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Edelstein Type L-fuzzy Fixed Point Theorems

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Abstract : In this manuscript, we introduced some L-fuzzy contractive mappings and established some L-fuzzy fixed points results for L-fuzzy contractive and L-fuzzy locally contractive mappings on a compact metric spaces and compact connected metric spaces respectively. Our results extend some interesting results in the literature, we also presented an example to support our findings.

Keywords : Fuzzy sets; L-fuzzy sets; Fixed points; L-fuzzy fixed points; Contractive mapping; L-fuzzy mapping.

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1 Introduction

In 1965, Zadeh [22] initiated the development of the modified set theory known as fuzzy set theory, which is a tool that makes possible the description of vagueness and imprecision. Later in 1967, Goguen [12] makes a generalization of the fuzzy set theory by replacing the interval $[0, 1]$ with a lattice L.

The study of fixed points theorems in fuzzy mathematics was initiated by Weiss [21]. Heilpern [13] introduced the concept of fuzzy contraction mappings and established a fixed point theorem for fuzzy contraction mappings in a complete metric linear spaces, which is a fuzzy extension of Banach contraction principle [8] and Nadler's fixed point theorem [14].

In [9], [10] Edelstein established a generalization of Banach contraction principle for contractive mappings as follows.

Theorem 1.1. *Let X be a compact metric space and $T : X \longrightarrow X$ satisfy*

$$d(Tx, Ty) < d(x, y), \text{ for all } x, y \in X \text{ with } x \neq y.$$

Then there exist a unique $x \in X$ such that $x = Tx$.

Afterwards, several authors (see [3, 6, 11, 15, 16, 17, 18] and references therein) studied fixed point theorems for fuzzy generalized contractive mappings.

Frigon and Regan [11] generalized the Heilpern theorem under a contractive condition for 1-level sets of a fuzzy contraction on a complete metric space, where the 1-level sets need not be convex or compact. And later Azam and Beg [7] obtained a common α-fuzzy fixed point of a pair of fuzzy mappings on a complete metric space under a generalized contractive condition for α-level sets via Hausdorff metric for fuzzy sets, which generalized the results proved by Azam and Arshad [4] among others.

In 2009, Azam et al. [5] presented some fixed point theorems for fuzzy mappings under Edelstein locally contractive conditions on a compact metric space using the d_∞-metric for fuzzy sets.

Recently, Rashid et al. [20] introduced the notions of d_L^∞-metric and Hausdorff distances for L-fuzzy sets to identify a contractive relation between L-fuzzy and crisp mappings, and also presented some fixed point and coincidence theorems. Rashid et al. [19] introduced the concept of β_{α_L}-admissible for a pair of L-fuzzy mappings and establish the existence of common L-fuzzy fixed point theorem.

In this paper, we will establish some fixed point theorems for an L-fuzzy mappings by using contractive conditions in compact connected metric spaces, which is a generalization of [5] and construct an example to support our results.

2 Preliminaries

Throughout this paper, we shall adopt the notations as being recorded in [1, 2, 5, 9, 10, 11, 13, 14, 19, 20]. Let (X, d) be a metric space. Define and denote.

$$CB(X) = \{A : A \text{ is closed and bounded subsets of } X\}$$

$$C(X) = \{A : A \text{ is compact subsets of } X\}$$

For $\epsilon > 0$ and $A, B \in CB(X)$. We define

$$d(x, A) = \inf_{y \in A} d(x, y),$$

$$d(A, B) = \inf_{x \in A, y \in B} d(x, y),$$

$$N(\epsilon, A) = \{x \in X : d(x, y) < \epsilon \text{ for some } a \in A\},$$

$$E_{A,B} = \{\epsilon > 0 : A \subseteq N(\epsilon, B), B \subseteq N(\epsilon, A)\}.$$

Then,

$$H(A, B) = \inf E_{A,B}.$$

Definition 2.1. *A fuzzy set in X is a function with domain X and range in $[0, 1]$. i.e A is a fuzzy set if $A : X \longrightarrow [0, 1]$.*

Let $\mathcal{F}(X)$ denotes the collection of all fuzzy sets of X. If A is a fuzzy set and $x \in X$, then $A(x)$ is called the grade of membership of x in A. The α-level set of A is denoted by $[A]_\alpha$ and is defined as below:

$$[A]_\alpha = \{x \in X : A(x) > 0\} \text{ for } \alpha \in (0, 1],$$
$$[A]_0 = \text{ closure of the set } \{x \in X : A(x) > 0\}.$$

Definition 2.2. (Abdullahi and Azam [2]). *A partially ordered set $(L, \preceq_L)$ is called*

(i) *a lattice: if $a \vee b \in L, a \wedge b \in L$ for any $a, b \in L$;*

(ii) *a complete lattice: if $\bigvee A \in L, \bigwedge A \in L$ for any $A \subseteq L$;*

(iii) *a distributive lattice: if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for any $a, b, c \in L$;*

(iv) *a complete distributive lattice: if $a \vee (\bigwedge b_i) = \bigwedge_i (a \wedge b_i)$, $a \wedge (\bigvee_i b_i) = \bigvee_i (a \wedge b_i)$ for any $a, b_i \in L$.*

Definition 2.3. (Goguen [12]). *An L-fuzzy set in X is a function whose domain is X and co-domain is L, where L is a complete distributive lattice with 1_L and 0_L i.e If $A : X \longrightarrow L$, then A is an L-fuzzy set.*

Definition 2.4. (Goguen [12]). *Let L be a lattice, the top and bottom elements of L are 1_L and 0_L respectively, and if $a, b \in L, a \vee b = 1_L$ and $a \wedge b = 0_L$ then b is a unique complement of a denoted by $\acute{a}$.*

Remark 2.5. (Goguen [12]). *If $L = [0, 1]$, then the L-fuzzy set is just the fuzzy set in the original sense by Zadeh [22], which shows that L-fuzzy set is bigger.*

Let $\mathcal{F}_L(X)$ denotes the class of all L-fuzzy sets of X. The α_L-level set of an L-fuzzy set A is denoted as A_{α_L} and defined as below:

$$A_{\alpha_L} = \{x \in X : \alpha_L \preceq_L A(x)\} \text{ for } \alpha_L \in L \backslash \{0_L\},$$

$$A_{0_L} = \overline{\{x \in X : 0_L \preceq_L A(x)\}},$$

where $\overline{B}$ denotes the closure of the set B (Crisp).

For $A, B \in \mathcal{F}(X), A \subset B$ means $A(x) \leq B(x)$ for each $x \in X$. If there exists an $\alpha \in [0, 1]$ such that $[A]_\alpha, [B]_\alpha \in CB(X)$. Then, we define

$$D_\alpha(A, B) = H([A]_\alpha, [B]_\alpha).$$

If $[A]_\alpha, [B]_\alpha \in CB(X)$ for each $\alpha \in [0, 1]$. Then, we define

$$d_\infty(A, B) = \sup_\alpha D_\alpha(A, B).$$

Let X be an arbitrary set, Y be a metric space. A mapping T is called a fuzzy mapping if T is a mapping from X to $\mathcal{F}(Y)$.

A fuzzy mapping T is a fuzzy subset on $X \times Y$ with membership function $T(x)(y)$. The function $T(x)(y)$ is the grade of membership of y in $T(x)$.

Definition 2.6. *A fuzzy set A in a metric linear space V is said to be an approximate quantity if and only if $[A]_\alpha$ is compact and convex in V for each $\alpha \in [0, 1]$ and $\sup_{x \in V} A(x) = 1$. Let $\mathcal{W}(V)$ denotes the collection of all approximate quantities in V.*

For $A, B \in \mathcal{F}_L(X), A \subset B$ means $A(x) \preceq_L B(x)$ for each $x \in X$. If there exists an $\alpha_L \in L \backslash \{0_L\}$ such that $A_{\alpha_L}, B_{\alpha_L} \in CB(X)$. Then, we define

$$D_{\alpha_L}(A, B) = H(A_{\alpha_L}, B_{\alpha_L}).$$

If $A_{\alpha_L}, B_{\alpha_L} \in CB(X)$ for each $\alpha_L \in L \backslash \{0_L\}$. Then, we define

$$d_L^\infty(A, B) = \sup_{\alpha_L} D_{\alpha_L}(A, B).$$

Let X be an arbitrary set, Y be a metric space. A mapping T is called an L-fuzzy mapping if T is a mapping from X to $\mathcal{F}_L(Y)$(i.e class of L-fuzzy subsets of Y).

An L-fuzzy mapping T is an L-fuzzy subset on $X \times Y$ with membership function $T(x)(y)$. The function $T(x)(y)$ is the grade of membership of y in $T(x)$.

Definition 2.7. *An L-fuzzy set A in a metric linear space V is said to be an approximate quantity if and only if A_{α_L} is compact and convex in V, for each $\alpha_L \in L\backslash\{0_L\}$ and $\sup_{x\in V} A(x) = 1_L$. Let $\mathcal{W}_L(V)$ denotes the collection of all approximate quantities in V.*

Definition 2.8. (Azam et. al. [5]). *A mapping $T : X \longrightarrow \mathcal{F}(X)$ is called fuzzy (globally) contraction if there exists $\lambda \in [0, 1)$ such that*

$$d_\infty(Tx, Ty) \leq \lambda\, d(x, y), \text{ for all } x, y \in X.$$

Definition 2.9. (Azam et. al. [5]). *A mapping $T : X \longrightarrow \mathcal{F}(X)$ is called (ϵ, λ) uniformly fuzzy locally contraction if for $x, y \in X, \lambda \in [0, 1)$*

$$d(x, y) < \epsilon \implies d_\infty(Tx, Ty) \leq \lambda\, d(x, y).$$

Definition 2.10. (Azam et. al. [5]). *A mapping $T : X \longrightarrow \mathcal{F}(X)$ is called fuzzy (globally) contractive if for $x, y \in X, x \neq y$*

$$d_\infty(Tx, Ty) < d(x, y).$$

Definition 2.11. (Azam et. al. [5]). *A mapping $T : X \longrightarrow \mathcal{F}(X)$ is called fuzzy locally contractive if to each $x \in X$ there exists an open set U containing x so that if $y, z \in U, y \neq z$*

$$d_\infty(Ty, Tz) < d(y, z).$$

Definition 2.12. *A mapping $T : X \longrightarrow \mathcal{F}_L(X)$ is called an L-fuzzy (globally) contraction if there exists $\lambda \in [0, 1)$ such that*

$$d_L^\infty(Tx, Ty) \leq \lambda\, d(x, y), \text{ for all } x, y \in X.$$

Definition 2.13. *A mapping $T : X \longrightarrow \mathcal{F}_L(X)$ is called (ϵ, λ) uniformly L-fuzzy locally contraction if for $x, y \in X, \lambda \in [0, 1)$*

$$d(x, y) < \epsilon \implies d_L^\infty(Tx, Ty) \leq \lambda\, d(x, y).$$

Definition 2.14. *A mapping $T : X \longrightarrow \mathcal{F}_L(X)$ is called an L-fuzzy (globally) contractive if for $x, y \in X, x \neq y$*

$$d_L^\infty(Tx, Ty) < d(x, y). \tag{2.1}$$

Definition 2.15. *A mapping $T : X \longrightarrow \mathcal{F}_L(X)$ is called an L-fuzzy locally contractive if to each $x \in X$ there exists an open set U containing x so that if $y, z \in U, y \neq z$*

$$d_L^\infty(Ty, Tz) < d(y, z). \tag{2.2}$$

Definition 2.16. (Abdullahi and Azam [2]). *A point $x \in X$ is said to be a fuzzy fixed point of a fuzzy mapping $T : X \longrightarrow \mathcal{F}(X)$ if $\{x\} \subseteq T(x)$.*

Definition 2.17. (Abdullahi and Azam [2]). *A point $z \in X$ is said to be an L-fuzzy fixed point of an L-fuzzy mapping $T : X \longrightarrow \mathcal{F}_L(X)$ if $z \in [Tz]_{\alpha_L}$ for some $\alpha_L \in L\backslash\{0_L\}$.*

Lemma 2.18. (Heilpern [13]). *Let $x \in X, A \in \mathcal{W}(X)$, and $\{x\}$ be a fuzzy set with membership function equal to characteristic function of set $\{x\}$. If $\{x\} \subset A$, then $p_\alpha(x, A) = 0$ for each $\alpha \in [0, 1]$.*

Lemma 2.19. *Let $x \in X, A \in \mathcal{W}_L(X)$, and $\{x\}$ be an L-fuzzy set with membership function equal to characteristic function of set $\{x\}$. If $\{x\} \subset A$, then $p_{\alpha_L}(x, A) = 0_L$ for $\alpha_L \in L\backslash\{0_L\}$.*

Lemma 2.20. (Heilpern [13]). *$p_\alpha(x, A) \leq d(x, y) + p_\alpha(y, A)$ for any $x, y \in X$.*

Lemma 2.21. *$p_{\alpha_L}(x, A) \leq d(x, y) + p_{\alpha_L}(y, A)$ for any $x, y \in X$.*

Lemma 2.22. (Heilpern [13]). *If $\{x_0\} \subset A$, then $p_\alpha(x_0, B) \leq D_\alpha(A, B)$ for each $B \in \mathcal{W}(X)$.*

Lemma 2.23. *If $\{x_0\} \subset A$, then $p_{\alpha_L}(x_0, B) \leq D_{\alpha_L}(A, B)$ for each $B \in \mathcal{W}_L(X)$.*

Lemma 2.24. (Nadler [14]). *Let (X, d) be a metric space and $A, B \in CB(X)$. Then for any $a \in A$ there exists $b \in B$ such that $d(a, b) \leq H(A, B)$.*

Definition 2.25. *Let (X, d) be a metric space, $x, y \in X$ and $\epsilon > 0$. An ϵ-chain from x to y is define as a finite set of points $x_0, x_1, x_2, \ldots, x_n$ such that $x = x_0, y = x_n$ and $d(x_j, x_{j+1}) < \epsilon$ for all $j = 0, 1, 2, \ldots, j - 1$.*

Lemma 2.26. *Let (X, d) be a compact connected metric space, then for each $\epsilon > 0$ and $x, y \in X$, there exists an ϵ-chain from x to y and the mapping $d^\epsilon : X \times X \longrightarrow \mathbb{R}$ define by:*

$$d^\epsilon(x, y) = \inf\{M\}$$

where $M = \{\sum_{j=0}^{n-1} d(x_j, x_{j+1}) : x_0, x_1, x_2, \ldots, x_n$ is an ϵ-chain from x to $y\}$, is a metric on X equivalent to d. Furthermore, for $x, y \in X, \epsilon > 0$ there exists an $\epsilon -$ chain $x = x_0, x_1, x_2, \ldots, x_n = y$ such that

$$d^\epsilon(x, y) = \sum_{j=0}^{n-1} d(x_j, x_{j+1})$$

3 Main Results

Theorem 3.1. *Let (X, d) be a compact metric space and $T : X \longrightarrow \mathcal{F}_L(X)$ be L-fuzzy contractive mapping. Then T has an L-fuzzy fixed point.*

Proof: For $x \in X$. $[Tx]_{\alpha_L}$ is non empty and compact.
Define a mapping $g : X \longrightarrow [0, \infty)$ by $g(x) = p_{\alpha_L}(x, Tx)$. It now implies that,

$$\begin{aligned} g(x) &= p_{\alpha_L}(x, Tx) \\ &\leq d(x, y) + p_{\alpha_L}(y, Tx) \\ &\leq d(x, y) + p_{\alpha_L}(y, Ty) + H([Tx]_{\alpha_L}, [Ty]_{\alpha_L}) \\ &\leq d(x, y) + p_{\alpha_L}(y, Ty) + D_{\alpha_L}(Tx, Ty) \\ &\leq d(x, y) + p_{\alpha_L}(y, Ty) + \sup_{\alpha_L} D_{\alpha_L}(Tx, Ty) \\ &\leq d(x, y) + p_{\alpha_L}(y, Ty) + d_L^{\infty}(Tx, Ty) \end{aligned}$$

Thus,

$$g(x) - g(y) \leq d(x, y) + d_L^{\infty}(Tx, Ty),$$

and by symmetry, we have

$$|g(x) - g(y)| \leq d(x, y) + d_L^{\infty}(Tx, Ty).$$

Hence g is continuous from (2.1) and the above inequality. Since X is compact g attains it's minimum say at a point $x^* \in X$.
By compactness of $[Tx^*]_{\alpha_L}$ we can choose $x_1 \in X$, such that $\{x_1\} \subset T(x^*)$ and $d(x^*, x_1) = p_{\alpha_L}(x^*, Tx^*) = g(x^*)$. Then, $\{x^*\} \subset T(x^*)$. Otherwise $g(x_1) = p_{\alpha_L}(x_1, Tx_1)$ along with Lemma 2.23 will imply that

$$\begin{aligned} g(x_1) &= p_{\alpha_L}(x_1, Tx_1) \\ &\leq D_{\alpha_L}(Tx^*, Tx_1) \\ &\leq \sup_{\alpha_L} D_{\alpha_L}(Tx^*, Tx_1) \\ &\leq d_L^{\infty}(Tx^*, Tx_1) \\ &< d(x^*, x_1) \\ &= p_{\alpha_L}(x^*, Tx^*) = g(x^*). \end{aligned}$$

A contradiction and therefore, $\{x^*\} \subset T(x^*)$. Hence,

$$x^* \in [T(x^*)]_{\alpha_L},\ \alpha_L \in L \backslash \{0_L\},$$

as required.

□

Example 3.2. *Let $X = [0, 1]$, $d(x, y) = |x - y|$ for all $x, y \in X$, then (X, d) is a compact metric space. Let $L = \{\delta, \gamma, \tau\}$ with $\delta \preceq_L \tau$ and $\gamma \preceq_L \tau$, where δ and γ are not comparable, then $(L, \preceq_L)$ is a complete distributive lattice. Define $T : X \longrightarrow \mathcal{F}_L(X)$ as below:*

$$T(x)(t) = \begin{cases} \tau, & \text{if } 0 \leq t \leq \frac{x}{6}; \\ \delta, & \text{if } \frac{x}{6} < t \leq \frac{x}{3}; \\ \gamma, & \text{if } \frac{x}{3} < t \leq 1. \end{cases}$$

For every $x \in X$, $\alpha_L = \tau$ *exists for which*

$$[Tx]_\tau = [0, \frac{x}{6}].$$

Now, for each $x, y \in X$, *we have*

$$\begin{aligned} d_L^\infty(Tx, Ty) &= \sup_{\alpha_L} D_{\alpha_L}(Tx, Ty) \\ &= \sup_{\alpha_L} H([Tx]_{\alpha_L}, [Ty]_{\alpha_L}) \\ &= \sup |\frac{x}{6} - \frac{y}{6}| \\ &\leq \frac{1}{6} \sup |x - y| \\ &\leq \frac{1}{6} \sup d(x, y) \\ &< d(x, y), \end{aligned}$$

which implies that all the conditions of Theorem 3.1 are satisfied. Hence, there exists $0 \in X$ *such that* $0 \in [T0]_\tau$.

Corollary 3.3. *Let* (X, d) *be a compact metric space and* $T : X \longrightarrow \mathcal{F}_L(X)$ *be L-fuzzy contractive mapping. If* $\alpha_L = 1_L$, *then* T *has a fixed point.*

Proof: Since $\alpha_L = 1_L \in L$, by theorem 3.1 it follows that there exists an $x^* \in X$ such that $x^* \in [T(x^*)]_{1_L}$. Thus, implying that $x^* = T(x^*)$ and hence T has a fixed point. □

Theorem 3.4. *Let* (X, d) *be a compact connected metric space and* $T : X \longrightarrow \mathcal{F}_L(X)$ *be L-fuzzy locally contractive mapping. Then* T *has an L-fuzzy fixed point.*

Proof: To begin with, definition 2.15 and Lemma 2.24, implies that for each $x \in X$ which belongs to an open set say U so that if $y, z \in U, y \neq z$, we have

$$H([Ty]_{\alpha_L}, [Tz]_{\alpha_L}) < d(y, z). \tag{3.1}$$

By Lemma 2.26, for each $\epsilon > 0$ and each pair of points say $u, v \in X$, there exists an ϵ-chain $u = x_0, x_1, x_2, \ldots, x_n = v$ from u to v. Also, as X is compact we find $\delta > 0$ so that for $y, z \in U, y \neq z,$ and $d(x, y) < \delta$, then

$$H([Tx]_{\alpha_L}, [Ty]_{\alpha_L}) < d(x, y).$$

Define $d^*(u, v) : X \times X \longrightarrow [0, \infty)$ by

$$d^*(u, v) = \inf\{M\},$$

where $M = \sum_{j=0}^{n-1} d(x_j, x_{j+1})$ such that $x_0, x_1, x_2, \ldots, x_n$ is a $\frac{\delta}{2}$-chain from u to v. Thus, $d^* = d^{\frac{\delta}{2}}$ and d^* is a metric on X which is equivalent to d and there exists a $\frac{\delta}{2}$-chain $u = x_0, x_1, x_2, \ldots, x_n = v$ from u to v such that

$$d^*(u, v) = \sum_{j=0}^{n-1} d(x_j, x_{j+1}). \tag{3.2}$$

Now, $d(x_j, x_{j+1}) \leq \frac{\delta}{2} < \delta$ implies that

$$H([Tx_j]_{\alpha_L}, [Tx_{j+1}]_{\alpha_L}) < d(x_j, x_{j+1}) < \delta,$$

and it further implies that

$$d(x_j, x_{j+1}) - H([Tx_j]_{\alpha_L}, [Tx_{j+1}]_{\alpha_L}) > 0.$$

Let

$$M_j = d(x_j, x_{j+1}) - H([Tx_j]_{\alpha_L}, [Tx_{j+1}]_{\alpha_L}) > 0,$$

for $j = 0, 1, 2, \ldots, j - 1$ and

$$H([Tx_j]_{\alpha_L}, [Tx_{j+1}]_{\alpha_L}) < d(x_j, x_{j+1}) - \frac{M_j}{2}. \tag{3.3}$$

To show that

$$[Tx_0]_{\alpha_L} \subset N^{d^*}(k, [Tx_n]_{\alpha_L}), \text{ for some } k > 0, \tag{3.4}$$

we consider an element (arbitrary) $y_0 \in [Tx_0]_{\alpha_L}$, using (3.3) we may choose $y_1 \in [Tx_1]_{\alpha_L}$ such that

$$d(y_0, y_1) < d(x_0, x_1) - \frac{M_0}{2}.$$

Similarly, we find $y_2 \in [Tx_2]_{\alpha_L}$ such that

$$d(y_1, y_2) < d(x_1, x_2) - \frac{M_1}{2}.$$

Continuing in this way, will yield a set of points $y_0, y_1, y_2, \ldots, y_n$ where $y_j \in [Tx_j]_{\alpha_L}$ such that

$$d(y_j, y_{j+1}) < d(x_j, x_{j+1}) - \frac{M_j}{2},$$

for $j = 0, 1, 2, \ldots, j - 1$. Clearly the points $y_0, y_1, y_2, \ldots, y_n$ forms a $\frac{\delta}{2}$-chain from y_0 to y_n. Hence,

$$\begin{aligned} d^*(y_0, y_n) &\leq \sum_{j=0}^{n-1} d(y_j, y_{j+1}) \\ &< \sum_{j=0}^{n-1} \left(d(x_j, x_{j+1}) - \frac{M_j}{2} \right) \\ &= \sum_{j=0}^{n-1} d(x_j, x_{j+1}) - \sum_{j=0}^{n-1} \frac{M_j}{2}. \end{aligned}$$

Thus, by (3.2) we have

$$d^*(y_0, y_n) < \sum_{j=0}^{n-1} d^*(u,v) - \sum_{j=0}^{n-1} \frac{M_j}{2}.$$

Letting

$$k = d^*(u,v) - \sum_{j=0}^{n-1} \frac{M_j}{2},$$

we have $k > 0$ and $y_0 \in N^{d^*}(k, [Tx_n]_{\alpha_L})$, hence (3.4) holds. We now only remain to show that

$$[Tx_n]_{\alpha_L} \subset N^{d^*}(k, [Tx_0]_{\alpha_L}). \tag{3.5}$$

So, consider an element (arbitrary) $z_n \in [Tx_n]_{\alpha_L}$, again using (3.3) we may choose $z_{n-1} \in [Tx_{n-1}]_{\alpha_L}$ such that

$$d(z_{n-1}, z_n) < d(x_0, x_1) - \frac{M_{n-1}}{2}.$$

Similarly, we find $z_{n-2} \in [Tx_{n-2}]_{\alpha_L}$ such that

$$d(z_{n-2}, z_{n-1}) < d(x_1, x_2) - \frac{M_{n-2}}{2}.$$

Continuing in this way, we have a set of points $z_0, z_1, z_2, \ldots, z_n$ where $z_j \in [Tx_j]_{\alpha_L}$ such that

$$d(z_j, z_{j+1}) < d(x_j, x_{j+1}) - \frac{M_j}{2}.$$

Therefore, $z_n \in N^{d^*}(k, [Tx_0]_{\alpha_L})$, hence (3.5) holds. It now follows from (3.4) and (3.5) that, $k \in E^{d^*}_{[Tx_0]_{\alpha_L}, [Tx_n]_{\alpha_L}}$.
Thus,

$$H^*([Tx_0]_{\alpha_L}, [Tx_n]_{\alpha_L}) < k,$$

and

$$\begin{aligned} H^*([Tu]_{\alpha_L}, [Tv]_{\alpha_L}) &< d^*(u,v) - \sum_{j=0}^{n-1} \frac{M_j}{2} \\ &< d^*(u,v). \end{aligned}$$

Hence

$$D^*_{\alpha_L}(Tx, Ty) < d^*(x,y), \text{ for any } x, y \in X. \tag{3.6}$$

Define a mapping $g : X \longrightarrow [0,\infty)$ by $g(x) = p^*_{\alpha_L}(x, Tx)$. It implies that

$$\begin{aligned} g(u) &= p^*_{\alpha_L}(u, Tu) \\ &\leq d^*(u, v) + p^*_{\alpha_L}(v, Tu) \\ &\leq d^*(u, v) + p^*_{\alpha_L}(v, Tv) + D^*_{\alpha_L}(Tu, Tv). \end{aligned}$$

It further implies that

$$g(u) \leq d^*(u, v) + p^*_{\alpha_L}(v, Tv) + d^*_{\alpha_L}(Tu, Tv). \tag{3.7}$$

Now, from (3.6) and (3.7), we have

$$g(u) \leq d^*(u, v) + p^*_{\alpha_L}(v, Tv) + d^*(u, v).$$

Thus,

$$g(u) - g(v) \leq 2\, d^*(u, v),$$

and by symmetry, we have

$$|g(u) - g(v)| \leq 2\, d^*(u, v).$$

Hence g is continuous. Since X is compact g attains it's minimum say at a point $x^* \in X$ such that $g(x^*) = m = p^*_{\alpha_L}(x^*, Tx^*)$. Otherwise, by compactness of $[Tx^*]_{\alpha_L}$ we can choose $u_1 \in [Tx^*]_{\alpha_L}$, such that $\{u_1\} \subset T(x^*)$, $d(x^*, u_1) = p^*_{\alpha_L}(x^*, Tx^*) = g(x^*) = m$ and

$$g(u_1) = p^*_{\alpha_L}(u_1, Tu_1) \leq D^*_{\alpha_L}(Tx^*, Tu_1).$$

Thus, using (3.6) and the above inequality, we have

$$g(u_1) < d^*(x^*, u_1) = m,$$

which contradicts the fact that m is minimal point of g. Therefore

$$x^* \in [T(x^*)]_{\alpha_L},\ \alpha_L \in L\backslash\{0_L\}.$$

□

Corollary 3.5. *Let (X, d) be a compact connected metric space and $T : X \longrightarrow \mathcal{F}_L(X)$ be L-fuzzy locally contractive mapping. If $\alpha_L = 1_L$, then T has a fixed point.*

Proof: Since $\alpha_L = 1_L \in L$, by theorem 3.4 it follows that T has a fixed point. □

4 Conclusion

In this manuscript, we utilized the idea of L-fuzzy set [12] to introduced some L-fuzzy contractive mappings and furthermore established some L-fuzzy fixed points theorems for L-fuzzy contractive and L-fuzzy locally contractive mappings on a compact metric spaces and a compact connected metric spaces respectively, our results generalized many results in the literature. We also gave an example to support our findings.

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