



On Global Existence of Solutions for Abstract Nonlinear Functional Neutral Integro- Differential Equations with Nonlocal Condition

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Abstract : The aim of the present paper is to establish the global existence of solutions of initial value problem for nonlinear neutral functional integro-differential equations with nonlocal condition in Banach spaces. Our analysis is based on an application of the topological transversality theorem known as Leray-Schauder alternative, rely on a priori bounds of solution and the Pachpatte's inequality.

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1 Introduction

The theory of functional differential equations with nonlocal conditions has been extensively studied in the literature, see [1–7] as they have applications in physics and many other areas of applied mathematics. The nonlocal condition is more precise for describing natural phenomena than the classical condition because more information is taken into account, thereby decreasing the negative effects

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incurred by a possibly single measurement taken at initial time.

Let X be a Banach space with the norm $\|\cdot\|$. Let $C = C([-r, 0], X)$, $0 < r < \infty$, be the Banach space of all continuous functions $\psi : [-r, 0] \rightarrow X$ endowed with supremum norm

$$\|\psi\|_C = \sup\{\|\psi(t)\| : -r \leq t \leq 0\}.$$

Let $B = C([-r, T], X)$, $T > 0$, be the Banach space of all continuous functions $x : [-r, T] \rightarrow X$ with the supremum norm $\|x\|_B = \sup\{\|x(t)\| : -r \leq t \leq T\}$. For any $x \in B$ and $t \in [0, T]$, we denote x_t the element of C given by $x_t(\theta) = x(t + \theta)$, for $\theta \in [-r, 0]$. Consider the nonlinear functional integro-differential equations with non local condition of the type

$$\frac{d}{dt}[x(t) - u(t, x_t)] + Ax(t) = f(t, x_t, \int_0^t k(t, s)h(s, x_s)ds), \quad t \in [0, T], \quad (1.1)$$

$$x(t) + (g(x_{t_1}, \dots, x_{t_p}))(t) = \phi(t), \quad -r \leq t \leq 0, \quad (1.2)$$

where $-A$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$, $t \geq 0$ in X , $f : [0, T] \times C \times X \rightarrow X$, $k : [0, T] \times [0, T] \rightarrow \mathbb{R}$, $h : [0, T] \times C \rightarrow X$, $u : [0, T] \times C \rightarrow X$ are continuous functions and ϕ is given element of C . The function $g : C^p \rightarrow C$, where $0 < t_1 < t_2 < \dots < t_p \leq T$, $p \in \mathbb{N}$ is given function. For example

$$g(x_{t_1}, \dots, x_{t_p})(\theta) = \sum_{k=1}^p c_k x(t_k + \theta), \quad x \in C([-r, T], X), \quad \theta \in [-r, 0].$$

where c_k , ($k = 1, 2, 3, \dots, p$) are constants.

Equations of the form (1.1)-(1.2) or their special forms serve as an abstract formulation of partial integro-differential equations which arise in the problems with memory visco-elasticity and many other physical phenomena, see [6, 8–10] and the references given therein. The problem of existence, uniqueness and other properties of solutions of the equations of the type (1.1)-(1.2) or their special forms have been studied by many authors by using different techniques, see [1, 5, 7, 11–15].

The aim of the present paper is to study the existence of global solutions to integral equations (1.1)-(1.2). The main tool used in our analysis is based on an application of the topological transversality theorem known as Leray-Schauder alternative, rely on a priori bounds of solutions and Pachpatte's inequality. The interesting and useful aspect of the method employed here is that it yields simultaneously the global existence of solutions and the maximal interval of existence. This paper is organized as follows. Section 2 presents the preliminaries and the statement of our main result. In Section 3 we prove our main result.

2 Preliminaries and Main Result

Before proceeding to the statement of our main result, we shall set forth some preliminaries and hypotheses that will be used in our subsequent discussion.

Definition 2.1. A function $x : (-r, T) \rightarrow X, T > 0$ is called the *mild solution* of the Cauchy problem (1.1)-(1.2) if $x(t) + (g(x_{t_1}, \dots, x_{t_p}))(t) = \phi(t), -r \leq t \leq 0$, the restriction of $x(\cdot)$ to the interval $[0, T)$ is continuous and for each $0 \leq t < T$, the function $AT(t-s)u(s, x_s), s \in [0, t)$, is integrable and the integral equation

$$x(t) = T(t)[\phi(0) - g(x_{t_1}, \dots, x_{t_p})(0) - u(0, \phi(0) - g(x_{t_1}, \dots, x_{t_p})(0))] + u(t, x_t) + \int_0^t AT(t-s)u(s, x_s)ds + \int_0^t T(t-s)f(s, x_s, \int_0^s k(s, \tau)h(\tau, x_\tau)d\tau)ds,$$

$t \in [0, T]$, is satisfied.

The following Pachpatte’s inequality plays the key role in our analysis.

Lemma 2.2. [16] *Let u, f and g be non-negative continuous functions defined on R_+ , for which the inequality*

$$u(t) \leq u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s)(\int_0^s g(\sigma)u(\sigma)d\sigma)ds, \quad t \in R_+,$$

holds, where u_0 is non-negative constant. Then

$$u(t) \leq u_0[1 + \int_0^t f(s)exp(\int_0^s [f(\sigma) + g(\sigma)]d\sigma)ds], \quad t \in R_+.$$

Our main result is based on the following version of the topological transversality theorem, known as Leray-Schauder alternative given by Granas [17].

Lemma 2.3. *Let S be a convex subset of a normed linear space E and assume $0 \in S$. Let $F : S \rightarrow S$ be a completely continuous operator, and let*

$$\varepsilon(F) = \{x \in S : x = \lambda Fx \text{ for some } 0 < \lambda < 1\}.$$

Then either $\varepsilon(F)$ is unbounded or F has a fixed point.

We list the following hypotheses for our convenience.

(H₁) – A is the infinitesimal generator of a compact semigroup of bounded linear operators $T(t), t \geq 0$ in X . Then there exist constants $K_0 \geq 1$ and $K_1 > 0$ such that

$$\|T(t)\| \leq K_0 \quad \text{and} \quad \|AT(t)\| \leq K_1.$$

(H₂) There exists a continuous function $p : [0, T] \rightarrow \mathbb{R}_+ = [0, \infty)$ such that

$$\|f(t, \psi, x)\| \leq p(t)(\|\psi\|_C + \|x\|),$$

for every $t \in [0, T], \psi \in C$ and $x \in X$.

(H₃) There exists a continuous function $q : [0, T] \rightarrow \mathbb{R}_+$ such that

$$\|h(t, \psi)\| \leq q(t)(\|\psi\|_C).$$

(H₄) There exists a constant $G \geq 0$ such that

$$\max \|g(x_{t_1}, x_{t_2}, \dots, x_{t_p})\| \leq G.$$

(H₅) There exist constants $c_1 < 1$ and $c_2 > 0$ such that

$$\|u(t, \phi)\| \leq c_1 \|\phi\| + c_2, \quad t \in [0, T], \quad \phi \in C.$$

(H₆) For every positive integer k there exists $h_k \in L^1(0, T)$

$$\sup_{\|\psi\|_C, \|x\| \leq k} \|f(t, \psi, x)\| \leq h_k(t), \text{ for } t \in [0, T] \quad a.e.$$

(H₇) For each $t \in [0, T]$ the function $f(t, \dots) : C \times X \rightarrow X$ is continuous and for each $(\psi, x) \in C \times X$ the function $f(\cdot, \psi, x) : [0, T] \rightarrow X$ is strongly measurable.

(H₈) For each $t \in [0, T]$ the functions $h(t, \cdot) : C \rightarrow X$ are continuous and for each $\psi \in C$ the function $h(\cdot, \psi) : [0, T] \rightarrow X$ is strongly measurable.

With these preparations we state our main result to be proved in the present paper.

Theorem 2.4. *Suppose that hypotheses (H₁)-(H₈) are satisfied then the Cauchy problem (1.1)-(1.2) has a mild solution x on $[-r, T]$.*

Remark 2.5. It is to be noted that J.P. Daur and K. Bhalchandran [11] and M. B. Dhakne and G. B. Lamb [14] have studied equations like (1.1)-(1.2) by using Schaefer's fixed point theorem and Leray Schauder alternative with local condition respectively. We further remark that here in our theorem, we are achieving results for nonlocal problem with less hypotheses only by applying Pachpatte's inequality in addition to topological transversality theorem known as Leray Schauder alternative.

3 Proof of Theorem

To prove the existence of mild solution of the initial-value problem (1.1)-(1.2), we apply topological transversality theorem given in Theorem 2.4. First we establish the priori bounds on the solutions to the initial value problem (1) _{λ} -(1.2) for $\lambda \in (0, 1)$, where

$$\frac{d}{dt}[x(t) - \lambda u(t, x_t)] + Ax(t) = \lambda f(t, x_t, \int_0^t k(t, s)h(s, x_s)ds), \quad t \in [0, T]. \quad (1.1)_\lambda$$

Let $x(t)$ be a solution of the problem (1.1) _{λ} -(1.2) then it satisfies the equivalent integral equation

$$\begin{aligned} x(t) = & \lambda T(t)[\phi(0) - g(x_{t_1}, \dots, x_{t_p})(0) - u(0, \phi(0) - g(x_{t_1}, \dots, x_{t_p})(0))] + \lambda u(t, x_t) \\ & + \lambda \int_0^t AT(t-s)u(s, x_s)ds + \lambda \int_0^t T(t-s)f(s, x_s, \int_0^s k(s, \tau)h(\tau, x_\tau)d\tau)ds, \end{aligned} \quad (3.1)$$

$t \in [0, T]$,

$$x(t) + (g(x_{t_1}, \dots, x_{t_p}))(t) = \phi(t), \quad -r \leq t \leq 0. \tag{3.2}$$

Since k is continuous on compact set $[0, T] \times [0, T]$, there is constant $L > 0$ such that $|k(t, s)| \leq L$, for $0 \leq s \leq t \leq T$. Also let $\|\phi\|_C = c$. Using hypotheses $(H_1) - (H_5)$ and equation (3.1), we obtain

$$\begin{aligned} \|x(t)\| &\leq K_0[c + G + c_1(c + G) + c_2] + c_1\|x_t\|_C + c_2 + \int_0^t K_1[c_1\|x_s\|_C + c_2]ds \\ &\quad + \int_0^t K_0p(s)[\|x_s\|_C + \int_0^s |k(s, t)|\|h(\tau, x_\tau)\|d\tau]ds \\ &\leq K_0[(c + G) + c_1(c + G) + c_2] + c_1\|x_t\|_C + c_2 \\ &\quad + \int_0^t K_1c_1\|x_s\|_C ds + K_1c_2 \int_0^t ds \\ &\quad + \int_0^t K_0p(s)[\|x_s\|_C + \int_0^s Lq(\tau)(\|x_\tau\|_C)d\tau]ds \\ &\leq c_3 + c_1\|x_t\|_C + \int_0^t [K_1c_1 + K_0p(s)]\|x_s\|_C ds \\ &\quad + \int_0^t K_0p(s) \int_0^s Lq(\tau)(\|x_\tau\|_C)d\tau ds \end{aligned} \tag{3.3}$$

where $c_3 = K_0[c + G + c_1(c + G) + c_2] + c_2 + k_1c_2T$.

Let $M(t) = \sup\{\frac{K_1c_1+K_0p(t)}{1-c_1}, \frac{K_0p(t)}{1-c_1}, Lq(t)\}$ and $M^* = \sup\{M(t) : t \in [0, T]\}$. Define the function $z : [0, T] \rightarrow \mathbb{R}$ by $z(t) = \sup\{\|x(s)\| : -r \leq s \leq t\}$, $t \in [0, T]$. Let $t^* \in [-r, t]$ be such that $z(t) = \|x(t^*)\|$. If $t^* \in [0, T]$, then from (3.3), we have

$$\begin{aligned} z(t) &\leq c_3 + c_1\|x_{t^*}\|_C + \int_0^{t^*} [K_1c_1 + K_0p(s)]\|x_s\|_C ds \\ &\quad + \int_0^{t^*} K_0p(s) \int_0^s Lq(\tau)\|x_\tau\|_C d\tau ds \\ &\leq \frac{c_3}{1-c_1} + \int_0^t M(s)z(s)ds + \int_0^t M(s) \int_0^s M(\tau)z(\tau)d\tau ds \end{aligned} \tag{3.4}$$

If $t^* \in [-r, 0]$, then $z(t) \leq \|\phi\|_C + G \leq c + G$ and the inequality (3.4) holds good. Thus for $t^* \in [-r, T]$, the inequality (3.4) holds good. Thanks to Pachpatte's inequality given in lemma 2.2 and applying it to inequality (3.4) we get,

$$\begin{aligned} z(t) &\leq \frac{c_3}{1-c_1} \left[1 + \int_0^t M(s) \exp\left(\int_0^s M(\tau) + M(\tau)d\tau\right) ds \right] \\ &\leq \frac{c_3}{1-c_1} [1 + M^* \exp(2M^*T)T] = Q, \end{aligned}$$

where Q is some constant.

Consequently, we have $\|x\|_B = \sup\{\|x(t)\| : t \in [-r, T]\} \leq Q$.

Now, we rewrite initial value problem (1.1)-(1.2) as follows: For $\phi \in C$, define $\widehat{\phi} \in B$ by

$$\widehat{\phi}(t) = \begin{cases} \phi(t) - (g(x_{t_1}, \dots, x_{t_p}))(t) & \text{if } -r \leq t \leq 0 \\ T(t)[\phi(0) - (g(x_{t_1}, \dots, x_{t_p}))(0)] & \text{if } 0 \leq t \leq T. \end{cases}$$

If $y \in B$ and $x(t) = y(t) + \widehat{\phi}(t)$, $t \in [-r, T]$, then it is easy to see that y satisfies

$$\begin{aligned} y(t) &= y_0 = 0; \quad -r \leq t \leq 0 \quad \text{and} \\ y(t) &= -T(t)[u(0, \phi(0) - (g(x_{t_1}, \dots, x_{t_p}))(0))] + u(t, y_t + \widehat{\phi}_t) \\ &\quad + \int_0^t AT(t-s)u(s, y_s + \widehat{\phi}_s)ds \\ &\quad + \int_0^t T(t-s)f\left(s, y_s + \widehat{\phi}_s, \int_0^s k(s, \tau)h(\tau, y_\tau + \widehat{\phi}_\tau)d\tau\right) ds, \quad t \in [0, T] \end{aligned}$$

if and only if $x(t)$ satisfies the following integral equations

$$\begin{aligned} x(t) &= T(t)[\phi(0) - g(x_{t_1}, \dots, x_{t_p})(0) - u(0, \phi(0) - g(x_{t_1}, \dots, x_{t_p})(0))] + u(t, x_t) \\ &\quad + \int_0^t AT(t-s)u(s, x_s)ds + \int_0^t T(t-s)f\left(s, x_s, \int_0^s k(s, \tau)h(\tau, x_\tau)d\tau\right)ds, \end{aligned} \tag{3.5}$$

$t \in [0, T]$,

$$x(t) + (g(x_{t_1}, \dots, x_{t_p}))(t) = \phi(t), \quad -r \leq t \leq 0. \tag{3.6}$$

We define the operator $F : B_0 \rightarrow B_0$, $B_0 = \{y \in B : y_0 = 0\}$ by

$$(Fy)(t) = \begin{cases} 0 & \text{if } -r \leq t \leq 0 \\ -T(t)[u(0, \phi(0) - (g(x_{t_1}, \dots, x_{t_p}))(0))] + u(t, y_t + \widehat{\phi}_t) \\ \quad + \int_0^t AT(t-s)u(s, y_s + \widehat{\phi}_s)ds \\ \quad + \int_0^t T(t-s)f\left(s, y_s + \widehat{\phi}_s, \int_0^s k(s, \tau)h(\tau, y_\tau + \widehat{\phi}_\tau)d\tau\right) ds & \text{if } t \in [0, T]. \end{cases} \tag{3.7}$$

From the definition of an operator F defined by the equation (3.7), it is to be noted that the equations (3.5)-(3.6) can be written as

$$y = Fy$$

and the integral equations (3.1)-(3.2) can be written as

$$y = \lambda Fy.$$

Now, we prove that F is completely continuous. First, we prove that $F : B_0 \rightarrow B_0$ is continuous. Let $\{v_n\}$ be a sequence of elements of B_0 converging to v in B_0 . Then there exists an integer N such that $\|v_n(t)\| \leq N$ for all n and $t \in [0, T]$. So $v_n \in B_N$ and $v \in B_N$. Then by using hypothesis (H_6) - (H_8) we have

$$f\left(t, v_{n_t} + \widehat{\phi}_t, \int_0^t k(t, s)h(s, v_{n_s} + \widehat{\phi}_s)ds\right) \rightarrow f\left(t, v_t + \widehat{\phi}_t, \int_0^t k(t, s)h(s, v_s + \widehat{\phi}_s)ds\right)$$

for each $t \in [0, T]$, since

$$\begin{aligned} & \left\| f\left(t, v_{n_t} + \widehat{\phi}_t, \int_0^t k(t, s)h(s, v_{n_s} + \widehat{\phi}_s)ds\right) - f\left(t, v_t + \widehat{\phi}_t, \int_0^t k(t, s)h(s, v_s + \widehat{\phi}_s)ds\right) \right\| \\ & \leq 2h_{N'}(t), \end{aligned}$$

where $N' = \max\{N + \|\widehat{\phi}\|, TM^*[N + \|\widehat{\phi}\|]\}$. Then by dominated convergence theorem, we have

$$\begin{aligned} \|(Fv_n)(t) - (Fv)(t)\| & \leq \|u(t, v_{n_t} + \widehat{\phi}_t) - u(t, v_t + \widehat{\phi}_t)\| \\ & \quad + \int_0^t \|AT(t-s)\| \|u(s, v_{n_s} + \widehat{\phi}_s) - u(s, v_s + \widehat{\phi}_s)\| ds \\ & \quad + \int_0^t \|T(t-s)\| \left\| f\left(s, v_{n_s} + \widehat{\phi}_s, \int_0^s k(s, \tau)h(\tau, v_{n_\tau} + \widehat{\phi}_\tau)d\tau\right) \right. \\ & \quad \left. - f\left(s, v_s + \widehat{\phi}_s, \int_0^s k(s, \tau)h(\tau, v_\tau + \widehat{\phi}_\tau)d\tau\right) \right\| ds \\ & \rightarrow 0 \quad \text{as } n \rightarrow \infty, \forall t \in [0, T]. \end{aligned}$$

Since $\|Fv_n - Fv\|_B = \sup_{t \in [-r, T]} \|(Fv_n)(t) - (Fv)(t)\|$, it follows that

$$\|Fv_n - Fv\|_B \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which implies $Fv_n \rightarrow Fv$ in B_0 as $v_n \rightarrow v$ in B_0 . Therefore, F is continuous.

To prove that F is completely continuous it remains to prove that F maps a bounded set of B_0 into a precompact set of B_0 . Let $B_m = \{y \in B_0 : \|y\|_B \leq m\}$ for $m \geq 1$. Now, we show that F_{B_m} is uniformly bounded. From the equation (3.7) and using hypotheses $(H_1) - (H_5)$ and the fact that $\|y\|_B \leq m, y \in B_m$ implies $\|y_t\|_C \leq m, t \in [0, T]$ we obtain

$$\begin{aligned} \|(Fy)(t)\| & \leq \|T(t)\| [c_1\|\phi(0) - (g(x_{t_1}, \dots, x_{t_p}))(0)\| + c_2] + c_1\|y_t + \widehat{\phi}_t\| + c_2 \\ & \quad + \int_0^t K_1 [c_1\|y_s + \widehat{\phi}_s\| + c_2] ds \\ & \quad + \int_0^t K_0 p(s) [\|y_s + \widehat{\phi}_s\|_C + \int_0^s |k(s, \tau)| \|h(\tau, y_\tau + \widehat{\phi}_\tau)\| d\tau] ds \\ & \leq K_0 [c_1(c + G) + c_2] + c_1(m + c + G) + c_2 + K_1 c_1 (m + c + G)T \\ & \quad + K_1 c_2 T + K_0 TM^* [m + c + G + \frac{TM^*}{2} (m + c + G)]. \end{aligned}$$

This implies that the set $\{(Fy)(t) : \|y\|_B \leq m, -r \leq t \leq T\}$ is uniformly bounded in X and hence F_{B_m} is uniformly bounded. Now we show that F maps B_m into an equicontinuous family of functions with values in X . Let $y \in B_m$ and $t_1, t_2 \in [-r, T]$. Then from the equation (3.7) and using the hypotheses $(H_1) - (H_5)$ we have

Case 1 : Suppose $0 \leq t_1 \leq t_2 \leq T$

$$\begin{aligned} & \| (Fy)(t_2) - (Fy)(t_1) \| \\ & \leq \| T(t_1) - T(t_2) \| \| u(0, \phi(0) - (g(x_{t_1}, \dots, x_{t_p}))(0)) \| \\ & \quad + \| u(t_2, y_{t_2} + \widehat{\phi}_{t_2}) - u(t_1, y_{t_1} + \widehat{\phi}_{t_1}) \| \\ & \quad + \int_0^{t_1} \| AT(t_2 - s) - AT(t_1 - s) \| \| u(s, y_s + \widehat{\phi}_s) \| ds \\ & \quad + \int_{t_1}^{t_2} \| AT(t_2 - s) \| \| u(s, y_s + \widehat{\phi}_s) \| ds \\ & \quad + \int_0^{t_1} \| T(t_2 - s) - T(t_1 - s) \| M^* [m + c + G + TM^*(m + c + G)] ds \\ & \quad + \int_{t_1}^{t_2} \| T(t_2 - s) \| M^* [m + c + G + TM^*(m + c + G)] ds. \end{aligned}$$

Case 2 : Suppose $-r \leq t_1 \leq 0 \leq t_2 \leq T$. Then $(Fy)(t_1) = 0$. Therefore proceeding as in Case 1, we get

$$\begin{aligned} & \| (Fy)(t_2) - (Fy)(t_1) \| = \| (Fy)(t_2) \| \\ & \leq \| -T(t_2)[u(0, \phi(0) - (g(x_{t_1}, \dots, x_{t_p}))(0))] + u(t_2, y_{t_2} + \widehat{\phi}_{t_2}) \| \\ & \quad + \int_0^{t_2} \| AT(t_2 - s)u(s, y_s + \widehat{\phi}_s) \| ds \\ & \quad + \int_0^{t_2} \| T(t_2 - s)f \left(s, y_s + \widehat{\phi}_s, \int_0^s k(s, \tau)h(\tau, y_\tau + \widehat{\phi}_\tau) d\tau \right) \| ds. \end{aligned}$$

As $t_2 - t_1 \rightarrow 0$ implies $t_2 \rightarrow t_1$ but $t_1 \leq 0$ and $t_2 \geq 0$ implies $t_2 \rightarrow 0$.

Therefore above equation becomes,

$$\begin{aligned} \| (Fy)(t_2) - (Fy)(t_1) \| & \leq \| -T(0)[u(0, \phi(0) - (g(x_{t_1}, \dots, x_{t_p}))(0))] + u(0, y_0 + \widehat{\phi}_0) \| \\ & = \| -[u(0, \phi(0) - (g(x_{t_1}, \dots, x_{t_p}))(0))] + u(0, x(0)) \| \\ & = \| -[u(0, \phi(0) - (g(x_{t_1}, \dots, x_{t_p}))(0))] \\ & \quad + [u(0, \phi(0) - (g(x_{t_1}, \dots, x_{t_p}))(0))] \|. \end{aligned}$$

Case 3 : Suppose $-r \leq t_1 \leq t_2 \leq 0$. Then $(Fy)(t_1) = 0$ and $(Fy)(t_2) = 0$. Therefore,

$$\| (Fy)(t_2) - (Fy)(t_1) \| = 0.$$

Since $T(t)$, $t > 0$ is compact, $T(t)$ is continuous in uniform operator topology. Therefore, as $t_2 - t_1 \rightarrow 0$, the right hand side in the cases 1 - 3 tends to zero. Thus F maps B_m into an equicontinuous family of functions with values in X .

We have already shown that F_{B_m} is an equicontinuous and uniformly bounded collection. To prove the set F_{B_m} is precompact in B , it is sufficient, by Arzela-Ascoli's argument, to show that F maps B_m into a precompact set in X . Let $0 < t \leq T$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $y \in B_m$, we define

$$\begin{aligned} (F_\epsilon y)(t) &= -T(t)[u(0, \phi(0) - (g(x_{t_1}, \dots, x_{t_p}))(0))] + u(t, y_t + \widehat{\phi}_t) \\ &\quad + \int_0^{t-\epsilon} AT(t-s)u(s, y_s + \widehat{\phi}_s)ds \\ &\quad + \int_0^{t-\epsilon} T(t-s)f\left(s, y_s + \widehat{\phi}_s, \int_0^s k(s, \tau)h(\tau, y_\tau + \widehat{\phi}_\tau)d\tau\right) ds \\ &= -T(t)[u(0, \phi(0) - (g(x_{t_1}, \dots, x_{t_p}))(0))] + u(t, y_t + \widehat{\phi}_t) \\ &\quad + T(\epsilon) \int_0^{t-\epsilon} AT(t-s-\epsilon)u(s, y_s + \widehat{\phi}_s)ds \\ &\quad + T(\epsilon) \int_0^{t-\epsilon} T(t-s-\epsilon)f\left(s, y_s + \widehat{\phi}_s, \int_0^s k(s, \tau)h(\tau, y_\tau + \widehat{\phi}_\tau)d\tau\right) ds. \end{aligned}$$

Since $T(t)$ is the compact operator, the set $Y_\epsilon(t) = \{(F_\epsilon y)(t) : y \in B_m\}$ is precompact in X for every ϵ , $0 < \epsilon < t$. Moreover for every $y \in B_m$, we have

$$\begin{aligned} (Fy)(t) - (F_\epsilon y)(t) &= \int_{t-\epsilon}^t AT(t-s)u(s, y_s + \widehat{\phi}_s)ds \\ &\quad + \int_{t-\epsilon}^t T(t-s)f\left(s, y_s + \widehat{\phi}_s, \int_0^s k(s, \tau)h(\tau, y_\tau + \widehat{\phi}_\tau)d\tau\right) ds. \end{aligned}$$

By making use of hypotheses $(H_1) - (H_5)$ and the fact that $\|y\|_B \leq m, y \in B_m$ implies $\|y_t\|_C \leq m, t \in [0, T]$, we have

$$\begin{aligned} &\|(Fy)(t) - (F_\epsilon y)(t)\| \\ &\leq \int_{t-\epsilon}^t \|AT(t-s)u(s, y_s + \widehat{\phi}_s)\|ds \\ &\quad + \int_{t-\epsilon}^t \|T(t-s)f\left(s, y_s + \widehat{\phi}_s, \int_0^s k(s, \tau)h(\tau, y_\tau + \widehat{\phi}_\tau)d\tau\right)\| ds \\ &\leq \int_{t-\epsilon}^t K_1[c_1(m+c+G) + c_2]ds \\ &\quad + \int_{t-\epsilon}^t \|T(t-s)\|p(s) \left[\|y_s + \widehat{\phi}_s\|_C + \int_0^s Lq(\tau)(\|y_\tau + \widehat{\phi}_\tau\|_C)d\tau \right] ds \\ &\leq \{K_1[c_1(m+c+G) + c_2] + K_0M^*[m+c+G+TM^*(m+c+G)]\}\epsilon. \end{aligned}$$

This shows that there exists precompact sets arbitrarily close to the set $\{(Fy)(t) : y \in B_m\}$. Hence the set $\{(Fy)(t) : y \in B_m\}$ is precompact in X . This shows that

F is completely continuous operator. Moreover, the set

$$\varepsilon(F) = \{y \in B_0 : y = \lambda Fy, \quad 0 < \lambda < 1\},$$

is bounded in B , since for every y in $\varepsilon(F)$, the function $x(t) = y(t) + \widehat{\phi}(t)$ is a solution of initial value problem (1) _{λ} -(1.2) for which we have proved that $\|x\|_B \leq Q$ and hence $\|y\|_B \leq Q + c + G$. Now, by virtue of Lemma 2.3, the operator F has a fixed point \tilde{y} in B_0 . Then $\tilde{x} = \tilde{y} + \widehat{\phi}$ is a solution of the initial value problem (1.1)-(1.2). This completes the proof of the Theorem 2.4.

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