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Ψ-Stability for Nonlinear Difference Equations

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Abstract : This paper deals with obtaining sufficient conditions for the Ψ-(uniform)stability of trivial solutions of linear and nonlinear difference equations on ℕ. And also we provide a way to construct (uniformly)stable difference equation from the given equation using the concept of Ψ-(uniform)stability.

Keywords : difference equations; fundamental matrix; Ψ-stable; Ψ-uniformly stable.
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1 Introduction

The purpose of this paper is to obtain sufficient conditions for the Ψ-(uniform) stability of trivial solution of nonlinear difference equation

$$x(n+1) = A(n)x(n) + f(n, x(n)) \tag{1.1}$$

and the linear difference equation

$$x(n+1) = A(n)x(n) + B(n)x(n), \tag{1.2}$$

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as a perturbed equation of

$$x(n+1) = A(n)x(n), \tag{1.3}$$

where $A(n)$, $B(n)$ are $m \times m$ matrix-valued functions and $f(n, x(n))$ is a vector-valued function of order m on $\mathbb{N} = \{0, 1, 2, \ldots\}$. Also, we develop new difference equations corresponding to (1.1)-(1.3) which are (uniformly)stable on $\mathbb{N}$, provided (1.1)-(1.3) are Ψ-(uniformly)stable on $\mathbb{N}$. We investigate conditions on the fundamental matrix $Y(n)$ for the linear equation (1.3) and on the function $f(n, x)$ under which the trivial solution of (1.1) is Ψ-(uniformly)stable on $\mathbb{N}$. Here, Ψ is a matrix function.

Difference equations play an important role in many scientific fields such as numerical analysis, finite element techniques, control theory, discrete mathematical structures and several problems of mathematical modeling [1–3]. The theory of difference equations is of immense use in the construction of discrete mathematical models, which can explain better when compared to continuous models. Marchalo [4] introduced the notions of Ψ-(uniform)stability for trivial solution of the nonlinear system $x' = f(t, x)$ and also obtained new sufficient conditions for the linear system $x' = A(t)x$. Recently, Han and Hong [5], Diamandescu [6, 7], Suresh Kumar, Rao and Murty [8, 9] extended the concept of Ψ-bounded solutions of differential equations to difference equations. The Ψ-(uniformly)stability for nonlinear difference equations are not yet studied. With the motivation of the above works, in this paper we obtain sufficient conditions for the Ψ-(uniformly)stable for the trivial solution of nonlinear difference equations.

2 Preliminaries

Let $\mathbb{R}^m$ denote the m-dimensional Euclidean space. For $x = (x_1, x_2, x_3, ..., x_m)^T$ in $\mathbb{R}^m$, let $\|x\| = \max\{|x_1|, |x_2|, \ldots, |x_m|\}$ be the norm of x. For an $m \times m$ matrix $A = [a_{ij}]$, we define the norm $|A| = \sup_{\|x\| \leq 1} \|Ax\|$.

For the existence of solution of (1.1), we assume that A is an invertible $m \times m$ matrix on $\mathbb{N}$ and $f : \mathbb{N} \times \mathbb{R}^m \to \mathbb{R}^m$, is a m-vector such that $f(n, 0) = 0$ for $n \in \mathbb{N}$.

Let Ψ be a diagonal matrix of order m defined by

$$\Psi = \text{diag}[\Psi_1, \Psi_2, \ldots \Psi_m],$$

where $\Psi_i : \mathbb{N} \to (0, \infty)$, $i = 1, 2 \ldots m$.

Definition 2.1. If the vector valued function $u(n) \in \mathbb{R}^m$ satiesfies difference equation (1.1), then $u(n)$ is called a *solution* of (1.1). It is clear that $u(n) = 0$ (zero vector) is always a solution of (1.1) and is called a *trivial solution* of (1.1). And also $u(n) = 0$ is the trivial solution of (1.2) and (1.3).

Definition 2.2. [1] Any $m \times m$ matrix $Y(n)$ whose columns are linearly independent solutions of the difference equation (1.3) is called a *fundamental matrix* of (1.3).

It is obvious that, $Y(n)$ is the solution of (1.3) and is nonsingular.

Definition 2.3. [1] The solution $u(n)$ of $x(n+1) = f(n, x(n))$ is said to be

(i) *stable*, if for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, n_0) > 0$ such that any solution $\bar{u}(n)$ of $x(n+1) = f(n, x(n))$, the inequality $\|u(n_0) - \bar{u}(n_0)\| < \delta$, implies $\|\bar{u}(n) - u(n)\| < \varepsilon$ for all $n \geq n_0$, where $n, n_0 \in \mathbb{N}$.

(ii) *uniformly stable*, if it is stable and δ independent of n_0.

Now, we define Ψ-stability and Ψ-uniform stability for trivial solutions of (1.1).

Definition 2.4. The trivial solution of (1.1) is said to be *Ψ-stable* on $\mathbb{N}$ if for every $\varepsilon > 0$ and every n_0 in $\mathbb{N}$, there exists $\delta = \delta(\varepsilon, n_0) > 0$ such that any solution $x(n)$ of (1.1) which satisfies the inequality $\|\Psi(n_0)x(n_0)\| < \delta$, also exists and satisfies the inequality $\|\Psi(n)x(n)\| < \varepsilon$ for all $n \geq n_0$.

Definition 2.5. The trivial solution of (1.1) is said to be *Ψ-uniformly stable* on $\mathbb{N}$ if it is Ψ-stable on $\mathbb{N}$ and in Definition 2.4 δ is independent of n_0.

Remark 2.6. For $\Psi_i = 1$, $i = 1, 2 \ldots n$, we obtain the notions of classical stability and uniform stability.

Now, we generate a difference equation, by multiplying $\Psi(n+1)$ on both sides of (1.1)

$$\Psi(n+1)x(n+1) = \Psi(n+1)A(n)x(n) + \Psi(n+1)f(n, x(n))$$

$$\Psi(n+1)x(n+1) = \Psi(n+1)A(n)\Psi^{-1}(n)\Psi(n)x(n) + \Psi(n+1)f(n, \Psi^{-1}(n)\Psi(n)x(n)).$$

Taking $z(n) = \Psi(n)x(n)$, the above equation can be written as

$$z(n+1) = A_\Psi(n)z(n) + F_\Psi(n, z(n)), \tag{2.1}$$

where $A_\Psi(n) = \Psi(n+1)A(n)\Psi^{-1}(n)$ and $F_\Psi(n, z(n)) = \Psi(n+1)f(n, \Psi^{-1}(n)z(n))$. The corresponding linear equation of (2.1) is

$$z(n+1) = A_\Psi(n)z(n). \tag{2.2}$$

From Definitions 2.4, 2.5 and Remark 2.6, we have the following lemma.

Lemma 2.7. *The trivial solution of* (1.1) *is Ψ-(uniformly)stable on* $\mathbb{N}$ *if and only if the trivial solution of* (2.1) *is (uniformly)stable on* $\mathbb{N}$.

Remark 2.8. From Lemma 2.7, we infer that if the difference equation (1.1) is not (uniformly)stable on $\mathbb{N}$, we can generate (uniformly)stable difference equation (2.1) from (1.1) with the help of Ψ-(uniform)stability of (1.1).

The following lemma represent the relationship between fundamental matrices of (1.3) and (2.2).

Lemma 2.9. *Let $Y(n)$ be the fundamental matrix of* (1.3)*, then $Z(n) = \Psi(n)Y(n)$ is the fundamental matrix of* (2.2)*.*

Proof. Since $Y(n)$ is the fundamental matrix of (1.3), then $Y(n)$ satisfies (1.3) and nonsingular. Consider

$$\begin{aligned} Z(n+1) &= \Psi(n+1)Y(n+1) \\ &= \Psi(n+1)A(n)Y(n) \\ &= \Psi(n+1)A(n)\Psi^{-1}(n)\Psi(n)Y(n) \\ &= A_\Psi(n)Z(n) \end{aligned}$$

Since $\Psi(n)$ and $Y(n)$ are nonsingular matrices, then $Z(n)$ also nonsingular. Therefore, $Z(n) = \Psi(n)Y(n)$ is the fundamental matrix of (2.2). □

3 Ψ-Stability for Linear Difference Equations

The purpose of this section is to obtain sufficient conditions for Ψ-(uniform) stability of trivial solution of linear difference equations (1.2) and (1.3). These conditions can be expressed in terms of the fundamental matrix of (1.3).

In the following theorem we obtain necessary and sufficient condition for Ψ-(uniform)stability of trivial solution of linear systems (1.2).

Theorem 3.1. *Let $Y(n)$ be a fundamental matrix for* (1.3)*. Then*

(i) The trivial solution of (1.3) *is Ψ-stable on $\mathbb{N}$ if and only if there exists a positive constant M such that $|\Psi(n)Y(n)| \leq M$ for all $n \in \mathbb{N}$.*

(ii) The trivial solution of (1.3) *is Ψ-uniformly stable on $\mathbb{N}$ if and only if there exists a positive constant M such that $|\Psi(n)Y(n)Y^{-1}(k)\Psi^{-1}(k)| \leq M$, for all $k \leq n$ and $k, n \in \mathbb{N}$.*

Proof. The solution of (1.3) with $x(n_0) = x_0$ is $x(n) = Y(n)Y^{-1}(n_0)x_0$ for $n \in \mathbb{N}$.

Suppose that there exist $M > 0$ such that $|\Psi(n)Y(n)| \leq M$ for $n \in \mathbb{N}$. For $\varepsilon > 0$ and $n_0 \in \mathbb{N}$, let $\delta(\varepsilon, n_0) = \dfrac{\varepsilon}{2M|Y^{-1}(n_0)\Psi^{-1}(n_0)|}$. For $\|\Psi(n_0)x(n_0)\| < \delta$ and $n \geq n_0$, we get

$$\begin{aligned} \|\Psi(n)x(n)\| &= \|\Psi(n)Y(n)Y^{-1}(n_0)\Psi^{-1}(n_0)\Psi(n_0)x(n_0)\| \\ &= M|Y^{-1}(n_0)\Psi^{-1}(n_0)|\delta < \varepsilon, \end{aligned}$$

which implies that the trivial solution of (1.3) is Ψ-stable on $\mathbb{N}$.

Conversely suppose that the trivial solution of (1.3) is Ψ-stable on $\mathbb{N}$. Then, for $\varepsilon = 1$ and $n_0 = 0$, there exists $\delta > 0$ such that any solution $x(n)$ of (1.3) which satisfies the inequality $\|\Psi(0)x(0)\| < \delta$ and

$$\|\Psi(n)Y(n)(\Psi(0)Y(0))^{-1}\Psi(0)x(0)\| < 1 \quad \text{for } n \in \mathbb{N}.$$

Let $v \in \mathbb{R}^n$ be such that $\|v\| \le 1$. If we take $x(0) = \frac{\delta}{2}\Psi^{-1}(0)v$, then we have $\|\Psi(0)x(0)\| < \delta$. Hence, $\|\Psi(n)Y(n)(\Psi(0)Y(0))^{-1}\frac{\delta}{2}v\| < 1$ for $n \in \mathbb{N}$. Therefore, $|\Psi(n)Y(n)(\Psi(0)Y(0))^{-1}| \le 2/\delta$ for $n \in \mathbb{N}$. Hence, $|\Psi(n)Y(n)| \le M$, a positive constant, for $n \in \mathbb{N}$.

Part (ii) is proved similarly. □

The following example illustrate Theorem 3.1.

Example 3.2. Consider the linear difference equation (1.3) with

$$A(n) = \begin{bmatrix} 0 & 1 \\ -\frac{2}{n+1} - 1 & 0 \end{bmatrix}.$$

Then its fundamental matrix is

$$Y(n) = \begin{bmatrix} (n+1)\cos\frac{n\pi}{2} & (n+1)\sin\frac{n\pi}{2} \\ -(n+2)\sin\frac{n\pi}{2} & (n+2)\cos\frac{n\pi}{2} \end{bmatrix}.$$

Clearly $Y(n)$ is unbounded on $\mathbb{N}$, it follows that the equation (1.3) is not stable on $\mathbb{N}$.

Now, we construct a difference equation (2.2) from (1.3), which is uniformly stable on $\mathbb{N}$ with the help of Theorem 3.1 and Lemma 2.7. Consider

$$\Psi(n) = \begin{bmatrix} \frac{1}{n+1} & 0 \\ 0 & \frac{1}{n+2} \end{bmatrix}.$$

Then for all $k \le n$, $k, n \in \mathbb{N}$, we get

$$\Psi(n)Y(n)Y^{-1}(k)\Psi^{-1}(k) = \begin{bmatrix} \cos\frac{(n-k)\pi}{2} & \sin\frac{(n-k)\pi}{2} \\ -\sin\frac{(n-k)\pi}{2} & \cos\frac{(n-k)\pi}{2} \end{bmatrix}$$

and $|\Psi(n)Y(n)Y^{-1}(k)\Psi^{-1}(k)| \le 2$. From Theorem 3.1, the trivial solution of linear difference equation (1.3) is Ψ-uniformly stable on $\mathbb{N}$.

By Lemma 2.7, the difference equation (2.2) with

$$A_\Psi(n) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

is uniformly stable on $\mathbb{N}$.

Remark 3.3. Ψ-uniform stability implies Ψ-stability but the converse need not be true. It is shown by the following example.

Example 3.4. Consider the linear difference equation (1.3) with

$$A(n) = \begin{bmatrix} \left(\frac{n+2}{n+1}\right)^2 & -\left(\frac{n+2}{n+1}\right)^2 (2n+3)e^n \\ 0 & \left(\frac{n+2}{n}\right)^2 e^{-1} \end{bmatrix}.$$

Then the fundamental matrix of (1.3) is

$$Y(n) = \begin{bmatrix} (n+1)^2 & 1 \\ 0 & \dfrac{e^{-n}}{(n+2)^2} \end{bmatrix}.$$

If

$$\Psi(n) = \begin{bmatrix} \frac{1}{(n+1)^2} & 0 \\ 0 & (n+2)^2 \end{bmatrix}, \quad \text{for all} \quad n \in \mathbb{N},$$

then

$$\Psi(n)Y(n) = \begin{bmatrix} 1 & \dfrac{1}{(n+1)^2} \\ 0 & e^{-n} \end{bmatrix}, \quad \text{for all} \quad n \in \mathbb{N}.$$

Clearly $|\Psi(n)Y(n)| \leq 2$, for all $n \in \mathbb{N}$. From Theorem 3.1, the equation (1.3) is Ψ-stable on $\mathbb{N}$. On the other hand

$$\Psi(n)Y(n)Y^{-1}(k)\Psi^{-1}(k) = \begin{bmatrix} 1 & e^k\left(\dfrac{1}{(n+1)^2} - \dfrac{1}{(k+1)^2}\right) \\ 0 & e^{k-n} \end{bmatrix}.$$

is unbounded for $0 \leq k \leq n$, $n, k \in \mathbb{N}$. Again from Theorem 3.1, the equation (1.3) is not Ψ-uniformly stable on $\mathbb{N}$.

Now, we consider various Ψ-stability problems connected with the linear difference equation (1.2) as a perturbed equation of (1.3). We seek conditions under which the Ψ-(uniform) stability of (1.3) implies the Ψ-(uniform)stability of (1.2).

Theorem 3.5. *Suppose that $B(n)$ is an $m \times m$ matrix function for $n \in \mathbb{N}$. If the linear difference equation* (1.3) *is Ψ-uniformly stable on $\mathbb{N}$ and*

$$\sum_{k=0}^{\infty} |\Psi(k+1)B(k)\Psi^{-1}(k)| < +\infty,$$

then the perturbed linear difference equation (1.2) *is also Ψ-uniformly stable on $\mathbb{N}$.*

Proof. Let $Y(n)$ be a fundamental matrix for the linear difference equation (1.3). Since the equation (1.3) is Ψ-uniformly stable on $\mathbb{N}$, there exists a positive constant M such that

$$|\Psi(n)Y(n)Y^{-1}(k)\Psi^{-1}(k)| \leq M \quad \text{for } 0 \leq k \leq n, \quad k, n \in \mathbb{N}.$$

The solution of (1.2) with initial condition $x(n_0) = x_0$ is unique and defined for all $n \in \mathbb{N}$. Therefore, by the variation of constants formula,

$$x(n) = Y(n)Y^{-1}(n_0)x_0 + \sum_{k=n_0+1}^{n} Y(n)Y^{-1}(k)B(k-1)x(k-1).$$

For $n_0 \le n$, $n, n_0 \in \mathbb{N}$,

$$\Psi(n)x(n) = \Psi(n)Y(n)Y^{-1}(n_0)\Psi^{-1}(n_0)\Psi(n_0)x_0 + \sum_{k=n_0+1}^{n} \Psi(n)Y(n)Y^{-1}(k)\Psi^{-1}(k)\Psi(k)B(k-1)\Psi^{-1}(k-1)\Psi(k-1)x(k-1).$$

From the above conditions, it results that

$$\|\Psi(n)x(n)\| \le M\|\Psi(n_0)x(n_0)\| + M\sum_{k=n_0}^{n-1} |\Psi(k+1)B(k)\Psi^{-1}(k)|||\Psi(k)x(k)||.$$

Therefore, by Gronwall's inequality,

$$\|\Psi(n)x(n)\| \le M\|\Psi(n_0)x_0\| \prod_{k=n_0}^{n-1} \big(1 + M|\Psi(k+1)B(k)\Psi^{-1}(k)|\big) \le M\|\Psi(n_0)x_0\| e^{M\sum_{k=n_0}^{n-1} |\Psi(k+1)B(k)\Psi^{-1}(k)|}.$$

The above inequality shows that the linear difference equation (1.2) is Ψ-uniformly stable on $\mathbb{N}$. □

If the linear difference equation (1.3) is only Ψ-stable, then its perturbed equation (1.2) need not be Ψ-stable. This is shown by the following example.

Example 3.6. Consider the linear difference equation (1.2) with $A(n)$ as in Example 3.4 and

$$B(n) = \begin{bmatrix} 0 & \left(\dfrac{(2n+1)(n+2)^2}{4} + \dfrac{2n+3}{(n+1)^2}\right)(n+2)^2e^n \\ 0 & 0 \end{bmatrix}.$$

From Example 3.4, equation (1.3) is Ψ-stable on $\mathbb{N}$. The fundamental matrix of (1.2) is

$$\widetilde{Y}(n) = \begin{bmatrix} (n+1)^2 & \dfrac{n^2(n+1)^2}{4} \\ 0 & \dfrac{e^{-n}}{(n+2)^2} \end{bmatrix}.$$

Let $\Psi(n) = \begin{bmatrix} \dfrac{1}{(n+1)^2} & 0 \\ 0 & (n+2)^2 \end{bmatrix}$. Then, we have

$$\Psi(n)\widetilde{Y}(n) = \begin{bmatrix} 1 & \dfrac{n^2}{4} \\ 0 & e^{-n} \end{bmatrix}.$$

Because $|\Psi(n)\widetilde{Y}(n)|$ is unbounded on $\mathbb{N}$, it follows that the equation (1.2) is not Ψ-stable on $\mathbb{N}$.

Remark 3.7. Theorem 3.5 is no longer true if we require that

$$\Psi(k+1)B(k)\Psi^{-1}(k) \to 0$$

as $k \to \infty$, instead of the condition

$$\sum_{k=0}^{\infty} |\Psi(k+1)B(k)\Psi^{-1}(k)| < +\infty.$$

This is shown by the following example.

Example 3.8. Consider the equation (1.2) with

$$A(n) = \begin{bmatrix} 0 & \frac{n+1}{n+2} \\ -\frac{n+1}{n+2} & 0 \end{bmatrix}, \quad B(n) = \begin{bmatrix} 0 & \frac{2}{n+2} \\ -\frac{2}{n+2} & 0 \end{bmatrix}.$$

Then

$$Y(n) = \frac{1}{n+1} \begin{bmatrix} \cos\left(\frac{n\pi}{2}\right) & \sin\left(\frac{n\pi}{2}\right) \\ -\sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right) \end{bmatrix} \text{ and } \widetilde{Y}(n) = (n+2) \begin{bmatrix} \cos\left(\frac{n\pi}{2}\right) & \sin\left(\frac{n\pi}{2}\right) \\ -\sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right) \end{bmatrix}$$

are fundamental matrices for the systems (1.3) and (1.2) respectively.

If $\Psi(n) = \begin{bmatrix} n+1 & 0 \\ 0 & n+1 \end{bmatrix}$, then

$$\Psi(n)Y(n)Y^{-1}(k)\Psi^{-1}(k) = \begin{bmatrix} \cos\left(\frac{(n-k)\pi}{2}\right) & \sin\left(\frac{(n-k)\pi}{2}\right) \\ -\sin\left(\frac{(n-k)\pi}{2}\right) & \cos\left(\frac{(n-k)\pi}{2}\right) \end{bmatrix},$$

for $0 \le k \le n < \infty$. It is easily seen that the equation (1.3) is Ψ-uniformly stable on $\mathbb{N}$. And also

$$\Psi(n)\widetilde{Y}(n) = (n+1)(n+2) \begin{bmatrix} \cos\left(\frac{n\pi}{2}\right) & \sin\left(\frac{n\pi}{2}\right) \\ -\sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right) \end{bmatrix}.$$

It follows that the equation (1.2) is not Ψ-uniformly stable on $\mathbb{N}$. Finally, we have

$$\sum_{n=0}^{\infty} |\Psi(n+1)B(n)\Psi^{-1}(n)| = \sum_{n=0}^{\infty} \frac{2}{n+1} = \infty$$

and

$$\lim_{n\to\infty} |\Psi(n+1)B(n)\Psi^{-1}(n)| = 0.$$

Theorem 3.9. *Suppose that:*

1. *The linear equation* (1.3) *is* Ψ*-stable on* $\mathbb{N}$.

2. *There exist a sequence $\varphi : \mathbb{N} \to (0, \infty)$ and a positive constant L such that the fundamental matrix $Y(n)$ of (1.3) satisfies the condition*

$$\sum_{k=0}^{n-1} \varphi(k)|\Psi(n)Y(n)Y^{-1}(k+1)\Psi^{-1}(k+1)| \leq L, \quad n \in \mathbb{N}. \tag{3.1}$$

3. *$B(n)$ is a $m \times m$ matrix function on $\mathbb{N}$ such that*

$$b = \sup_{n \in \mathbb{N}} \varphi^{-1}(n)|\Psi(n+1)B(n)\Psi^{-1}(n)| \tag{3.2}$$

is a sufficiently small number and $Lb < 1$.

Then the linear perturbed equation (1.2) *is Ψ-stable on $\mathbb{N}$.*

Proof. From the assumption (1) and Theorem 3.1, there exists a constant M such that $|\Psi(n)Y(n)| \leq M$, for all $n \in \mathbb{N}$. By using variation of constant formula the solution of (1.2) with $x(n_0) = x_0$ is given by

$$x(n) = Y(n)Y^{-1}(n_0)x_0 + \sum_{k=n_0}^{n-1} Y(n)Y^{-1}(k+1)B(k)x(k), \quad n \in \mathbb{N}.$$

For a given $\varepsilon > 0$, $n_0 \in \mathbb{N}$, we choose $\delta(n_0, \varepsilon) = \dfrac{(1-Lb)\varepsilon}{2M|Y^{-1}(n_0)\Psi^{-1}(n_0)|}$ such that $\|\Psi(n_0)x_0\| < \delta$. Consider

$$\begin{aligned} \|\Psi(n)x(n)\| \leq & |\Psi(n)Y(n)| \ |Y^{-1}(n_0)\Psi^{-1}(n_0)| \ \|\Psi(n_0)x(n_0)\| \\ & + \sum_{k=n_0}^{n-1} \varphi(k)|\Psi(n)Y(n)Y^{-1}(k+1)\Psi^{-1}(k+1)|\varphi^{-1}(k) \\ & \qquad |\Psi(k+1)B(k)\Psi^{-1}(k)|\|\Psi(k)x(k)\| \\ \leq & M|Y^{-1}(n_0)\Psi^{-1}(n_0)|\delta + \sum_{k=n_0}^{n-1} \varphi(k)|\Psi(n)Y(n)Y^{-1}(k+1)\Psi^{-1}(k+1)| \\ & \left(\sup_{k \in \mathbb{N}} \varphi^{-1}(k)|\Psi(k+1)B(k)\Psi^{-1}(k)|\right) \|\Psi(k)x(k)\|. \end{aligned}$$

From the hypothesis $b = \sup_{k \in \mathbb{N}} \varphi^{-1}(k)|\Psi(k+1)B(k)\Psi^{-1}(k)|$ is sufficently small and $Lb < 1$.

Therefore, for $n_0 \leq n$, $n_0 \in \mathbb{N}$, we get

$$\|\Psi(n)x(n)\| \leq M|Y^{-1}(n_0)\Psi^{-1}(n_0)|\delta + Lb\|\Psi(n)x(n)\|.$$

It implies that

$$\sup_{n_0\le n}\|\Psi(n)x(n)\| \le (1-Lb)^{-1}M|Y^{-1}(n_0)\Psi^{-1}(n_0)|\delta$$
$$=\frac{M}{1-Lb}|Y^{-1}(n_0)\Psi^{-1}(n_0)|\delta$$
$$=\frac{\varepsilon}{2}<\varepsilon.$$

Hence the equation (1.2) is Ψ-stable on $\mathbb{N}$. □

Example 3.10. Consider the linear perturbed difference equation (1.2) with

$$A(n)=\begin{bmatrix}\frac{n+2}{n+1} & 0\\ 0 & 2\end{bmatrix},\quad B(n)=\begin{bmatrix}0 & 0\\ 0 & \frac{1}{2(n+1)}\end{bmatrix}.$$

Then the fundamental matrix of (1.3) is

$$Y(n)=\begin{bmatrix}n+1 & 0\\ 0 & 2^n\end{bmatrix}.$$

If

$$\Psi(n)=\begin{bmatrix}\frac{1}{n+1} & 0\\ 0 & 3^{-n}\end{bmatrix},\quad \varphi(n)=1, \forall\ n\in\mathbb{N},$$

then

$$\Psi(n)Y(n)=\begin{bmatrix}1 & 0\\ 0 & \left(\frac{2}{3}\right)^n\end{bmatrix},\quad |\Psi(n)Y(n)|=\left(\frac{2}{3}\right)^n\le 1, \forall\ n\in\mathbb{N}.$$

From Theorem 3.1, the linear difference equation (1.3) is Ψ-stable on $\mathbb{N}$. And also

$$\Psi(n)Y(n)Y^{-1}(k+1)\Psi^{-1}(k+1)=\begin{bmatrix}1 & 0\\ 0 & \left(\frac{2}{3}\right)^n\left(\frac{3}{2}\right)^{k+1}\end{bmatrix},$$

$$\sum_{k=0}^{n-1}\varphi(k)|\Psi(n)Y(n)Y^{-1}(k+1)\Psi^{-1}(k+1)|=\left(\frac{2}{3}\right)^n\sum_{k=0}^{n-1}\left(\frac{3}{2}\right)^{k+1}$$
$$=3\left(1-\left(\frac{2}{3}\right)^n\right)\le 3,$$

for $n\in\mathbb{N}$.

Consider

$$\Psi(n+1)B(n)\Psi(n)=\begin{bmatrix}0 & 0\\ 0 & \frac{1}{6(n+1)}\end{bmatrix},$$

we have

$$b=\sup_{n\in\mathbb{Z}_+}\varphi^{-1}(n)|\Psi(n+1)B(n)\Psi^{-1}(n)|=\frac{1}{6(n+1)}\le\frac{1}{6}.$$

All conditions of Theorem 3.9 are satisfied with $M=1$, $L=3$ and $b=\frac{1}{6}$. Therefore, the linear perturbed equation (1.2) is Ψ-stable on $\mathbb{N}$.

4 Ψ-Stability for Nonlinear Difference Equations

In this section, we consider the difference equation (1.1) as the perturbed equation of (1.3) and provide sufficient conditions on the nonlinear function $f(n, x(n))$, we obtain Ψ-(uniform)stability of the trivial solution of (1.1).

Theorem 4.1. *Suppose that*

(i) The linear equation (1.3) *is* Ψ*-stable on* $\mathbb{N}$.

(ii) There exist a sequence $\varphi : \mathbb{N} \to (0, \infty)$ *and a positive constant* L *such that the fundamental matrix* $Y(n)$ *of the equation* (1.3) *satisfies the condition* (3.1).

(iii) The nonlinear function f *satisfies the condition*

$$|\Psi(n+1)f(n, x(n))| \leq \frac{\alpha(n)}{\varphi(n)} \|\Psi(n)x(n)\|, \tag{4.1}$$

where $\alpha(n)$ *is a nonnegative sequence such that*

$$\sup_{n \geq n_0} \frac{\alpha(n)}{\varphi(n)} < \frac{1}{L}, \tag{4.2}$$

for all $n, n_0 \in \mathbb{N}$ *and* $x(n) \in \mathbb{R}^n$.

Then, the trivial solution of nonlinear difference equation (1.1) *is* Ψ*-stable on* $\mathbb{N}$.

Proof. From condition (i) and Threorem 3.1, it follows that there exists a positive constant M such that $|\Psi(n)Y(n)| \leq M$, for all $n \in \mathbb{N}$.

From (4.2), there exists β such that

$$\frac{\alpha(n)}{\varphi(n)} \leq \beta < \frac{1}{L}, \quad \text{for all} \quad n \in \mathbb{N}.$$

For a given $\varepsilon > 0$ and $n_0 \leq n$, $n_0 \in \mathbb{N}$, we choose

$$\delta = \min \left\{ \frac{\varepsilon}{2}, \frac{1 - \beta L}{2M|Y^{-1}(n_0)\Psi^{-1}(n_0)|} \right\}$$

such that $\|\Psi(n_0)x_0\| < \delta$. By variation of constants formula, the solution of (1.1) with $x(n_0) = x_0$ is given by

$$x(n) = Y(n)Y^{-1}(n_0)x(n_0) + \sum_{k=n_0}^{n-1} Y(n)Y^{-1}(k+1)f(k, x(k)), \tag{4.3}$$

for all $n_0 \le n \in \mathbb{N}$. Consider

$$\begin{aligned}\|\Psi(n)x(n)\| &\le |\Psi(n)Y(n)||Y^{-1}(n_0)\Psi^{-1}(n_0)|\|\Psi(n_0)x(n_0)\| \\ &\quad + \sum_{k=n_0}^{n-1} \varphi(k)|\Psi(n)Y(n)Y^{-1}(k+1)\Psi^{-1}(k+1)||\Psi(k+1)f(k,x(k))| \\ &\le M|Y^{-1}(n_0)\Psi^{-1}(n_0)|\delta + \sum_{k=n_0}^{n-1}\varphi(k)|\Psi(n)Y(n)Y^{-1}(k+1)\Psi^{-1}(k+1)| \\ &\qquad \left(\sup_{k\in\mathbb{N}} \frac{\alpha(k)}{\varphi(k)}\right)\|\Psi(k)x(k)\| \\ &\le M|Y^{-1}(n_0)\Psi^{-1}(n_0)|\delta + \beta L\|\Psi(n)x(n)\|.\end{aligned}$$

It implies that

$$\begin{aligned}\|\Psi(n)x(n)\| &\le \frac{M}{1-\beta L}|Y^{-1}(n_0)\Psi^{-1}(n_0)|\delta \\ &\le \frac{\varepsilon}{2} < \varepsilon.\end{aligned}$$

From Definition 2.4, the trivial solution of nonlinear difference equation (1.1) is Ψ-stable on $\mathbb{N}$. □

The following example illustrate above theorem.

Remark 4.2. Theorem 3.9 is the particular case of Theorem 4.1.

Example 4.3. Consider the nonlinear difference equation (1.1) with

$$A(n) = \begin{bmatrix} \frac{n+2}{n+1} & 0 \\ 0 & 2 \end{bmatrix}, \quad f(n,x(n)) = \begin{bmatrix} 5^{-n}\left(\frac{n+2}{n+1}\right)\sin(x_1(n)) \\ 3(5^{-n})x_2(n) \end{bmatrix}.$$

If

$$\Psi(n) = \begin{bmatrix} \frac{1}{n+1} & 0 \\ 0 & 3^{-n} \end{bmatrix}, \quad \phi(n) = 1, \forall \ n \in \mathbb{N}$$

then from Example 3.10, conditions (i) and (ii) of Theorem 4.1 are satisfied. Consider

$$\Psi(n+1)f(n,x(n)) = 5^{-n}\begin{bmatrix} \frac{\sin(x_1(n))}{n+1} \\ 3^{-n}x_2(n) \end{bmatrix},$$

we have

$$\|\Psi(n+1)f(n,x(n))\| \le 5^{-n}\|\Psi(n)x(n)\|,$$

for all $n \in \mathbb{N}$. If $\alpha(n) = 5^{-n}$, then (4.1) and (4.2)are satisfied. Therefore, all conditions of Theorem 4.1 are satisfied. Thus, the nonlinear equation (1.1) is Ψ-stable on $\mathbb{N}$.

From Lemma 2.7, the difference equation (2.1) with

$$A_\Psi = \begin{bmatrix} 1 & 0 \\ 0 & 2/3 \end{bmatrix} \quad \text{and} \quad F_\Psi(n, z(n)) = \frac{1}{5^n} \begin{bmatrix} \dfrac{\sin((n+1)z_1(n))}{n+1} \\ z_2(n) \end{bmatrix}$$

is stable on $\mathbb{N}$.

Theorem 4.4. *Suppose that :*

(i) The fundamental matrix $Y(n)$ of the equation (1.3) satisfies

$$|\Psi(n)Y(n)Y^{-1}(k)\Psi^{-1}(k)| \leq M,$$

for all $n_0 \leq k \leq n$, where M is a positive constant.

(ii) The nonlinear function f satisfies the condition

$$\|\Psi(n+1)f(n, x(n))\| \leq \alpha(n)\|\Psi(n)x(n)\|,$$

where $\alpha(n)$ is a nonnegative sequence on $\mathbb{N}$ such that $N = \sum_{k=0}^{\infty} \alpha(n) < \infty$.

Then, the trivial solution of the nonlinear difference equation (1.1) is Ψ-uniformly stable on $\mathbb{N}$.

Proof. Let $n_0 \in \mathbb{N}$ and $x(n_0) = x_0 \in \mathbb{N}$. Then the solution of (1.1) with $x(n_0) = x_0$ is given by (4.3). Let $\varepsilon > 0$ and $\delta(\varepsilon) = \dfrac{e^{-MN}}{2M}\varepsilon > 0$ such that $\|\Psi(n_0)x_0\| < \delta$. Consider

$$\begin{aligned} \|\Psi(n)x(n)\| \leq & |\Psi(n)Y(n)Y^{-1}(n_0)\Psi^{-1}(n_0)|\|\Psi(n_0)x_0\| \\ & + \sum_{k=n_0+1}^{n} |\Psi(n)Y(n)Y^{-1}(k)\Psi^{-1}(k)|\|\Psi(k)f(k-1, x(k-1))\| \\ & \leq M\delta + M\sum_{k=n_0}^{n-1} \|\Psi(k+1)f(k, x(k))\| \\ & \leq M\delta + M\sum_{k=n_0}^{n-1} \alpha(k)\|\Psi(k)x(k)\|. \end{aligned}$$

By Grownwall's inequality

$$\begin{aligned} \|\Psi(n)x(n)\| & \leq M\delta + \prod_{k=n_0}^{n-1}(1 + M\alpha(n)) \\ & \leq M\delta e^{M\sum_{k=n_0}^{n-1}\alpha(n)} \\ & < M\delta e^{MN} \leq \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

From Definition 2.5, the trivial solution of nonlinear difference equation (1.1) is Ψ-uniformly stable on $\mathbb{N}$. □

Remark 4.5. Theorem 3.5 is the particular case of Theorem 4.4.

Example 4.6. Consider the nonlinear difference system (1.1) with

$$A(n) = \begin{bmatrix} 0 & 1 \\ -\frac{n+3}{n+1} & 0 \end{bmatrix} \text{ and } f(n, x(n)) = \begin{bmatrix} 2^{-n}\left(\frac{n+2}{n+1}\right) x_1(n) \\ 2^{-n}\left(\frac{n+3}{n+2}\right)\sin(x_2(n)) \end{bmatrix}, \text{ for all } n \in \mathbb{N}.$$

The fundamental matrix of (1.3) is

$$Y(n) = \begin{bmatrix} (n+1)\cos\frac{n\pi}{2} & (n+1)\sin\frac{n\pi}{2} \\ -(n+2)\sin\frac{n\pi}{2} & (n+2)\cos\frac{n\pi}{2} \end{bmatrix}.$$

If

$$\Psi(n) = \begin{bmatrix} \frac{1}{n+1} & 0 \\ 0 & \frac{1}{n+2} \end{bmatrix} \text{ for } n \in \mathbb{N},$$

then from Example 3.2, we have

$$|\Psi(n)Y(n)Y^{-1}(k)\Psi^{-1}(k)| \le 2,$$

for $n_0 \le k \le n$. Consider

$$\Psi(n+1)f(n, x(n)) = 2^{-n}\begin{bmatrix} \frac{x_1}{n+1} \\ \frac{\sin(x_2(n))}{n+2} \end{bmatrix},$$

it follows that, condition (ii) of Theorem 4.4 satisfied with $\alpha(n) = 2^{-n}$ (nonnegative sequence) and

$$N = \sum_{k=0}^{\infty}\alpha(k) = \sum_{k=0}^{\infty} 2^{-n} = 2 < \infty.$$

Therefore, from Theorem 4.4 the trivial solution of (1.1) is Ψ-uniformly stable on $\mathbb{N}$.

From Lemma 2.7, the difference equation (2.1) with

$$A_\Psi = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } F_\Psi(n, z(n)) = \frac{1}{2^n}\begin{bmatrix} z_1(n) \\ \frac{\sin((n+2)z_2(n))}{n+2} \end{bmatrix}$$

is uniformly stable on $\mathbb{N}$.

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