 3) $P'_{4l+1} \cup P'_{4m+1}$ where $l + m + 1 = j$ and $l, m \geq 0$.

If $n \equiv 2$ (mod 4) or $n = 4j+2$, where $j > 0$, there are two cases: 1) $P'_{4l} \cup P'_{4m+3}$ where $l + m + 1 = j$ and $l, m \geq 0$; and 2) $P'_{4l+1} \cup P'_{4m+2}$ where $l + m + 1 = j$ and $l, m \geq 0$.

If $n \equiv 3$ (mod 4) or $n = 4j+3$, where $j > 0$, there are three cases: 1) $P'_{4l} \cup P'_{4m}$ where $l + m = j$ and $l, m \geq 0$; 2) $P'_{4l+1} \cup P'_{4m+3}$ where $l + m + 1 = j$ and $l, m \geq 0$; and 3) $P'_{4l+2} \cup P'_{4m+2}$ where $l + m + 1 = j$ and $l, m \geq 0$.

By applying Lemmas 2.6 and 2.7 to consider each cases, the result follows. □

Acknowledgement(s) : The authors would like to thank the referees for their comments and suggestions on the manuscript. This research project was supported by the Development and Promotion of Science and Technology Talent Project Scholarship (DPST), the Thailand Research Fund (TRF grant MRG6080201), and Faculty of Science, Mahidol University.

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(Received 24 June 2018)
(Accepted 22 November 2018)