Thai Journal of Mathematics
Volume 16 (2018) Number 3 : 817–830
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Approximating Common Fixed Points for Two G-Asymptotically Nonexpansive Mappings with Directed Grahps

Manakorn Wattanataweekul

Department of Mathematics, Statistics and Computer, Faculty of Science
Ubon Ratchathani University, Ubon Ratchathani 34190, Thailand
e-mail : manakorn.w@ubu.ac.th

Abstract : In this paper, we introduce an iterative method for finding a common fixed points of two G-asymptotically nonexpansive mappings with directed graph in a uniformly convex Banach space. Weak and strong convergence of the proposed method are established under some suitable control conditions. We also give some numerical example for supporting our main theorem and compare convergence rate between the studied iteration and the modified Ishikawa iteration.

Keywords : G-asymptotically nonexpansive mapping; directed graph; common fixed point; Banach space.
2010 Mathematics Subject Classification : 47H09; 47H10.

1 Introduction

This class of asymptotically nonexpansive mappings was to introduced by Goebel and Kirk [1] in 1972. They proved that, if C is a nonempty bounded closed convex subset of a uniformly convex Banach space X, then every asymptotically nonexpansiveself-mapping T of C has a fixed point. The fixed point iteration process for asymptotically nonexpansive mapping in Banach spaces including Mann and Ishikawa iterations processes have been studied extensively by many authors; see ([1]-[5]). Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers and X be a uniformly convex Banach space, C be a nonempty closed convex subset of X and $F(T) : Tx = x$. A mapping $T : C \to C$ is said to be asymptotically nonexpansive if for a sequence $\{k_n\}$, $k_n \geq 1$ wih $\lim_{n\to\infty} k_n = 1$, if $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in C$ and for all $n \in \mathbb{N}$.

In 2003, Chang et al. [6] introduced the following iteration process: $x_0 \in X$ and

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \end{cases} \tag{1.1}$$

for all $n \geq 0$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. The iterative schemes (1.1) are called the modified Ishikawa iteration.

In 2008, by combination of the concepts in fixed point theory and graph theory, Jachymski [7] generalized the Banach's contraction principle in a complete metric space endowed with a directed graph. In 2015, Tiammee et al. [8] proved Browders convergence theorem for G-nonexpansive mapping in a Hilbert space with a directed graph. They also proved the strong convergence of the Halpern iteration for a G-nonexpansive mapping. Recently, Suparatulatorn et al. [9] proved the weak and strong convergence of a sequence generated by a modified S-iteration process for finding a common fixed point of two G-nonexpansive mappings in a uniformly convex Banach space with a directed graph.

On this basis, we have introduced a new iterative scheme as follows: $x_0 \in C$ and

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T_2^n x_n, \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T_1^n y_n, \end{cases} \tag{1.2}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ and $T_1, T_2 : C \to C$ are two G-asymptotically nonexpansive mappings. We prove, under some certain conditions, weak and strong convergence theorem of a new iterative scheme for approximating common fixed points of two G-asymptotically nonexpansive mappings in a uniformly convex Banach space X endowed with a directed graph. Moreover, we present numerical example for the new iterative scheme to compare with the modified Ishikawa iteration.

2 Preliminaries

In this section, we provide and recall some definitions and lemmas which will be used in the next sections.

Let C be a nonempty subset of a real Banach space X. Let $\triangle$ denote the diagonal of the cartesian product $C \times C$, i.e., $\triangle = \{(x, x) : x \in C\}$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with C, and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \triangle$. We assume G has no parallel edges. Thus we can identify the graph G with the pair $(V(G), E(G))$. A mapping $T : C \to C$ is said to be *G-asymptotically nonexpansive* if T satisfies the following conditions:

(i) T preserves edges of G (or T is edge-preserving), i.e.,

$$(x, y) \in E(G) \implies (Tx, Ty) \in E(G).$$

(ii) if there exists a sequence $\{k_n\}$, $k_n \geq 1$ with $\lim_{n\to\infty} k_n = 1$ such that

$$||T^n x - T^n y|| \leq k_n ||x - y||,$$

whenever $(x, y) \in E(G)$ and each $n \geq 1$.

Definition 2.1. The *conversion* of a graph G is the graph obtained from G by reversing the direction of edges denoted by G^{-1} and

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

Definition 2.2. Let x and y be vertices of a graph G. A *path* in G from x to y of length $N (N \in \mathbb{N} \cup \{0\})$ is a sequence $\{x_i\}_{i=0}^N$ of $N + 1$ vertices for which $x_0 = x$, $x_N = y$, and $(x_i, x_{i+1}) \in E(G)$ for $i = 0, 1, \ldots, N - 1$.

Definition 2.3. A graph G is said to be *connected* if there is a path between any two vertices of the graph G.

Definition 2.4. Let $x_0 \in V(G)$ and $A \subseteq V(G)$. We say that

(i) *A is dominated by x_0* if $(x_0, x) \in E(G)$ for all $x \in A$.

(ii) *A dominates* x_0 if for each $x \in A$, $(x, x_0) \in E(G)$.

Definition 2.5. A directed graph $G = (V(G), E(G))$ is said to be *transitive* if, for any $x, y, z \in V(G)$ such that (x, y) and (y, z) are in $E(G)$, then $(x, z) \in E(G)$.

Definition 2.6 ([7])**.** A mapping $T : X \to X$ is called *G-continuous* if given $u \in X$ and a sequence $\{u_n\}$ for $n \in \mathbb{N}$, $u_n \to u$ and $(u_n, u_{n+1}) \in E(G)$ imply $Tu_n \to Tu$.

Definition 2.7. A mapping $T : C \to C$ is called *G-semicompact* if for a sequence $\{x_n\}$ in C with $(x_n, x_{n+1}) \in E(G)$ and $\lim_{n\to\infty} ||Tx_n - x_n|| = 0$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in C$ as $j \to \infty$.

Definition 2.8. Let C be a nonempty subset of a Banach space X and let $T : C \to X$ be a mapping. Then, T is said to be *G-demiclosed* at $y \in X$ if, for any sequence $\{x_n\}$ in C such that $\{x_n\}$ converges weakly to $x \in C$, $\{Tx_n\}$ converges strongly to y and $(x_n, x_{n+1}) \in E(G)$ imply $Tx = y$.

Definition 2.9 ([10])**.** A Banach space X is said to satisfy *Opial's condition* if for any sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y||$$

for all $y \in X$ with $x \neq y$.

Property G ([8]) Let C be a nonempty subset of a normed space X and let $G = (V(G), E(G))$ be a directed graph with $V(G) = C$. We said that C has the *Property G* if for each sequence $\{x_n\}$ in C converging weakly to $x \in C$ with $(x_n, x_{n+1}) \in E(G)$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.

Lemma 2.10 ([11]). *Suppose X is a Banach space satisfying Opial's condition and C is a nonempty weakly compact convex subset of X and $T : C \to C$ is an asymptotically nonexpansive mapping. Suppose also $\{x_n\}$ is a sequence in C converging weakly to x and for which the sequence $\{x_n - Tx_n\}$ converges strongly to 0. Then $\{T^n x\}$ converges weakly to x.*

Lemma 2.11 ([5]). *Let $\{a_n\}, \{b_n\}$and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n,$$

for all $n = 1, 2, \ldots$. If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then

(*i*) $\lim\limits_{n\to\infty} a_n$ *exists;*

(*ii*) $\lim\limits_{n\to\infty} a_n = 0$ *whenever* $\liminf\limits_{n\to\infty} a_n = 0$.

Lemma 2.12 ([12]). *Let $p > 1, r > 0$ be two fixed numbers. Then a Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty), g(0) = 0$ such that*

$$||\lambda x + (1 - \lambda)y||^p \leq \lambda||x||^p + (1 - \lambda)||y||^p - w_p(\lambda)g(||x - y||),$$

for all x, y in $B_r = \{x \in X : ||x|| \leq r\}, \lambda \in [0, 1]$, where

$$w_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda).$$

Lemma 2.13 ([13]). *Suppose C has Property G : $\{x_n\} \rightharpoonup x$ and $(x_n, x_{n+1}) \in E(G)$, there exists a subsequence $\{x_{n_k}\}$ such that for each k, $(x_{n_k}, x) \in E(G)$. Let T be a G-asymptotically nonexpansive mapping on C with asymptotic coefficient $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Then $I - T$ is G-demiclosed at 0.*

Lemma 2.14 ([14]). *Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim\limits_{n\to\infty} ||x_n - u||$ and $\lim\limits_{n\to\infty} ||x_n - v||$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converges weakly to u and v, respectively, then $u = v$.*

3 Weak and Strong Convergence Theorems

In this section, we prove weak and strong convergence theorems of a new iteration for two G-asymptotically nonexpansive mappings in a Banach space endowed with a directed graph. Thoughtout of this section, let C be a nonempty closed, bounded and convex subset of a Banach space X with a directed graph $G = (V(G), E(G))$ such that $V(G) = C$ and $E(G)$ is convex. We also suppose the graph G is transitive. Suppose $T_1, T_2 : C \to C$ are two G-asymptotically nonexpansive mappings with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, and $F = F(T_1) \cap F(T_2) \neq \emptyset$. For arbitrary $x_0 \in C$, define

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T_2^n x_n, \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T_1^n y_n \end{cases} \tag{3.1}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

First, we need the following propositons and lemmas.

Proposition 3.1. *Let $z_0 \in F$ and $x_0 \in C$ be such that (x_0, z_0), (z_0, x_0) are in $E(G)$. Then (x_n, z_0), (y_n, z_0), (z_0, x_n), (z_0, y_n), (x_n, y_n) and (x_n, x_{n+1}) are in $E(G)$.*

Proof. We proceed by induction. Since T_1, T_2 are edge-preserving, it can be easily seen that T_1^n and T_2^n are also edge-preserving for all $n \in \mathbb{N}$. From $(x_0, z_0) \in E(G)$, we get $(T_2^n x_0, z_0) \in E(G)$ and so $(y_0, z_0) \in E(G)$ because $E(G)$ is convex. Then, since T_1^n is edge-preserving, $n \geq 1$ and $(y_0, z_0) \in E(G)$, we obtain $(T_1^n y_0, z_0) \in E(G)$, we get $(x_1, z_0) \in E(G)$. Thus, by edge-preserving of T_2^n, $(T_2^n x_1, z_0) \in E(G)$. Again, by the convexity of $E(G)$ and $(T_2^n x_1, z_0), (x_1, z_0) \in E(G)$, we have $(y_1, z_0) \in E(G)$ and hence $(T_1^n y_1, z_0) \in E(G)$. Next, we assume that $(x_k, z_0) \in E(G)$. Since T_2^n is edge-preserving, we get $(T_2^n x_k, z_0) \in E(G)$ and hence $(y_k, z_0) \in E(G)$. Since T_1^n is edge-preserving, we have $(T_1^n y_k, z_0) \in E(G)$. By the convexity of $E(G)$, we get $(x_{k+1}, z_0) \in E(G)$. Hence, by edge-preserving of T_2^n, we obtain $(T_2^n x_{k+1}, z_0) \in E(G)$ and so $(y_{k+1}, z_0) \in E(G)$. Therefore $(x_n, z_0), (y_n, z_0) \in E(G)$ for all $n \geq 1$. Using a similar argument, we can show that $(z_0, x_n), (z_0, y_n) \in E(G)$ under the assumption that $(z_0, x_0) \in E(G)$. By the transitivity of G, we obtain (x_n, y_n), $(x_n, x_{n+1}) \in E(G)$. This completes the proof. □

Proposition 3.2. *Let X be a Banach space with a directed graph G and let $T : C \to C$ be G-asymptotically nonexpansive mapping. If X has the Property G, then T is G-continuous.*

Proof. Let $\{x_n\}$ be a sequence in X such that $x_n \to x$. We show that $Tx_n \to Tx$. To show this, let $\{Tx_{n_k}\}$ be a subsequence of $\{Tx_n\}$. Since $(x_n, x_{n+1}) \in E(G)$ and G is transitive, we obtain $(x_{n_k}, x_{n_{k+1}}) \in E(G)$. Since $x_{n_k} \to x$ and $(x_{n_k}, x_{n_{k+1}}) \in E(G)$, by Property G, there is a subsequence $\{x_{n'_k}\}$ of $\{x_{n_k}\}$ such that $(x_{n'_k}, x) \in E(G)$ for all $k \in \mathbb{N}$. Since T is G-asymptotically nonexpansive mapping and $(x_{n'_k}, x) \in E(G)$, we obtain

$$||Tx_{n'_k} - Tx|| \leq k_1 ||x_{n'_k} - x||$$

as $k \to \infty$. Thus $Tx_{n'_k} \to Tx$. By the double extract subsequence principle, we include that $Tx_n \to Tx$. Then T is G-continuous. □

Lemma 3.3. *If X is a uniformly convex Banach space and $(x_0,z_0),(z_0,x_0)\in E(G)$ for arbitrary $x_0\in C$ and $z_0\in F$, then*

(i) $\lim_{n\to\infty}||x_n-z_0||$ *exists.*

(ii) *If* $0<\liminf_{n\to\infty}\alpha_n\le\limsup_{n\to\infty}\alpha_n<1$, *then* $\lim_{n\to\infty}||T_1^ny_n-y_n||=0$.

(iii) *If* $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$, *then* $\lim_{n\to\infty}||T_2^nx_n-x_n||=0$.

(iv) $\lim_{n\to\infty}||T_1^nx_n-x_n||=0$.

Proof. Let $z_0\in F$. By Proposition 3.1, $(x_n,z_0),(y_n,z_0)\in E(G)$. Choose a number $r>0$ such that $C\subseteq B_r$ and $C-C\subseteq B_r$. By Lemma 2.12, there exists a continuous, strictly increasing and convex function $g:[0,\infty)\to[0,\infty), g(0)=0$ such that

$$||\lambda x+(1-\lambda)y||^2\le\lambda||x||^2+(1-\lambda)||y||^2-w_2(\lambda)g(||x-y||) \tag{3.2}$$

for all $x,y\in B_r,\lambda\in[0,1]$, where $w_2(\lambda)=\lambda(1-\lambda)^2+\lambda^2(1-\lambda)$. It follows from (3.2) and G-asymptotically nonexpansiveness of T_2 that

$$\begin{aligned}||y_n-z_0||^2&=||(1-\beta_n)(x_n-z_0)+\beta_n(T_2^nx_n-z_0)||^2\\&\le(1-\beta_n)||x_n-z_0||^2+\beta_n||T_2^nx_n-z_0||^2-w_2(\beta_n)g(||T_2^nx_n-x_n||)\\&\le(1-\beta_n)||x_n-z_0||^2+\beta_nk_n^2||x_n-z_0||^2-w_2(\beta_n)g(||T_2^nx_n-x_n||)\\&=(1-\beta_n+\beta_nk_n^2)||x_n-z_0||^2-w_2(\beta_n)g(||T_2^nx_n-x_n||).\end{aligned} \tag{3.3}$$

It follows from (3.2) and G-asymptotically nonexpansiveness of T_1 that

$$\begin{aligned}||x_{n+1}-z_0||^2&=||(1-\alpha_n)(y_n-z_0)+\alpha_n(T_1^ny_n-z_0)||^2\\&\le(1-\alpha_n)||y_n-z_0||^2+\alpha_n||T_1^ny_n-z_0||^2-w_2(\alpha_n)g(||T_1^ny_n-y_n||)\\&\le(1-\alpha_n)||y_n-z_0||^2+\alpha_nk_n^2||y_n-z_0||^2-w_2(\alpha_n)g(||T_1^ny_n-y_n||)\\&=(1-\alpha_n+\alpha_nk_n^2)||y_n-z_0||^2-w_2(\alpha_n)g(||T_1^ny_n-y_n||)\\&\le(1-\alpha_n+\alpha_nk_n^2)((1-\beta_n+\beta_nk_n^2)||x_n-z_0||^2\\&\quad-w_2(\beta_n)g(||T_2^nx_n-x_n||))-w_2(\alpha_n)g(||T_1^ny_n-y_n||)\\&\le(1+\beta_n(k_n^2-1)+\alpha_n(k_n^2-1)+\alpha_n\beta_nk_n^2(k_n^2-1))||x_n-z_0||^2\\&\quad-(1-\alpha_n+\alpha_nk_n^2)w_2(\beta_n)g(||T_2^nx_n-x_n||)\\&\quad-w_2(\alpha_n)g(||T_1^ny_n-y_n||)\\&\le||x_n-z_0||^2+(\beta_n(k_n^2-1)+\alpha_n(k_n^2-1)+\alpha_n\beta_nk_n^2(k_n^2-1))\\&\quad||x_n-z_0||^2-(1-\alpha_n+\alpha_nk_n^2)w_2(\beta_n)g(||T_2^nx_n-x_n||)\\&\quad-w_2(\alpha_n)g(||T_1^ny_n-y_n||)\\&=||x_n-z_0||^2+(k_n^2-1)(\beta_n+\alpha_n+\alpha_n\beta_nk_n^2)||x_n-z_0||^2\\&\quad-(1-\alpha_n+\alpha_nk_n^2))w_2(\beta_n)g(||T_2^nx_n-x_n||)\\&\quad-w_2(\alpha_n)g(||T_1^ny_n-y_n||).\end{aligned}$$

Since $\{k_n\}$ and C are bounded, there exists a constant $M > 0$ such that

$$(\beta_n + \alpha_n + \alpha_n\beta_n k_n^2)||x_n - z_0||^2 \leq M$$

for all $n \geq 1$. It follows that

$$(1 - \alpha_n(1 - k_n^2))w_2(\beta_n)g(||T_2^n x_n - x_n||)) \leq ||x_n - z_0||^2 - ||x_{n+1} - z_0||^2 + M(k_n^2 - 1) \quad (3.4)$$

and

$$w_2(\alpha_n)g(||T_1^n y_n - y_n||) \leq ||x_n - z_0||^2 - ||x_{n+1} - z_0||^2 + M(k_n^2 - 1). \quad (3.5)$$

(i) From (3.4), we get $||x_{n+1}-z_0||^2 \leq ||x_n-z_0||^2+M(k_n^2-1)$. Since $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$, it follows from Lemma 2.11 that $\lim_{n\to\infty} ||x_n - z_0||$ exists.

(ii) If $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, there exist some real number $\delta > 0$ and a positive integer n_0 such that

$$w_2(\alpha_n) = \alpha_n(1 - \alpha_n)^2 + \alpha_n^2(1 - \alpha_n) \geq \delta > 0,$$

for all $n \geq n_0$. It follows from (3.5) that for any natural number $m \geq n_0$,

$$\begin{aligned}\sum_{n=n_0}^{m} g(||T_1^n y_n - y_n||) &\leq \sum_{n=n_0}^{m} w_2(\alpha_n)g(||T_1^n y_n - y_n||)\\ &\leq ||x_{n_0} - z_0||^2 - ||x_{m+1} - z_0||^2 + M\sum_{n=n_0}^{m}(k_n^2 - 1)\\ &\leq ||x_{n_0} - z_0||^2 - M\sum_{n=n_0}^{m}(k_n^2 - 1). \end{aligned} \quad (3.6)$$

Since $0 \leq t^2 - 1 \leq 2t(t - 1)$ for all $t \geq 1$, the assumption $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ implies that $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $m \to \infty$ in inequality (3.6). Then

$$\sum_{n=n_0}^{\infty} g(||T_1^n y_n - y_n||) < \infty,$$

and therefore $\lim_{n\to\infty} g(||T_1^n y_n-y_n||) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\to\infty} ||T_1^n y_n - y_n|| = 0$.

(iii) If $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ and $\liminf_{n\to\infty} \alpha_n > 0$, then by using a similar method, together with inequality (3.4), it can be shown that $\lim_{n\to\infty} ||T_2^n x_n - x_n|| = 0$.

(iv) From $y_n = (1 - \beta_n)x_n + \beta_n T_2^n x_n$, we have

$$\begin{aligned}||y_n - x_n|| &= ||(1 - \beta_n)x_n + \beta_n T_2^n x_n - x_n||\\ &\leq \beta_n||T_2^n x_n - x_n||. \end{aligned} \quad (3.7)$$

By (iii) and (3.7), it follows that

$$\lim_{n\to\infty} ||y_n - x_n|| = 0. \tag{3.8}$$

Observe that

$$\begin{aligned} ||T_1^n x_n - x_n|| &\leq ||T_1^n x_n - T_1^n y_n|| + ||T_1^n y_n - y_n|| + ||y_n - x_n|| \\ &\leq k_n||x_n - y_n|| + ||T_1^n y_n - y_n|| + ||y_n - x_n|| \\ &= (k_n + 1)||x_n - y_n|| + ||T_1^n y_n - y_n||. \end{aligned}$$

This together with (3.8) and (ii) imply that

$$\lim_{n\to\infty} ||T_1^n x_n - x_n|| = 0.$$

□

Lemma 3.4. *Let X be a uniformly convex Banach space and $(x_0, z_0), (z_0, x_0) \in E(G)$ for arbitrary $x_0 \in C$ and $z_0 \in F$. If*

(*i*) $\lim_{n\to\infty} ||T_1^n x_n - x_n|| = 0$

(*ii*) $\lim_{n\to\infty} ||T_1^n y_n - y_n|| = 0$

(*iii*) $\lim_{n\to\infty} ||T_2^n x_n - x_n|| = 0,$

then

$$\lim_{n\to\infty} ||T_1 x_n - x_n|| = 0 = \lim_{n\to\infty} ||T_2 x_n - x_n||.$$

Proof. Let $z_0 \in F$ be such that (x_0, z_0), (z_0, x_0) are in $E(G)$. By Proposition 3.1, we have $(x_n, x_{n+1}) \in E(G)$. Observe that

$$\begin{aligned} ||x_{n+1} - T_1^n x_{n+1}|| &\leq ||x_{n+1} - x_n|| + ||T_1^n x_n - T_1^n x_{n+1}|| + ||T_1^n x_n - x_n|| \\ &\leq ||x_{n+1} - x_n|| + k_n||x_n - x_{n+1}|| + ||T_1^n x_n - x_n|| \\ &= (1 + k_n)||x_{n+1} - x_n|| + ||T_1^n x_n - x_n||. \end{aligned}$$

Since $x_{n+1} - x_n = (y_n - x_n) + \alpha_n(T_1^n y_n - y_n)$, we obtain

$$||x_{n+1} - T_1^n x_{n+1}|| \leq (1 + k_n)||y_n - x_n|| + (1+k_n)\alpha_n||T_1^n y_n - y_n|| + ||T_1^n x_n - x_n||.$$

This together with (3.8) and the assumption that

$$\lim_{n\to\infty} ||T_1^n y_n - y_n|| = 0 = \lim_{n\to\infty} ||T_1^n x_n - x_n||$$

imply that

$$\lim_{n\to\infty} ||x_{n+1} - T_1^n x_{n+1}|| = 0.$$

Since

$$||x_{n+1} - T_1 x_{n+1}|| \le ||x_{n+1} - T_1^{n+1} x_{n+1}|| + ||T_1 x_{n+1} - T_1^{n+1} x_{n+1}||$$
$$\le ||x_{n+1} - T_1^{n+1} x_{n+1}|| + k_1 ||x_{n+1} - T_1^{n} x_{n+1}||$$

as $n \to \infty$, we obtain

$$\lim_{n\to\infty} ||T_1 x_n - x_n|| = 0.$$

Similarly,

$$||x_{n+1} - T_2^n x_{n+1}|| \le ||x_{n+1} - x_n|| + ||T_2^n x_n - T_2^n x_{n+1}|| + ||T_2^n x_n - x_n||$$
$$\le ||x_{n+1} - x_n|| + k_n ||x_n - x_{n+1}|| + ||T_2^n x_n - x_n||$$
$$\le (1 + k_n)||y_n - x_n|| + (1 + k_n)\alpha_n ||T_1^n y_n - y_n|| + ||T_2^n x_n - x_n||.$$

Again, by (3.8) and $\lim_{n\to\infty} ||T_1^n y_n - y_n|| = 0 = \lim_{n\to\infty} ||T_2^n x_n - x_n||$, we have

$$\lim_{n\to\infty} ||x_{n+1} - T_2^n x_{n+1}|| = 0.$$

Thus

$$||x_{n+1} - T_2 x_{n+1}|| \le ||x_{n+1} - T_2^{n+1} x_{n+1}|| + ||T_2 x_{n+1} - T_2^{n+1} x_{n+1}||$$
$$\le ||x_{n+1} - T_2^{n+1} x_{n+1}|| + k_1 ||x_{n+1} - T_2^{n} x_{n+1}||$$

as $n \to \infty$, which implies

$$\lim_{n\to\infty} ||T_2 x_n - x_n|| = 0.$$

□

Theorem 3.5. *Let X be a uniformly convex Banach space satisfying the Opial's condition and let C be a nonempty closed and convex subset of X. Let T_1, T_2 be two G-asymptotically nonexpansive mappings on C with the nonempty common fixed point set $F = F(T_1) \cap F(T_2)$. Let $\{\alpha_n\}, \{\beta_n\}$ be sequences in $[0,1]$ satisfying*

(*i*) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$, *and*

(*ii*) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Assume that C has the Property G. Let $x_0 \in C$ be fixed so that (x_0, z_0) and (z_0, x_0) are in $E(G)$ for some $z_0 \in F$. If $\{x_n\}$ is a sequence defined by recursion (3.1), then $\{x_n\}$ converges weakly to a common fixed point of T_1 and T_2.

Proof. Let $z_0 \in F(T_1) \cap F(T_2)$ be such that $(x_0, z_0), (z_0, x_0) \in E(G)$. It follows from Lemma 3.3 (i) that $\lim_{n\to\infty} ||x_n - z_0||$ exists. So $\{x_n\}$ is bounded, hence it has a weakly convergent subsequence. We prove that $\{x_n\}$ has a unique weak subsequential limit in $F(T_1) \cap F(T_2)$. For, let u and v be weak limits of the subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$, respectively. By Lemma 3.4, we have $\lim_{n\to\infty} ||T_1x_n - x_n|| = 0$ and $I - T_1$ is G-demiclosed with respect to zero by Lemma 2.13, therefore we obtain $T_1u = u$. Similarly, $T_2u = u$. Again in the same fashion, we can prove that $v \in F(T_1) \cap F(T_2)$. By Lemma 2.14, we have $u = v$. Thus $\{x_n\}$ converges weakly to a common fixed point in $F(T_1) \cap F(T_2)$. □

Theorem 3.6. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X. Let T_1, T_2 be two G-asymptotically nonexpansive mappings on C with the nonempty common fixed point set $F = F(T_1)\cap F(T_2)$. Let $\{\alpha_n\}, \{\beta_n\}$ be sequences in $[0,1]$ satisfying*

(*i*) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, *and*

(*ii*) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Assume that C has the Property G and one of T_1 and T_2 is G-semicompact. Let $x_0 \in C$ be fixed so that (x_0, z_0) and (z_0, x_0) are in $E(G)$ for some $z_0 \in F$. If $\{x_n\}$ is a sequence defined by recursion (3.1), then $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.

Proof. We may assume that T_1 is G-semicompact. By Lemma 3.3, we obtain $\{x_n\}$ is bounded. From Lemma 3.4, we get

$$\lim_{n\to\infty} ||x_n - T_1x_n|| = 0 = \lim_{n\to\infty} ||x_n - T_2x_n||.$$

Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to z_0$ as $k \to \infty$.Thus

$$\lim_{k\to\infty} ||x_{n_k} - T_1x_{n_k}|| = 0 = \lim_{k\to\infty} ||x_{n_k} - T_2x_{n_k}||.$$

By Proposition 3.2, we obtain T_1 and T_2 are G-continuous. It follows that

$$||z_0 - T_1z_0|| = \lim_{k\to\infty} ||x_{n_k} - T_1x_{n_k}|| = 0$$

and

$$||z_0 - T_2z_0|| = \lim_{k\to\infty} ||x_{n_k} - T_2x_{n_k}|| = 0.$$

This yield $z_0 \in F$ so that $\{x_{n_k}\}$ converges strongly to $z_0 \in F$. But again by Lemma 3.4, $\lim_{n\to\infty} ||x_n - p||$ exists for all $p \in F$ therefore $\{x_n\}$ must itself converge to $z_0 \in F$. This completes the proof. □

4 Numerical Example

In this section, we give an example and its numerical experiments for supporting our main theorem. In 1976, Rhoades [15] gave the idea of how to compare the rate of convergence between two iterative methods as follows:

Definition 4.1 ([15])**.** Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a mapping. Suppose $\{x_n\}$ and $\{m_n\}$ are two iterations which converge to a fixed point q of T. Then $\{x_n\}$ is said to *converge faster than* $\{m_n\}$ if

$$||x_n - q|| \leq ||m_n - q||$$

for all $n \geq 1$.

In order to study the order of convergence of a real sequence $\{a_n\}$ converging to a, we usually use the well-known terminology in numerical analysis, see [16], for example.

Definition 4.2 ([16])**.** Suppose $\{a_n\}$ is a sequence that converges to a, with $a_n \neq a$ for all n. If positive constants λ and α exist with

$$\lim_{n\to\infty} \frac{|a_{n+1} - a|}{|a_n - a|^{\alpha}} = \lambda,$$

then we say that $\{a_n\}$ *converges to a of order α, with asymptotic error constant* λ. If $\alpha = 1$ (and $\lambda < 1$), the sequence is *linearly convergent*, and if $\alpha = 2$, the sequence is *quadratically convergent.*

In 2002, Berinde [17] employed the above concept for comparing the rate of convergence between the two iterative methods as follows:

Definition 4.3 ([17])**.** Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive numbers that converge to a, b, respectively. Assume there exists

$$\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = l.$$

(i) If $l = 0$, then it is said that the sequence $\{a_n\}$ *converges to a faster than* the sequence $\{b_n\}$ to b.

(ii) If $0 < l < \infty$, then we say that the sequences $\{a_n\}$ and $\{b_n\}$ *have the same rate of convergence.*

Definition 4.4 ([17, 18])**.** Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a mapping. Suppose $\{x_n\}$ and $\{m_n\}$ are two iterations which converge to a fixed point q of T. We say that $\{x_n\}$ *converges faster than* $\{m_n\}$ to q if

$$\lim_{n\to\infty} \frac{||x_n - q||}{||m_n - q||} = 0.$$

We now give an example which shows numerical experiment for supporting our main results and comparing the rate of convergence of the studied method and the modified Ishikawa iteration.

Example 4.5. Let $X = \mathbb{R}$ and $C = [0, 2]$. Let $G = (V(G), E(G))$ be a directed graph defined by $V(G) = C$ and $(x, y) \in E(G)$ if and only if $0.75 < x,\ y \leq 1.70$. Define a mapping $T_1, T_2 : C \to C$ by

$$T_1 x = \begin{cases} \frac{5}{8}\arcsin(x-1) + 1 & \text{if } x \neq \sqrt{3} \\ 0 & \text{if } x = \sqrt{3} \end{cases}$$

and

$$T_2 x = \begin{cases} x^{\log(2x)} & \text{if } x \neq \sqrt{2} \\ 2 & \text{if } x = \sqrt{2} \end{cases}$$

for all $x \in C$. Let $1 \leq k_n \leq 1.36$. Then T_1 and T_2 are G-asymptotically nonexpansive mappings. Let $x = \sqrt{3}, u = \sqrt{2}$ and $y = 1 = v$. Then $||T_1^n x - T_1^n y|| > k_n ||x - y||$ and $||T_2^n u - T_2^n v|| > k_n ||u - v||$ for all $n \in \mathbb{N}$. Let $\alpha_n = \frac{n+1}{5n+3}$ and $\beta_n = \frac{n+4}{10n+7}$. Choose $x_0 = 1.4$. Let $\{x_n\}$ be a sequence generated by *(3.1)* and $\{m_n\}$ be a sequence generated by the modified Ishikawa iteration *(1.1)*. We obtain the following numerical experiments for common fixed point of T_1 and T_2 and rate of convergence of $\{x_n\}$ and $\{m_n\}$.

We note that $x = 1$ is a common fixed point of T_1 and T_2.

Fig.1 Numerical errors of modified Ishikawa and new iteration

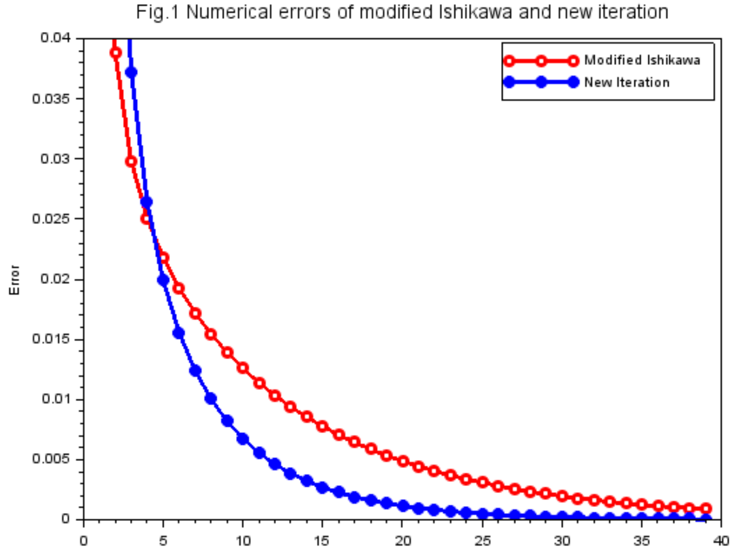

Table 1. Numerical experiments of Example 4.5

n	Modified Ishikawa	new itaretion	Rate of convergence		
	m_n	x_n	$\lvert m_n - 1\rvert$	$\lvert x_n - 1\rvert$	$\frac{\lvert x_n-1\rvert}{\lvert m_n-1\rvert}$
1	1.3224	1.2318	0.3224	0.2318	0.7192
2	1.2836	1.1711	0.2836	0.1712	0.6036
3	1.2538	1.1339	0.2538	0.1339	0.5276
4	1.2287	1.1074	0.2287	0.1075	0.4699
5	1.2069	1.0875	0.2069	0.0875	0.4231
...	...	...	...	...	...
34	1.0146	1.0007	0.0146	0.0007	0.0416

Table 2. Numerical errors of modified Ishikawa and new iteration

n	Modified Ishikawa		new itaretion	
	m_n	$\lvert m_n - m_{n-1}\rvert$	x_n	$\lvert x_n - x_{n-1}\rvert$
1	1.3224	0.0776	1.2318	0.1682
2	1.2836	0.0388	1.1711	0.0607
3	1.2538	0.0298	1.1339	0.0372
4	1.2287	0.0251	1.1074	0.0264
5	1.2069	0.0218	1.0875	0.0199
...	...	...	...	...
34	1.0146	0.0014	1.0007	0.0001

From Tables 1 and 2, we see that both $\{m_n\}$ and $\{x_n\}$ converge to $1 \in F$ and observe that $|x_n - 1| \leq |m_n - 1|$ and $\lim_{n\to\infty} \frac{|x_n - 1|}{|m_n - 1|} = 0$, so the sequence $\{x_n\}$ converges faster than $\{m_n\}$ generated by the modified Ishikawa iteration (Fig.1)

Acknowledgements : The author does deeply appreciate and would like to convey a profound gratitude to Professor Suthep Suantai for his invaluable advices on this research.

References

[1] K. Goebel, W.A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 35 (1972) 171-174.

[2] H. Fukhar-ud-din, S.H. Khan, Convergence of iterate with errors of asymptotically quasi-nonexpansive mappings and application, J. Math. Anal. Appl. 328 (2007) 821-829.

[3] H.K. Xu, Existence and convergence for fixed points of asymptotically nonexpansive type, Nonlinear Anal. 16 (1991) 1139-1146.

[4] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull. Aust. Math. Soc. 43 (1991) 153-159.

[5] K.K. Tan, H.K. Xu, Approximating fixed points of nonexpansive mapping by the Ishikawa iteration process, J. Math. Anal. Appl. 178 (1993) 301-308.

[6] S.S. Chang, Y.J. Cho, J.K. Kim, The equivalence between the convergence of modified Picard, modified Mann, and modified Ishikawa iterations, Mathematical and Computer Modelling 37 (2003) 985-991.

[7] J. Jachymski, The contraction principle for mappings on a metric space with a graph, Proc. Amer. Math. Soc. 136 (4) (2008) 1359-1373.

[8] J. Tiammee, A. Kaewkhao, S. Suantai, On Browder's convergence theorem and Halpern iteration process for G-nonexpansive mappings in Hilbert spaces endowed with graphs, Fixed Point Theory Appl. (2015); DOI 10.1186/s13663-015-0436-9.

[9] R. Suparatulatorn, W. Cholamjiak, S. Suantai, A modified S-iteration process for G-nonexpansive mappings in Banach spaces with graphs, Numer Algor 77 (2018) 479-490.

[10] Z. Opial, Weak convergence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967) 591-597.

[11] P.K. Lin, K.K. Tan, H.K. Xu, Demiclosedness principle and asymptotic behavior for asymptotically nonexpansivemappings, Nonlinear Anal. 24 (6) (1995) 929-946.

[12] H.K. Xu, Inequalities in Banach spaces with applications. Nonlinear Anal. 16 (1991) 1127-1138.

[13] M.G. Sangago, T.W. Hunde, H.Z. Hailu, Demicloseness and fixed points of G-asymptotically nonexpansive mapping in Banach spaces with graph, Adv. Fixed Point Theory 8 (3) (2018) 313-340.

[14] S. Suantai, Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings, J. Math. Anal. Appl. 331 (2005) 506-517.

[15] B.E. Rhoades, Comments on two xed point iteration methods, J. Math. Anal. Appl. 56 (1976) 741-750.

[16] R.L. Burden, J.D. Faires, Numerical Analysis, 9th edn. Brooks/Cole Cengage Learning, Boston (2010).

[17] V. Berinde, Iterative Approximation of Fixed Points. Editura Efemeride, Baia Mare (2002).

[18] W. Phuengrattana, S. Suantai, Comparison of the rate of convergence of various iterative methods for the class of weak contractions in Banach spaces, Thai J. Math. 11 (2013) 217-226.

(Received 27 August 2018)
(Accepted 31 December 2018)