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A Panorama of Applied Mathematical Problems in Economics

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Abstract : We present a variety of issues in modern economics where applied mathematics is needed. These include economic equilibrium problem where fixed point theory for Markov operators, and for ordered sets is needed; the emergent field of behavioral economics (Nobel Memorial Prize in Economics 2017) where more mathematical research is needed, for example, to capture cognitive decision-making, affecting economic data, and hence econometric modeling processes. We will focus on a promising venue, namely, quantum probability as a measure of uncertainty, since, unlike Kolmogorov probability, quantum probability, borrowed from quantum physics, is non-additive and non-commutative, thus avoiding Ellsberg's paradox (of von Neumann's expected utility) and capturing order effects of human judgements, respectively. Evoking the development of the framework of quantum probability as behavioral probability, as well as a quantum stochastic calculus for, say, financial econometrics, we will discuss various important mathematical research problems which can be classified as applied mathematics.

Keywords : Behavioral economics; economic equilibrium; fixed point theorems; Markov operators, Markov processes; quantum probability.

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1 Introduction

Upfront, this talk is about applied mathematics for mathematicians. Here, by "Applied Mathematics", where the emphasis (or distinction) is on the noun "Mathematics" (not the adjective "Applied"), we mean mathematical research problems arising from specific application domains, and aiming at solving problems The classical example is Newton's calculus for describing quantitatively his second law of motion (in which, when a force is applied to a moving object, the object changes its velocity) and coming back to derive his explicit second law of motion for us to build airplanes!

Another example of developimg new mathematics in science came from the failure of Newton mechanics (based on his calculus and differential equations) to explain the phenomenon of "blackbody radiation" at the beginning of the 20th century, leading von Neumann to develop what we know now as functional analysis, which, in turn, provide us with our i-phones!

This is in sharp constrast with "Mathematical Applications" research, where the emphasis is on "Applications", and the adjective "Mathematical" is just used to remind us that we simply use existing mathematical knowledge to solve some probkems of interest, such as shoppers of groceries using arithmetics.

Needed to say, as G. Hardy once said, pure mathematics has its own grandeur! Just look at the *actual Langlands program!*

How to do applied mathematics? Well, first, we need to have new (open) applications in mind. Remember, in an ideal sense, once the new mathematics has been developed appropriately, we need to come back to solve the original problem!

Where the applied problems came from? Well, as N. Wiener once said, you need to be a scientist, i.e., someone who knows well an application domain (e.g., economics) AND a "tool", e.g., Fourier analysis.

In the following, biased by my own research interests at present, I will only talk about applied mathematics for economics.

The purpose of my talk is simply indicating some potentially useful applied mathematical research issues in economics for mathematicians.

2 A past panorama of applied mathematics

Before talking about what we face ahead in this 21th century, let me entertain you a bit with history which, I think, could be useful for young mathematicians as a guideline to conduct significant applied mathematics.

The point is this. Essentially, mathematics was started as applied mathematics, again, in the sense that it was developed to provide a scientific language to fullfil some applications in mind, especially in physics. This will bring out the obvious fact that today econometrics (a combination of economic theories, mathematics and statistics) should "borrow" concepts and methods from physics, since, like quantum mechanics, economic fluctuations are precisely uncertain, dynamical systems.

While it was said that "*There are two kinds of mathematicians: John von Neumann and the rest of us",* von Neumann gave a good example of doing applied mathematics: from transforming Hilbert's pure mathematics (abstract generalization of Euclidean spaces to an infinitely dimensional setting) to provide the language for quantum mechnanics, around 1927 (Linear operators on Hilbert spaces and their spectral theory), to inventing Monte Carlo method (Statistical simulations) for computing integrals in physics, to suggesting game theory for analyzing economics behavior (social sciences), inaugurating a "rational" human decision theory based upon his Expected Utility Theory, see von Neumann and Morgenstern, 1944. Here is a short list of applied mathematical achievements in the past.

Isaac Newton (1643-1727)

It started out when humans seriously tried to understand things around them, in a scientific way (i.e., using measurements as observations, rather than just "look" and argue philosophically!). The legend mentioned questions such as "why apples fall to the ground, but the moon does not, and does not fly away from earth?"

Newton started wondering about how objects move, and realizing that they move because of forces acting upon them, and when they move, they move with speed, and speed can change, so the "acceleration" exists. His genius second law of motion is "Force = mass times acceleration". Of course, it is not a mathematical theorem (we cannot prove it). It is just an excellent guess. It became a "law" since "it works" as testified by experiments (making good predictions from his law). How to express his law in a "scientific language?" (i.e., mathematics). First, he had to develop what you learn in your first year at universities: The *calculus* (limits, derivatives); next, once an equation like $F = mx''(t)$ is rigorously written in mathematical language, we need to solve it, leading to what you call "*differential equations"*. With all that we have a theory of rational mechanics applying to motions on earth as well as on heaven.

Joseph Louis Lagrange (1736-1813)

Together with other giants (mathematicians) like Euler and Hamilton, Lagrange reformulated Newton's mechanics program. Can we study Newton mechanics without carrying out Newton's program? i.e., without solving differential equations. Of course, there are reasons for asking a such question.

Now, for a moving object, started at an initial state $(t_o, x(t_o) = a)$ (time, position) to be at a future state $(T, x(T) = b)$, the object must take some path $x(t)$ (a continuous function of time) joining the points a and b during the time interval $[t_o, T]$. Which one? Well, Newton will tell us: Solve my differential equation to get it.

Is there some other ways to get the object's trajectory without solving Newton's equation?

Focusing on energy rather than force, Lagrange got the answer: the trajectory of the object is a stationary solution to an optimization problem. Specifically, Let $L = K - V$, where K, V are kinetic and potential energies, respectively of the moving object, and the difference L is called the Lagrangian (in his honor). Note that K is a function of $x'(.)$, whereas V is a function of $x(.)$, alone, so that in "generalized coordinates", $L(x, x')$ is a function of the position and velocity.

An *action* is a functional: $S(.) : \mathcal{P}([t_o, T])$ (space of all paths joining a to b) $\to \mathbb{R}$:

$$S(x) = \int_{t_o}^{T} L(x(t), x'(t))dt$$

The path which makes this functional unchanged (stationary), i.e., an extremum, is precisely Newton's solution. This is referred to as the *Least Action Principle (LAP)*.

The mathematical problem is how to solve this optimization problem (in the 18th century)? Clearly, it is not an optimization on euclidean spaces that you are aware off in Calculus. It is a *functional optimization* problem, where the domain of optimization $\mathcal{P}([t_o, T])$ is an infinitely dimensional set (a function space), where we did not have a concept of derivative with respect to function" yet, i.e., a concept of *functional derivative*. They need to invent one. Together with Euler, Lagrange developed what we know now as the *Calculus of Variations*. Roughly speaking, this new tool implies that stationary solutions are solutions of the Euler-Lagrange's equation:

$$\frac{dS}{dx} = \frac{\partial L}{\partial x} - \frac{d}{dt}(\frac{\partial L}{\partial x'}) = 0$$

i.e., the trajectory of the moving object is the path $x_*(.)$ such that $\frac{dS}{dx}|_{x_*} = 0$.

John von Neumann (1903-1957)

Studied under Hilbert, Von Neumann gave a pleasant surprise to Hilbert. As a "pure" mathematician, Hilbert loved axiomatizations and generalizations. He generalized familiar Euclidean spaces to infinitely dimensional spaces which were called Hilbert spaces by Von Neumann (not Hilbert himself, of course!). They are purely mathematical structures without any applications in his mind. It was simply the transition from vectors to functions. Since a Hilbert space is a vector space, elements of it are called vectors.

Being living in a time where quantum mechanics (motion of particles/ micro-objects) was in full developments (around 1927), Von Neumann realized that Hilbert spaces can be used to represent "quantum states" in Schrodinger's theory, and hence developed *spectral theory of sell adjoint operators on Hilbert spaces*, providing the needed foundations for quantum mechanics, as we know today, and in your course that you call *functional analysis*. Note that Schrodinger's equation is the best "theory" of quantum physics we have, which "works", as you have your

smart phones to use today! Specifically, Schrodinger's equation is a PDE equation of the form

$$ih\frac{\partial\psi(x,t)}{\partial t} = -\frac{h^2}{2m}\Delta_x\psi(x,t) + V(x)\psi(x,t)$$

The solutions (wave functions) $\psi(.,t) \in L^2(\mathbb{R}^3, \mathcal{B}(\mathbb{R}^3), dx)$, a separable, complex Hilbert space. Von Neumann's functional analysis provides all necessary ingredients for quantum mechanics. See, e.g., Dirac [4].

In your elementary calculus course, you learn both derivatives and integral. You did learn functional derivatives (in the Calculus of Variations, e.g. $\frac{dS}{dx}$ above) in your course on Functional Analysis.

One question: How about derivative of set-functions? Of course, I am not talking about Radon-Nikodym derivatives of measures, but a "variational calculus of set-functions", i.e., a notion of derivative of a set function with respect to set to be used for optimization purposes (similar to the functional derivative via the calculus of variations). Well, see Nguyen [11].

This was also an "applied mathematical research" problem, since the motivation for doing it was to find a way to maximize an excess mass function in the process of estimating, nonparametrically, a probability density function.

But did you learn also *functional integrals***?** i.e., integrals over a space of functions? an infinitely dimensional space, as opposed to finitely dimensional euclidean space $\mathbb{R}^d$, in Functional Analysis? Maybe yes, maybe no! Yes, perhaps, e.g., for those who know probability, Wiener's (1894-1964) functional integral based on his Gaussian measure (for assigning a probability value to a set of paths in a Brownian motion).

As you know, integrals can be defined either based on a measure (like Lebesgue) or without a measure (like Riemann). Wiener integral is defined from its Wiener Gaussian measure. A *Riemann-type functional integral* was defined by Richard Feynman in 1948, providing an alternative way to obtain Schrodinger's wave function.

Richard Feynman (1918-1988)

The situation is similar to the reformulation of classical mechanics (of Newton) by Lagrange, via the crucial notion of an "action".

The law of quantum mechanics is given as solutions of the Schrodinger's equation, where a solution $\psi(x,t)$, the wave function, is the "probability amplitude" for a future state (x,t), i.e., the probability density for the position $x \in \mathbb{R}^3$, at a given time t, is $|\psi(x,t)|^2$. Note that, unlike Newtonian mechanics (for macro-objects), there is no concept of trajectories for particles, since, as shown by experiments, under the same initial conditions, a particle can move in different paths. at random. i.e., randomness is intrinsic in quantum mechanics (as opposed to Einstein "Does God play dice?"). As such, any path joining two points a and b in $\mathbb{R}^3$can be taken by the particle, so that the best you can ask is "what is the probability that

the particle will land at x, at a future time t?". The answer given by Schrodinger is the absolute square of its probability amplitude.

The Schrodinger's equation plays the role of Newton's equation in classical mechanics. What is its counterpart of Lagrangian? Can a reformulation, based on Lagrangian, be found to provide a solution of the Schrodinger's equation, without solving it?

Recall that the Lagrangian $L = K - V$ of a system "determines" the time evolution of the object via the actions $S(x)$ upon possible paths $x(.)$. Now, by the nature of quantum mechanics, we are not seeking to identify just one path, but rather we consider all possible paths, i.e., the space $\mathcal{P}([t_o, T])$. Since the particle could take any path, but at random, so that there should be a probability (in fact, a probability amplitude) attached to each path, and considering all of them could provide the overall probability amplitude for a state (T, b), similar to the value provided by Schrodinger wave function $\psi(T, b)$.

That was Feynman's idea. How to carry out his program? The first step is to find out what is local probability amplitude (contributed by each path)? Without going in details, we simply say this. By physical considerations, each path $x(,) \in \mathcal{P}([t_o, T])$ contributes the "amount" $\exp\{\frac{i}{h}S(x)\}$ to the total probability amplitude. The second step is "sum them up". The third step is verifying that that "sum" is correct, i.e., satisfying Schrodinger's equation. When this Feynman's program is done, we explore its implications, mostly with respect to applications, both in quantum mechanics and econometrics. It can be shown that Schrodinger's equation

$$ih\frac{\partial \psi}{\partial t} = -\frac{h^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi(x,t)$$

follows from Feynman's path integral formalism. Thus, Feynman's path integral is an equivalent formalism for quantum mechanics. For more details, see Feynman and Hibbs [6], Mazzucchi [9]. For the application of Path integrals to financial econometrics, see e.g., Baaquie [1[.

3 Economic equilibrium issues

Leaving aside econometric analyses in quantitative economics, equilibrium issues (for competive economies and economic dynamics) are central in economic studies. See a text like Stokey and Lucas,[13].

It is well known that Fixed Point Theory is an useful tool to investigate these issues, at least in the context of von Neumann's expected utility theory. With respect to stochastic dynamical economies, say, using Markov models, there is a need to provide various forms of FPT to establish the existence of equilibrium of economies, noting that an invariante (stationary) probability measure is the fixed point of a Markov operator. Specifically, the search for FPT, under various conditions, is in a setting such as:

Let the state space be an arbitrary set Ω, and $\mathcal{A}$ be a $\sigma-$ field of its subsets. A Markov process on $(\Omega, \mathcal{A})$ is specified by an initial probability on $(\Omega, \mathcal{A})$ and a transition function $Q : \Omega \times \mathcal{A} \to [0, 1]$ such that

a) For each $\omega \in \Omega$, $Q(\omega, .)$ is a probability measure on $(\Omega, \mathcal{A})$,

b) For each $A \in \mathcal{A}$, $Q(., A)$ is a $\mathcal{A}-$ measurable function,

where $Q(\omega, A)$ is the (conditional) probability that the process moves from state ω in one time period to a state in the set A in the next time period.

Let $\mathcal{P}(\Omega, \mathcal{A})$ denote the space of all probability measures on $(\Omega, \mathcal{A})$, and consider the self map (a Markov operator) $T : \mathcal{P}(\Omega, \mathcal{A}) \to \mathcal{P}(\Omega, \mathcal{A})$, $P \to TP$, defined by

$$TP(A) = \int_{\Omega} Q(\omega, A) dP(\omega)$$

where TP is the probability measure in the next time period given that the current values of the state are drawn according to the probability measure P. Then a probability measure P is called "invariant" if it is a *fixed point* of T, i.e., for all $A \in \mathcal{A}$, we have

$$P(A) = \int_{\Omega} Q(\omega, A) dP(\omega)$$

meaning that if an economy governed by a Markov process begins with an invariant law, then it will maintain that same law in all future time periods.

Note that the existence, uniqueness and stability of equilibrium of economic dynamical systems requires a *fixed point argument* for a self map T on some metric space (U, d), such as a Markov operator T and U is the space $D(\mu)$ of probability densities with respect to μ, equiped with the L^1 metric, we need to figure out when T has a fixed point.

The emphasis is on stochastic ordering and metrics on spaces ,such as the Hellinger metric

$$d(P, Q) = [\frac{1}{2} \int_{\Omega} (\sqrt{p} - \sqrt{q})^2 d(P + Q)(\omega)]^{\frac{1}{2}}$$

of probability measures for establihing appropriate fixed point theorems for applications to economics.

In view of the application of Tarski fixed point theorem to equilibria of economies and Knaster-Tarski fixed point theorem to stochastic dynamic economies,.more work is needed for fixed points of self maps on ordered, compact spaces of measures.

Remark. The set $\mathcal{M}(\Omega, \mathcal{A})$ of finite signed measures on $(\Omega, \mathcal{A})$, equiped with total variation norm is a *Banach lattice*, i.e., a real Banach space with an order relation $\leq$ such that $(\mathcal{M}, \leq)$ is a vector lattice and the norm is a lattice norm.

$(\mathcal{M}, \leq)$ is a vector lattice if it is an ordered vector space such that for any $\mu, \nu \in \mathcal{M}$, $\mu \vee \nu$ and $\mu \wedge \nu$ exist and $\mu \leq \nu \Longrightarrow \mu + \gamma \leq \nu + \gamma$ for all $\mu, \nu, \gamma \in \mathcal{M}$ (additivity); and $0 \leq \mu \Longrightarrow 0 \leq t\mu$ for all $\mu \in \mathcal{M}$, and $t \in \mathbb{R}^+$(positive homogeneous)

A norm on a vector lattice is called a lattice norm if $|\mu| \leq |\nu| \Longrightarrow ||\mu|| \leq ||\nu||$ where $|\mu| = (\mu \vee 0) + (-\mu \vee 0).(\mu^+ = \mu \vee 0,\ \mu^- = -(\mu \wedge 0))$.

It could be "interesting" to investigate PFT on these Banach lattices. This is so since the mathematical setting for commodity spaces is *topological vector lattices* (Riesz spaces). A topological vector lattice is a vector lattice where the lattice operations $(\mu, \nu) \to \mu \vee \nu, \mu \wedge \nu$ are (uniformly) continuous. The most important case is Banach lattices. Those are Banach spaces having a lattice order $\leq$ such that $|\mu| \leq |\nu| \Longrightarrow ||\mu|| \leq ||\nu||$ where $|\mu| = (\mu \vee 0) + (-\mu \vee 0).(\mu^+ = \mu \vee 0,$ $\mu^- = -(\mu \wedge 0))$. For example, L^p is a Banach lattice with their natural norm and order.

In economic applications, lattice structures of probability measures (e.g., arising from *stochastic dominance*) is useful for "stochastic optimization" since we can reduce a constraint involving a number of "inequalities" among probability measures (a constraint consisting of an ordered set of probability measures) to a single one (by taking the "minimum" of them). Also, in *robust Bayesian statistics*, where probability measures are prior ones, the minimum of them is not a probability measure, but some *nonadditive set function*, called *capacity*, or *fuzzy measure* (a situation termed "ambiguity" in decision theory). Here, the search for stochastic orders which can lead to lattice structures is thus useful. In this search, we will "run into" *copulas*!

Let me elaborate a bit on *lattice structure of* $\mathcal{M}(\Omega, \mathcal{A})$. Recall that the TV norm on $\mathcal{M}(\Omega, \mathcal{A})$ is

$$||\mu||_{TV} = ||\frac{d\mu}{d\lambda}||_1 = \int_\Omega |\frac{d\mu}{d\lambda}| d\lambda = \sup_{|f| \leq 1} |\int_\Omega f(\omega) d\mu(\omega)| = \sup_{A \in \mathcal{A}} (|\mu(A)| + |\mu(A^c)|)$$

for some dominating measure λ.

Note that the topology induced by TV norm on $\mathcal{P}(\Omega, \mathcal{A})$ is strictly stronger than the "weak" topology, namely

$P_n \to P$ weakly iff $P_n(A) \to P(A)$ when $n \to \infty$, for all $A \in \mathcal{A}$ such that $P(\delta A) = 0$, where δA is the boundary of A, and Ω is a metric space. With respect to TV norm, $\mathcal{P}(\Omega, \mathcal{A})$ is a subset of the Banach lattice $\mathcal{M}(\Omega, \mathcal{A})$.

The use of arbitrary chosen dominating measure λ allows us to define lattice structure for $\mathcal{M}(\Omega, \mathcal{A})$. The order relation on $\mathcal{M}(\Omega, \mathcal{A})$ is induced by the usual order $\leq$ on real numbers: for $\mu, \nu \in \mathcal{M}(\Omega, \mathcal{A})$, $\mu \leq \nu$ when $\mu(A) \leq \nu(A)$ for all $A \in \mathcal{A}$. We are going to show that $(\mathcal{M}, \leq)$ is a lattice.

First, observe that $A \to \max(\mu, \nu)(A) = \max\{\mu(A), \nu(A)\}$ is not a measure since it need not be additive (same for $\min\{\mu(.), \nu(.)\}$) so that it is not an upper bound for $\{\mu, \nu\}$ in $\mathcal{M}(\Omega, \mathcal{A})$

For μ, ν both $\ll \lambda$, let γ be the measure defined by $\gamma(A) = \int_A (\frac{d\mu}{d\lambda} \vee \frac{d\nu}{d\lambda}) d\lambda = \int_A (f(\omega) \vee g(\omega)) d\lambda(\omega)$, where $(f \vee g)(\omega) = \max\{f(\omega), g(\omega)\}$.

Clearly, for all $A \in \mathcal{A}$.

$$\gamma(A) = \int_A (\frac{d\mu}{d\lambda} \vee \frac{d\nu}{d\lambda}) d\lambda \geq \max\{\int_A \frac{d\mu}{d\lambda} d\lambda, \int_A \frac{d\nu}{d\lambda} d\lambda\} = \max\{\mu(A), \nu(A)\}$$

we have that $\gamma(.) \geq \max\{\mu(.), \nu(.)\}$.

Moreover, $\gamma(.)$ is the smallest measure greater than both μ and ν. Indeed, if $\tau(A) = \int_A hd\lambda$ such that $\tau \geq \max\{\mu(.), \nu(.)\}$, then, for all $A \in \mathcal{A}$,

$$\int_{A\cap(f\geq g)} fd\lambda = \mu[A \cap (f \geq g)] \leq \tau A \cap (f \geq g)] = \int_{A\cap(f\geq g)} hd\lambda$$

i.e.,

$$\int_{A\cap(f\geq g)} (h - f)d\lambda \geq 0$$

implying that $h \geq f$ on $(f \geq g)$ $\lambda-$ a.e. Similarly, $h \geq g$ on $(f < g)$ $\lambda-$ a.e, so that $h \geq f \vee g$, $\lambda-$ a.e, and hence $\tau \geq \gamma$.

Thus, for any $\{\mu, \nu\}$ in $\mathcal{M}(\Omega, \mathcal{A})$, there exist a smallest $\mu \vee \nu \geq \max\{\mu, \nu\}$.

Similarly, there is a largest $\mu \wedge \nu \geq \min\{\mu, \nu\}$, so that $\mathcal{M}(\Omega, \mathcal{A})$ is a lattice wrt $\leq$. The signed measure, denoted as $\mu \wedge \nu$, has density $\min\{f, g\}$ with respect to λ.

Fixed point theorems on ordered sets (for applications to economics)

If you seek more "applied rationale" to do more theoretical research on FPT or "areas" to apply FPT, then it seems the most promising one is quantitative economics, in which, besides econometrics, the investigation of equilibria, as "solutions" for (stochastic or deterministic) economic dynamics (systems), often requires "Fixed Point Arguments".

For example, in economics context, the commodity spaces are locally convex, Hausdorff, ordered vector spaces (e.g. Banach lattices), price functionals are elements of the topological dual of the commodity space (continuous linear functionals). One important problem is the existence of price equilibrium in exchange economies in which the commodity spaces could be infinitely dimensional such as L^∞ (Banach space), L^2 (Hilbert), the space of signed measures on a compact metric space. The finite demensional existence theory is when the commodity space is $\mathbb{R}^d$.

For a flavor of economic research involving the above mathematical structures, see e.g.,

The price equilibrium existence problem in topological vector lattices
A. Mas-Colell, *Econometrica* 54(5), 1039-1053, 1986

By "fixed point arguments" we mean looking at an equilibrium (of an economic dynamics) at a fixed point of some self map on appropriate spaces. From an application viewpoint, having a variety of available fix point theorems, we examine whether some of these can be applied to "solve" the economic problem. This consists essentially of figuring out whether sufficient conditions for the validity of some FPT are satisfied.

Typically, suppose the "solution" of an economic problem is formulated as a fixed point of a map $T : X \to X$. If X is a complete metric space and T is a

contraction, then, by Banach FPT, the solution is approximated by $\lim_{n\to\infty} T^n x_o$ for any $x_o \in X$, providing an algorithm for locating the solution (fixed point of T) computationally (with a geometric convergence rate). If we only have sufficient conditions for applying, say, Brouwer or Schauder's theorems, then the solutions, while existed, might not be unique, and the orbit $T^n x_o$ might not be convergent.

Are there additional conditions (to sufficient conditions for Brouwer or Schauder's theorems) to obtain similar (desirable) results as in the setting of Banach's theorem?

The basic additional condition is when X is *an ordered set*, and T is monotone increasing.

This reminds us of Tarski's fixed point theorem: "If X is a complete lattice and T is monotone then the set of fixed points forms a non-empty complete lattice" (i.e., a monotone self map on a complete lattice has a least fix point) or.

Knaster-Tarski's theorem : For a monotone self map T on a complete lattice

(i) The least fixed and the prefixed points of T exist, and they are identical,

(ii) The greatest fixed and postfixed points of T exist, and they are identical,

(iii) The fix points of T form a complete lattice

For $T : (X, \leq) \to (X, \leq)$, a point $x \in X$ is a prefixed point on T if $Tx \leq x$, and is a postfixed point if $x \leq Tx$. Note that a fixed point $x = Tx$ is both a prefixed and postfixed point. Knaster-Tarski Theorem is simply this: Let $T : (X, \leq) \to (X, \leq)$ where $(X, \leq)$ is a complete lattice and T monotone (order-preserving) . Then T has at least a fixed point. And, in fact, the set of fixed points of T is a complete lattice. Note that this theorem is non-constructive.

Historically, the above theorem, for a special case, was in

Un theoreme sur les fonctions d'ensembles
B. Knaster, *Ann. Soc. Polon. Math. (6), 133-134, 1928*

with the general version in

A lattice-theoretical fix point theorem and its applications
A. Tarski, *Pacific J. Math. (5), 285-309, 1955*

The proof of Tarski's theorem is non-constructive. A *constructive proof* (i.e., with a procedure for finding a fixed point) for it, as well as its extension to set-valued maps (due to L. Zhou, 1994) is in

A short and constructive proof of Tarski's fixed point theorem
F. Echenique, *Intern. J. Game Theory 32(2), 215-218, 2005*

We will elaborate on using Knaster-Tarski's fixed point theorem to show the existence of fixed points of self maps on *compact sets of probability measures equiped with stochastic ordering* (i.e., existence of stationary distributions of dynamic economies).

First, a simple example. The *stochastic order* relation on probability measures on the real line, $\mathcal{P}(\mathbb{R}, \mathcal{B})$, is this.

$\mu \leq_{st} \nu$ iff $\int_{\mathbb{R}} f(x)d\mu(x) \leq \int_{\mathbb{R}} f(x)d\nu(x)$ for any measurable $f : \mathbb{R} \to \mathbb{R}$, nondecreasing.

If we let $F_\mu : \mathbb{R} \to [0,1]$, $F_\mu(x) = P((-\infty, x])$, then $\mu \leq_{st} \nu$ iff $F_\nu(.) \leq F_\mu.)$, or equivalently

$$F^*_\mu(.) = 1 - F_\mu(.) \leq F^*_\nu(.) = 1 - F_\nu(.)$$

For a proof, see p. 67 in

Stochastic Dominance and Applications to Finance, Risk and Economics
S. Sriboonchitta, W-K. Wong, S. Dhompongsa & H.T. Nguyen
Chapman and Hall/CRC Press, 2010

Now, observe that the function

$$F(x) = 1 - \min\{F^*_\mu(x), F^*_\nu(x)\}$$

is a "distribution function".

As such, by Lebesgue-Stieltjes characterization theorem, there exists a probability measure $\gamma \in \mathcal{P}(\mathbb{R}, \mathcal{B})$ determined by $\gamma((-\infty, x]) = F(x)$.

Clearly $\gamma \leq_{st} \mu, \nu$ since $F^*_\gamma(x) = \min\{F_\mu(x), F^*_\nu(x)\}$. Thus, γ is the smallest glb of μ, ν, i.e., $\gamma = \mu \wedge_{st} \nu$. Similarly, $\mu \vee_{st} \nu$ is the probability measure with distribution $1 - \max\{F^*_\mu(x), F^*_\nu(x)\}$. Therefore, the stochastic order leads to a lattice structure for $\mathcal{P}(\mathbb{R}, \mathcal{B})$.

In economics, the above stochastic order is termed *first order stochastic dominance* when referring to (random) economic variables A real-valued random variable X is stochastic larger than another Y when $X \geq Y$ which is equivalent to $F_X(x) = P(X \leq x) \leq P(Y \leq x) = F_Y(x)$, for all $x \in \mathbb{R}$.

The "converse" can be seen as follows. Supose $F_X(.) \leq F_Y(,)$. What can be said about X and Y? Well, we can construct a pair of random variables X', Y', possibly defined on different probability spaces, having (marginal) distributions F_X, F_Y, respectively and more over $X' \geq Y'$ (a.s.). This is the method of *coupling*.

Here, we take $X' = F^{-1}_X(U)$ and $Y' = F^{-1}_Y(U)$, where $F^{-1}_X(.) : (0,1) \to \mathbb{R}$ is the quantile function of $F_X(.)$, i.e., the left continuous inverse defined as

$$F^{-1}_X(\alpha) = \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\}$$

and U is a random variable uniformly distributed on $(0,1)$.

Note that $F_X(.) \leq F_Y(,)$ is equivalent to $F^{-1}_Y(.) \leq F^{-1}_X(.)$, we do have that $X' \geq Y'$ (a.s.). The "coupling" (X', Y') is a pair of random variables whose "joint probability measure" (on a product space) has as marginals the probability measures associated with $F_X(.), F_Y(,)$. It's right here that the notion of *copulas* could be useful.

For example, the multivariate stochastic dominance order, i.e., on $\mathcal{P}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, is defined just like the one dimension case, namely

$$\mu \leq_{st} \nu \quad \text{iff} \quad \int_{\mathbb{R}} f(x)d\mu(x) \leq \int_{\mathbb{R}} f(x)d\nu(x)$$

for all measurable $f(.) : \mathbb{R}^d \to \mathbb{R}$ increasing (with respect to the usual componentwise order on $\mathbb{R}^d$). But unlike the unidimensional case, the order set $\mathcal{P}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \leq_{st})$ is not a lattice. However, some subsets of it could be. For example, the following $\mathcal{P}_C(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$:

A probability measure μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is like a "joint" probability measure which has n "marginal" measures on the factor spaces, as $\mathbb{R}^d = \mathbb{R} \times \mathbb{R} \times ... \times \mathbb{R}$ (n times), namely the projections of μ on each factor space. In view of Lebesgue-Stieltjes' theorem on $\mathbb{R}^d$, each μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is characterized by a multivariate distribution function

$$F_\mu : \mathbb{R}^d \to [0, 1], \qquad F_\mu(x) = \mu(\prod \lim_{i=1}^{d}(-\infty, x_i])$$

And as such, according to *Sklar's theorem*, there is a multivariate *copula* C joining the marginals μ_i (univariate distributions, or marginal measures) with the "joint" measure μ. Each copula describes a type of dependence between variables.

So let $\mathcal{P}_C(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ be the subspace of probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)$ with a common copula C, i.e., with the same type of dependence. Specifically, for all $\mu \in \mathcal{P}_C(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$,

$$F_\mu(x) = \mu(\prod_{i=1}^{d}(-\infty, x_i]) = C[\mu_i((-\infty, x_i]), i = 1, 2, ..., d]$$

It can be shown that the ordered set $(\mathcal{P}_C(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)), \leq_{st})$ is in fact a lattice. For a proof, see

Stochastic order relations and lattices of probability measures
A. Muller & M. Scarsini (Google)

Since economic dynamics could be often modeled by Markov processes, I will say more about copulas and Markov operators in the next Lecture.

The stochastic order on $\mathcal{P}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ should be extended to $\mathcal{M}(\Omega, \mathcal{A})$ for economic applications.

Markov operators

Markov processes enter economic modeling as stochastic exogenuous "shocks" to economic systems (e.g., in determining equilibrium asset prices in exchange economies).

A stochastic process is a collection of random elements indexed by an index set : $\mathbf{X} = (X_t, t \in T)$. The random elements $X_t : (\Omega, \mathcal{A}, P) \to (\mathcal{X}, \sigma(\mathcal{X}))$ (the state space). While one random variable X_t is governed by its probability law P_{X_t}, the whole process $\mathbf{X}$ is governed by its system of *finitely dimentional distributions*, by Kolmogorov consistency theorem.

Let's start out with a simple type of stochastic processes: *Discrete-time Markov chains on a finite state space.*

Here, $\mathbf{X} = (X_n, n \geq 0)$, and $(\mathcal{X}, \sigma(\mathcal{X})) = (\{1, 2, ..., k\}, 2^{\{1,2,...,k\}})$.

The so-called Markov property says that the future depends only on the present, and not the past.Specifically,

$$P(X_{n+1} = j | X_0, X_1, ..., X_n = i) = P(X_{n+1} = j | X_n = i) = p_{ij}$$

If the initial distribution of X_o is π_o (as a density on $\mathcal{X} = \{1, 2, ..., k\}$ or equivalently a probability measure of $2^{\mathcal{X}}$), then all finitely dimensional distributions are determined via the transistion matrix $p = [p_{ij}]_{k \times k}$. Note that p is a stochastic matrix, i.e., $p_{ij} \geq 0$, and for each row i, $\sum_j p_{ij} = 1$ which is the probability that, if at present, the chain is in state i, then the next state will be in some state.

$$\begin{aligned} P(X_o &= i_o, X_1 = i_1, ..., X_n = i_n) = \\ P(X_n &= i_n | X_o = i_o, ..., X_{n-1} = i_{n-1}) P(X_o = i_o, ..., X_{n-1} = i_{n-1}) \end{aligned}$$

$$\begin{aligned} &= P(X_n = i_n | X_{n-1} = i_{n-1}) P(X_o = i_o, X_1 = i_1, ..., X_{n-1} = i_{n-1}) \\ &= p_{i_{n-1} i_n} P(X_o = i_o, X_1 = i_1, ..., X_{n-1} = i_{n-1}) \end{aligned}$$

= ....(by Markov property)

$$= p_{i_{n-1} i_n} .... p_{i_o i_1} \pi_o(i_o)$$

We are interested in knowing whether or not, after a long time, the chain will attain a (steady) stationary mode. Note that, as a stochastic process, the chain will be strictly stationary when all X_n have the same distribution. What is the limiting distribution of the chain?, i.e., $\lim_{n \to \infty} P(X_n = j) = \pi_j$, $j \in \mathcal{X}$? noting that

$$P(X_n = j) = \sum_{i \in \mathcal{X}} \pi_o(i) p_{ij}^n$$

where $p_{ij}^n = P(X_n = j | X_0 = i)$ (the $n-$ step transition).

To determine the limiting distribution $\pi(.)$, we observe that it is "invariant" (with respect to the operator (a self map) $T : \pi(.) \to \sum_{i \in \mathcal{X}} \pi(i) p_{ij}$ on the space of probability measures on the finite set $\mathcal{X}$), i.e., for each $j \in \mathcal{X}$, $T\pi = \pi$:

$$\pi(j) = \sum_{i \in \mathcal{X}} \pi(i) p_{ij}$$

Indeed,

$$\begin{aligned}\pi(j) &= \lim_{n\to\infty} P(X_{n+1}=j) = \lim_{n\to\infty it}\sum_i P(X_n=i)p_{ij}\\ &= \sum_i [\lim_{n\to\infty} P(X_n=i)p_{ij} = \sum_{i\in\mathcal{X}} \pi(i)p_{ij}\end{aligned}$$

It is in fact unique, using a FPT on contraction (viewing the simplex as a metric space with the TV metric), noting that the condition $p_{ij} > 0$ is satisfied by replacing p with p^n, for some n large, where $p^n_{ij} > 0$.

Without going into technical details on general Markov processes $X_t : (\Omega, \mathcal{A}, P) \to (\mathcal{X}, \sigma(\mathcal{X}))$, we simply replace the stochastic matrix $p = [p_{ij}]_{k\times k}$ by a "transition function" $Q : \mathcal{X}\times\sigma(\mathcal{X}) \to [0,1]$ where $Q(x, A)$ is the probability that the next state will be in A given that it is at state x at present.

Remark. Let $(X(t), t \geq 0)$ be a stochastic process, defined on $(\Omega, \mathcal{A}, P)$ with values in $\mathcal{X}$, and let the filtration $\mathcal{F}_t = \sigma(X(s) : s \leq t)$. Then $(X(t), t \geq 0)$ is called a Markov process if

$$P(X(t+s) \in A|\mathcal{F}_t) = P(X(t+s) \in A|X(t))$$

The transition function is

$$Q(s, X(t), A) = P(X(t+s) \in A|\mathcal{F}_t)$$

Technically, for each $A \in \sigma(\mathcal{X})$, $x \to Q(x, A)$ is measurable, and for each $x \in \sigma(\mathcal{X})$, the set-function $A \to Q(x, A)$ is a probability measure on $(\mathcal{X}, \sigma(\mathcal{X}))$, denoted as $Q(x, dy)$.

We can define two operators associated with Q:

(i) $T : f \to \int_{\mathcal{X}} f(y)Q(x, dy) = (Tf)(x)$

(ii) $T^* : \mu \in \mathcal{P}(\mathcal{X}, \sigma(\mathcal{X})) \to (T^*\mu)(A) = \int_{\mathcal{X}} Q(x, A)d\mu(x) \in \mathcal{P}(\mathcal{X}, \sigma(\mathcal{X}))$.

Of course, in investigating fixed points, we need to ensure that the above operators are self maps.

With appropriate function spaces for f , e.g., the space of bounded, continuous functions on $\mathcal{X}$ (say, a metric space) $C(\mathcal{X})$, a Banach space with the sup norm, the above (Markov) operators could be self maps*, depending on* Q.

Similarly to the finite case, a probability measure $\mu \in \mathcal{P}(\mathcal{X}, \sigma(\mathcal{X}))$ is invariant with respect to T^* (a fixed point) when, for all $A \in \sigma(\mathcal{X})$,

$$\mu(A) = \int_{\mathcal{X}} Q(x, A)d\mu(x)$$

The economic interpretation is this. If an economy governed by a Markov process begins where the probability distribution of states is invariant, then it will maintain that same distribution in all future periods.

For example, the transition function Q is said to satisfy the Feller property if T on $C(\mathcal{X})$ is a self map. For example, if $\mathcal{X}$ is a compact subset of $\mathbb{R}^d$, and Q has

the Feller property, then there exists a probability measure on $(\mathcal{X}, \sigma(\mathcal{X}))$ which is invariant under Q.

The monotonicity of Q, in the sense that its associated T is monotone: $f \leq g \Longrightarrow Tf \leq Tg$, could be useful to investigate invariant probability measures under Q.

Also, some other conditions should be used for this purpose, such as

Condition D: the transition function Q is said to satisfy the condition D if there exist a finite measure γ on $(\mathcal{X}, \sigma(\mathcal{X}))$, an integer $N \geq 1$, and a number $\varepsilon > 0$ such that if $\gamma(A) \leq \varepsilon$ then $Q(x, A) \leq 1 - \varepsilon$ for all $x \in \mathcal{X}$.

For example, when a Q satisfies the condition D, and in addition, if $\gamma(A) > 0$, then for each $x \in \mathcal{X}$, there exists $n \geq 1$ such that $Q^n(x, A) > 0$, then T^* has only one invariant probability measure.

Of course, in applications, we need to address:

(a) Does an invariant probability measure exist?

(b) If so, is it unique?

(c) How can an invariant probability measure be calculated?

As such, more research on FPT for Markov transition functions seems "interesting"!

4 Social decision issues in economics

Clearly economic activities and data are affected by economic agents' decisions, among other uncertain information. Thus, technically, while economic time series data can be regarded as uncertain and dynamical systems (i.e., modeled as stochastic processes), they are somewhat different than "natural" stochastic processes by the fact that economic outcomes are influenced by human (e.g., customers) decisions, in the form of choices. Thus, since the birth of quantitative economics, understanding human (cognitive) decisions is essential, leading to a theory of decision (of course, under uncertainty, or incomplete information). Taking human decision-making process into the analysis of economic data is what we know now as *Behavioral Economics*, the topic of the 2017 Nobel Memorial Prize in Economics, awarded to Richard Thaler.

How humans make decisions under uncertainty? It's a question in social science, not natural science. However, it should bear some analogies? Right after providing the new mathematics for quantum mechanics, von Neumann embarked on another new mathematics for social science, with his game theory in 1928 [14]. It was precisely to answer this basic question, leading to the foundations of quantitative economics in 1944. It is his theory of rational decision-making based upon his concept of expected utility (where uncertainty is modeled as Kolmogorov probability, 1933, and not quantum probability!). See, e.g., Kreps [8].

Putting "expected utility" as a way (or a "law") that humans are supposed to follow to make decisions is just a "hypothesis" which, like with physics, need to be confirmed (or validated) by facts. But this is social science, so that the question of whether people actually make decisions in the way predicted by von

Neumann is left to *psychologists*! As we will see, the situation is completely similar to mechanics: While Newtonian mechanics is good for macro-objects, we should turn to quantum mechanics when dealing with micro-objects (particles). The basic assumption of von Neumann's decision theory is rationality of people. But are people always rational? And if they are not, why? Technically speaking, to carry out von Neumann's decision program, people need to have probabilities available to them, which they do not have in general. This is the main reason for a Bayesian approach: just "guess" prior probabilities and then use the Bayesian updating machinery to come up with necessary probabilities (noting that, as far as calculus is concerned, Bayesian probability follows the same calculus of frequentist probability, in particular, obeys the additivity axiom). Psychologists such as Allais (1953), Ellsberg (1961) and Edwards (1968), showed that von Neumann's expected utility as well as Bayesian updating are violated.

To be complete, psychologists added more situations showing that "likelihood" should not be additive. Here is one, the so-called "conjunction fallacy" showing that people seem to model uncertainty by set-functions which are not monotone increasing, and hence cannot be additive probability measures. This is serious for current approaches of "non-additive probabilities" which are still monotone set-functions.

Here is the well-known "Conjunction Fallacy" from the literature (Tversky & Kahneman, 1983). A lady named Linda was known as an active feminist in the past. consider now two events: $A =$ "She is acitive in the feminist movement", $B =$"she is a bank teller". Subjects are asked to guess the likelihoods of A, B , $A \cup B$ and $A \cap B$. It turns out that subjects judged $A \cap B$ is more likely than B.

In conclusion, the research question could be: How to model quantitatively the intuitive concept of "likelihood"? The current attempt in behavioral decision-making focuses only on relaxing the additivity axiom of Kolmogorov probability, leading to a host of non-additive probability approaches in the literature. However, at least two main characteristics in cognitive decision seems missing: the semantics of likelihood (needed to obtain likelihoods from empirical data) and the so-called "order effects" of information (in human's judgements towards decisions), which, in mathematical terms, means that information (as supplied by consecutive "events") is non commutative.

5. Quantum probability for behavioral economics

In view of the issues raised in the previous section, current research efforts turn to a concept of probability in quantum mechanics which seems to possess all desirable properties of likelihood used by humans. See e.g., Busemeyer and Bruza [2] and Haven and Khrennikov [7].

Discovering the laws of quantum mechanics was the best achievement of scientists of the $20th$ century. It was about how nature behaves. Although, at the atomic level, nature behaves randomly, physicists arrived at revealing how to compute probabilities of quantum events leading to technological advances as we witnessed in our actual societies.

It is about modeling the uncertainty involved in nature behavior. The concept

of chance is the same as for games of chance, i.e., viewed in a frequentist (objective) interpretation. What is different is the "rules" to compute probabilities of quantum events, due to the very physical observed facts. These differences are mainly the non additivity of the probability measures, and the non commutative of observables (quantum "random variables"). As Richard Feynman has said "and yet, these do not lead to logical inconsistencies", i.e., it is possible to define a concept of probability (to model the type of uncertainty encountered in quantum mechanics) based on some other set of axioms (different from those of Kolmogorov) but keeping the same meaning of chance that we are all familiar off. Specifically, a non addtive probability measure can be axiomatized. Moreover. such a non additive probability measure can be incorporated into a non commutative framework of observables. All this is for the most important thing in our quest to understand the world around us.

As we may have a good idea about the physical world around us, we may not have such an idea bout how people around us behave! Why we are interested about human behavior? Well, many reasons, but, practically, it's because of economics, the thing wich truely affects all of us. It is clear that such a topic should belong to psychology? For sure, it belongs to social sciences. So, having physical sciences aside, we are facing social sciences for various reasons, especially for economics, as self evidence! Understanding how investors will make their financial decisions should be a dream for economists!

The emergence of quantum probability (1927 or so) is a striking phenomenon. Not only quantum probability is non additive, it is also non commutative, a property "stranger" to all probability concepts we had so far (since it's hard to imagine such a property in measuring or assessing uncertainties).

In 1951, at a Berkeley Symposium on Mathematical Statistics and Probability, Richard Feynman gave a talk on quantum probability to probabilists, statisticians and mathematicians about "The concept of probability in quantum mechanics", see Feynman [5], pointing out precisely the "strange" (but natural, i.e., its axioms are not imposed by men, but they are the way nature designed our world!) properties of probability in quantum mechanics (as discovered by physicists): While quantum probability has a frequentist interpretation, it is not additive. It did not ring the bell! Perhaps, probabilists have simply reacted like "It's interesting, it's good to know, but it's not our business, we are concerned with uncertainty not in quantum mechanics". Some time later, in Strasbourg, France, Paul Andre Meyer, appearently motivated by pure mathematical interests, pursued deep research into non commutative probability, specifically addressing probabilists, in Meyer [10]. See also Parthasarathy [12], and Connes [3].

It is striking that, much earlier, physicists were facing the same problem as psychologists, namely the uncertainty encountered in quantum mechanics, as testified by observations, cannot be modeled as Kolmogorov probability since it is non additive and non commutative (in a sense to be specified shortly). What physicists have found is that a new theory of probability, called quantum probability, is nothing else than a natural generalization of Kolmogorov theory. In terms of cognition, this simply means that the axioms of ordinary probability measures are

too strong for behavioral decision-making.

How to generalize a commutative probability theory to a non commutative theory (which will entails the non additivity)? Perhaps, inspired by this problem that later Alain Connes pursued a general theory of *non commutative geometry.*

Following David Hilbert's advice "What is clear and easy to grasp attracts us, complications deter", let's first consider the simplest case of Kolmogorov probability, namely the finite sample space, representing a random experiment with a finite number of possible outcomes, e.g., a roll of a pair of dice. A finite probability space is a triple $(\Omega, \mathcal{A}, P)$ where $\Omega = \{1, 2, ..., n\}$, say, i.e., a finite set with cardinality n, $\mathcal{A}$ is the power set of Ω (events), and $P : \mathcal{A} \to [0, 1]$ is a probability measure ($P(\Omega) = 1$, and $P(A \cup B) = P(A) + P(B)$ when $A \cap B = \varnothing$). Note that since Ω is finite, the set-function P is determined by the density $\rho : \Omega \to [0, 1]$, $\rho(j) = P(\{j\})$, with $\sum_{j=1}^{n} \rho(j) = 1$. A real-valued random variable is $X : \Omega \to \mathbb{R}$. In this finite case, of course $X^{-1}(\mathcal{B}(\mathbb{R})) \subseteq \mathcal{A}$. The domain of P is the $\sigma-$field $\mathcal{A}$ of subsets of Ω (events) which is Boolean (commutative: $A \cap B = B \cap A$), i.e., *events are commutative*, with respect to intersection of sets. We wish to generalize this setting to a non commutative one, where "extended" events could be, in general, non commutative, with respect to an "extension" of $\cap$.

For this, we need some appropriate equivalent representation for all elements in this finite probability setting. Now since $\Omega = (1, 2, ..., n\}$, each function $X : \Omega \to \mathbb{R}$ is identified as a point in the (finitely dimensional Hilbert) space $\mathbb{R}^n$, namely $(X(1), X(2), ..., X(n))^t$, which, in turn, is equivalent to a $n \times n$ diagonal matrix with diagonal terms $X(1), X(2), ..., X(n)$. and zero outside (a special symmetric matrix), i.e.,

$$X \Longleftrightarrow [X] = \begin{bmatrix} X(1) & & & & 0 \\ & X(2) & & & \\ & 0 & . & 0 & \\ & & 0 & . & \\ 0 & & & & X(n) \end{bmatrix}$$

The set of such matrices is denoted as $\mathcal{D}_o$ which is a commutative (with respect to matrix multiplication) subalgebra of the algebra of all $n \times n$ matrices with real entries. As matrices act as (bounded, linear) operators from $\mathbb{R}^n \to \mathbb{R}^n$, we have transformed (equivalently) random variables into operators on a Hilbert space.

In particular, for each event $A \subseteq \Omega$, its indicator function $1_A : \Omega \to \{0, 1\}$ is identified as an element of $\mathcal{D}_o$ with diagonal terms $1_A(j) \in \{0, 1\}$.As such, each event A is identified as a (orthogonal) projection on $\mathbb{R}^n$, i.e., an operator T such that $T = T^2 = T^*$ (its transpose/ adjoint). Finally, the density $\rho : \Omega \to [0, 1]$ is identified with the element $[\rho]$ of $\mathcal{D}_o$ with nonnegative diagonal terms, and with trace $tr([\rho]) = 1$. An element of $\mathcal{D}_o$ with nonnegative diagonal terms is a positive operator, i.e., an operator T such that $< Tx, x > \geq 0$, for any $x \in \mathbb{R}^n$ (where $< ., . >$ denotes the scalar product of $\mathbb{R}^n$). Such an operator is necessarily symmetric (self adjoint). Thus, a probability density is a positive operator with unit trace. Thus, we have transformed the standard (Kolmogorov) probability

space $(\Omega, \mathcal{A}, P)$, with $\#(\Omega) = n$, into the triple $(\mathbb{R}^n, \mathcal{P}_o, \rho)$, where $\mathcal{P}_o$ denotes the subset of projections represented by elements of $\mathcal{D}_o$ (i.e., with $0-1$ diagonal terms) which represent "ordinary" events; and ρ (or $[\rho]$), an element of $\mathcal{D}_o$, is a positive operator with unit trace.

Now, keeping $\mathbb{R}^n$ as a finitely dimensional Hilbert space, we will proceed to extend $(\mathbb{R}^n, \mathcal{P}_o, \rho)$ to a non commutative "probabilty space". It suffices to extend $\mathcal{D}_0$, a special set of symmetric matrices, to the total set of all $n \times n$ symmetric matrices, denoted as $\mathcal{S}(\mathbb{R}^n)$, so that a random variable becomes an "observable", i.e., a self-adjoint operator on $\mathbb{R}^n$; an "quantum event" is simply an arbitrary projection on $\mathbb{R}^n$, i.e., an element of $\mathcal{P}$ (the set of all projections); and the probability density ρ becomes an arbitrary positive operator with unit trace. The triple $(\mathbb{R}^n, \mathcal{P}, \rho)$ is called a (finitely dimensional) *quantum probability space.* We recognize that quantum probability is based upon a new language, not real analysis, but functional analysis (i.e., not on the geometry of $\mathbb{R}^n$, but on its non commutative geometry, namely linear operators on it).

The extension of the above to arbitrary $(\Omega, \mathcal{A}, P)$ essentially involves the replacement of $\mathbb{R}^n$ by an infinitely dimensional, complex and separable Hilbert space H. For details, see texts like Dirac [4], Meyer [10], Parthasarathy [12].

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