Thai Journal of Mathematics
Volume 16 (2018) Number 3 : 775–787

http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Common Fixed Point Results on Complex Valued G_b-Metric Spaces

Ozgur Ege[1] and Ismet Karaca

Department of Mathematics, Faculty of Science, Ege University
Bornova, 35100, Izmir, Turkey
e-mail : ozgur.ege@ege.edu.tr (O. Ege)
ismet.karaca@ege.edu.tr (I. Karaca)

Abstract : In this study, we introduce some notions such as coincidence point, compatible and occasionally weakly compatible maps in complex valued G_b-metric spaces. Then using these notions we prove some new common fixed point theorems in complex valued G_b-metric spaces.

Keywords : fixed point; complex valued G_b-metric space; common fixed point theorem.
2010 Mathematics Subject Classification : 47H10; 54H25.

1 Introduction

Fixed point theory is a very active branch of nonlinear analysis. It is very famous due to its variety of applications in numerous areas such as engineering, computer science, economics, etc. The contractive-type conditions plays an important role in the fixed point theory. Many researchers have extended and generalized Banach contraction principle because it is the heart of this theory.

After introducing b-metric spaces and generalized Banach contraction principle in b-metric spaces in [1], many works such as [2–9] have been given. In 2011, Azam et al. [10] defined the new notion of complex valued metric spaces and obtained common fixed point theorems. Rao et al. [11] presented the complex valued b-metric spaces. Mustafa and Sims [12] introduced the notion of G-metric spaces

[1]Corresponding author.

and [13] proved some fixed point theorems in complete G-metric spaces. For other important papers in G-metric spaces, see [14–20].

The notion of G_b-metric space was presented by Aghajani et al. [21]. Then some coupled coincidence fixed point theorems for nonlinear (ψ, φ)-weakly contractive mappings in partially ordered G_b-metric spaces were proved in [22]. Lately, Ege [23] introduced the concept of complex valued G_b-metric spaces, proved Banach contraction principle and Kannan's fixed point theorem in this space. In [24], Ege proved a common fixed point theorem via α-series and obtained new results in the same space. For other studies on G_b-metric spaces, we refer to reader to [25, 26].

This paper is organized as follows: In section 2, we give the required information about complex valued G_b-metric spaces. In section 3, we define some notions such as coincidence point, compatible and occasionally weakly compatible maps and prove some common fixed point theorems in this space.

2 Preliminaries

To begin with, we give some basic definitions, notations and theorems which will be used later. Let's start with the definition of complex valued metric space. Azam et al. [10] introduced the notion of complex valued metric space. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(C_1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(C_2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(C_3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,

(C_4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

Note that we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of $(C_2), (C_3)$ and (C_4) is satisfied and we write $z_1 \prec z_2$ if only (C_4) is satisfied. The following statements hold:

(1) If $a, b \in \mathbb{R}$ with $a \leq b$, then $az \prec bz$ for all $z \in \mathbb{C}$.

(2) If $0 \precsim z_1 \precnsim z_2$, then $|z_1| < |z_2|$.

(3) If $z_1 \precsim z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$.

Definition 2.1. [23] Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a mapping $G : X \times X \times X \to \mathbb{C}$ satisfies:

(CG_b1) $G(x, y, z) = 0$ if $x = y = z$;

(CG_b2) $0 \prec G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;

(CG_b3) $G(x,x,y) \precsim G(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$;

(CG_b4) $G(x,y,z) = G(p\{x,y,z\})$, where p is a permutation of x,y,z;

(CG_b5) $G(x,y,z) \precsim s(G(x,a,a) + G(a,y,z))$ for all $x,y,z,a \in X$.

Then, G is called a *complex valued G_b-metric* and (X,G) is called a *complex valued G_b-metric space.*

Proposition 2.2. [23] *Let (X,G) be a complex valued G_b-metric space. Then for any $x,y,z \in X$,*

(i) $G(x,y,z) \precsim s(G(x,x,y) + G(x,x,z))$,

(ii) $G(x,y,y) \precsim 2sG(y,x,y)$.

Definition 2.3. [23] Let (X,G) be a complex valued G_b-metric space and $\{x_n\}$ be a sequence in X.

(i) $\{x_n\}$ is *complex valued G_b-convergent* to x if for every $a \in \mathbb{C}$ with $0 \prec a$, there exists $k \in \mathbb{N}$ such that $G(x,x_n,x_m) \prec a$ for all $n,m \geq k$.

(ii) A sequence $\{x_n\}$ is called *complex valued G_b-Cauchy* if for every $a \in \mathbb{C}$ with $0 \prec a$, there exists $k \in \mathbb{N}$ such that $G(x_n,x_m,x_l) \prec a$ for all $n,m,l \geq k$.

(iii) If every complex valued G_b-Cauchy sequence is complex valued G_b-convergent in (X,G), then (X,G) is said to be *complex valued G_b-complete.*

Proposition 2.4. [23] *Let (X,G) be a complex valued G_b-metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is complex valued G_b-convergent to x if and only if $|G(x,x_n,x_m)| \to 0$ as $n,m \to \infty$.*

Theorem 2.5. [23] *Let (X,G) be a complex valued G_b-metric space, then for a sequence $\{x_n\}$ in X and point $x \in X$, the following are equivalent:*

(i) *$\{x_n\}$ is complex valued G_b-convergent to x.*

(ii) $|G(x_n,x_n,x)| \to 0$ *as* $n \to \infty$.

(iii) $|G(x_n,x,x)| \to 0$ *as* $n \to \infty$.

(iv) $|G(x_m,x_n,x)| \to 0$ *as* $m,n \to \infty$.

Theorem 2.6. [23] *Let (X,G) be a complex valued G_b-metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is complex valued G_b-Cauchy sequence if and only if $|G(x_n,x_m,x_l)| \to 0$ as $n,m,l \to \infty$.*

3 Main Results

In this section, we first give a definition of compatible maps.

Definition 3.1. Let f and g be maps from a complex valued G_b-metric space (X, G) into itself. The maps f and g are called *compatible maps* if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0 \quad \text{or} \quad \lim_{n\to\infty} G(gfx_n, fgx_n, fgx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$.

Example 3.2. Let $X = [-1, 1]$ and a complex valued G_b-metric on X be given as follows:

$$G(x, y, z) = |x - y|^2 + |y - z|^2 + |x - z|^2$$

where $s = 2$ [23]. Define two self mappings $f, g : X \to X$ by $f(x) = x$ and $g(x) = \frac{x}{3}$. If we consider a sequence $\{x_n\} = \frac{1}{2n}$, we obtain the following results:

$$\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = \lim_{n\to\infty} G(\frac{1}{6n}, \frac{1}{6n}, \frac{1}{6n}) = 0$$

and

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} \frac{1}{2n} = 0 \quad \text{and} \quad \lim_{n\to\infty} gx_n = \lim_{n\to\infty} \frac{1}{6n} = 0.$$

Therefore f and g are compatible maps.

Theorem 3.3. *Let f and g be compatible maps of a complex valued G_b-metric space (X, G) satisfying*

(3.1) $f(X) \subseteq g(X)$,

(3.2) $G(fx, fy, fz) \precsim \alpha \max\{G(fx, gy, gz), G(gx, fy, gz), G(gx, gy, fz)\}$, *where* $\alpha \in [0, \frac{1}{2})$,

(3.3) one of f or g is continuous.

Then f and g have a unique common fixed point in X.

Proof. Let x_0 be a point in X. We can choose a point x_1 in X such that $fx_0 = gx_1$ using (3.1). More generally, a point x_{n+1} can be chosen such that

$$y_n = fx_n = gx_{n+1}, \quad n = 0, 1, 2, \ldots$$

By (3.2), we obtain

$$\begin{aligned} G(fx_n, fx_{n+1}, fx_{n+1}) \precsim& \alpha \max\{G(fx_n, gx_{n+1}, gx_{n+1}), G(gx_n, fx_{n+1}, gx_{n+1}), \\ & G(gx_n, gx_{n+1}, fx_{n+1})\} \\ =& \alpha \max\{G(fx_n, fx_n, fx_n), G(fx_{n-1}, fx_{n+1}, fx_n), \\ & G(fx_{n-1}, fx_n, fx_{n+1})\} \\ =& \alpha \max\{0, G(fx_{n-1}, fx_{n+1}, fx_n), G(fx_{n-1}, fx_n, fx_{n+1})\} \\ =& \alpha G(fx_{n-1}, fx_n, fx_{n+1}). \end{aligned}$$

By the rectangular inequality of complex valued G_b-metric space, we get

$$\begin{aligned} G(fx_{n-1}, fx_n, fx_{n+1}) &\precsim s(G(fx_{n-1}, fx_n, fx_n) + G(fx_n, fx_n, fx_{n+1})) \\ &\precsim s(G(fx_{n-1}, fx_n, fx_n) + 2G(fx_n, fx_{n+1}, fx_{n+1})). \end{aligned}$$

Hence by the above inequality shows that

$$G(fx_n, fx_{n+1}, fx_{n+1}) \precsim \frac{s\alpha}{1-2s\alpha} G(fx_{n-1}, fx_n, fx_n),$$

i.e.,

$$G(fx_n, fx_{n+1}, fx_{n+1}) \precsim q.G(fx_{n-1}, fx_n, fx_n)$$

where $q = \frac{s\alpha}{1-2s\alpha} < 1$. If we continue the same procedure, we obtain

$$G(fx_n, fx_{n+1}, fx_{n+1}) \precsim q^n G(fx_0, fx_1, fx_1).$$

Therefore, for all $n, m \in \mathbb{N}$, $n < m$, we have the followings by the rectangular property:

$$\begin{aligned} G(y_n, y_m, y_m) \precsim& s(G(y_n, y_{n+1}, y_{n+1})) + s^2 G(y_{n+1}, y_{n+2}, y_{n+2})) \\ & + s^3 G(y_{n+2}, y_{n+3}, y_{n+3}) + \ldots + s^m G(y_{m-1}, y_m, y_m) \\ \precsim& (q^n + q^{n+1} + \ldots + q^{m-1}) G(y_0, y_1, y_1) \\ \precsim& \frac{q^n}{1-q} G(y_0, y_1, y_1). \end{aligned}$$

Taking limits as $n, m \to \infty$, we have $\lim_{n\to\infty} G(y_n, y_m, y_m) = 0$.

Therefore, $\{y_n\}$ is a complex valued G_b-Cauchy sequence in X. Since (X, G) is a complex valued complete G_b-metric space, there is a point $z \in X$ such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1} = z.$$

Since f or g is continuous, we can assume that g is continuous, so

$$\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_{n+1} = gz.$$

Moreover, by the fact that f and g are compatible maps,

$$\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$$

implies that $\lim_{n\to\infty} fgx_n = gz$. If we set $x = gx_n$, $y = x_n$ and $z = x_n$ in (3.2), we obtain

$$G(fgx_n, fx_n, fx_n) \precsim \alpha \max\{G(fgx_n, gx_n, gx_n), G(ggx_n, fx_n, gx_n), G(ggx_n, gx_n, fx_n)\}.$$

Letting as $n \to \infty$, we get

$$G(gz, z, z) \precsim \alpha \max\{G(gz, z, z), G(gz, z, z), 0\}$$

implies $gz = z$. Using (3.2), we have

$$G(fx_n, fz, fz) \precsim \alpha \max\{G(fx_n, gz, gz), G(gx_n, fz, gz), G(gx_n, gz, fz)\}.$$

If we take limit as $n \to \infty$, we conclude that $fz = z$. This shows that $fz = gz = z$, i.e., z is a common fixed point of f and g.

For the uniqueness, we assume that $z_1 \neq z$ be another common fixed point of f and g. Then $G(z, z_1, z_1) \succ 0$ and

$$\begin{aligned} G(z, z_1, z_1) &= G(fz, fz_1, fz_1) \\ &\precsim \alpha \max\{G(fz, gz_1, gz_1), G(gz, fz_1, gz_1), G(gz, gz_1, fz_1)\} \\ &= \alpha G(z, z_1, z_1) \\ &< G(z, z_1, z_1), \end{aligned}$$

which is a contradiction. So we have $z = z_1$. □

Definition 3.4. A complex valued G_b-metric space (X, G) is called a *symmetric* if $G(x, y, y) = G(y, x, x)$ for all x, y in X.

Definition 3.5. A point x in a complex valued G_b-metric space (X, G) is called a *coincidence point* or *common fixed point* of f and g if $f(x) = g(x)$.

Definition 3.6. Two self-mappings f and g of a symmetric complex valued G_b-metric space (X, G) are said to be *occasionally weakly compatible* if and only if there is a point x in X which is coincidence point of f and g at which f and g commute.

We are now ready to give an example for our new definitions.

Example 3.7. Let $X = [0, \infty)$ and a complex valued G_b-metric on X be given by

$$G(x, y, z) = \frac{1}{9}(|x - y|^2 + |y - z|^2 + |x - z|^2)$$

where $s = 2$.

- (X, G) is symmetric since for all $x, y \in X$,

$$G(x, y, y) = G(y, x, x) = \frac{4}{9}|x - y|^2.$$

• Define $f, g : X \to X$ by $f(x) = x^3$ and $g(x) = \frac{x^2}{2}$ for $x \in X$. It is clear that there are two coincidence points for f and g in X. They are 0 and $\frac{1}{2}$.

• f and g are occasionally weakly compatible maps since 0 is a coincidence point of f and g and

$$fg(0) = gf(0) = 0.$$

Theorem 3.8. *Let (X, G) be a symmetric complex valued G_b-metric space. If f and g are occasionally weakly compatible self-maps on X satisfying*

$$G(fx, fy, fz) \precsim \alpha \max\{G(fx, gy, gz), G(gx, fy, gz), G(gx, gy, fz)\},$$

where $\alpha \in [0, \frac{1}{2})$. Then f and g have a unique common fixed point in X.

Proof. There exists a point u in X such that $fu = gu$ and $fgu = gfu$ because f and g are occasionally weakly compatible maps. We will show that fu is a fixed point of f. If $ffu \neq fu$, then we have

$$\begin{aligned} G(fu, ffu, ffu) &\precsim \alpha \max\{G(fu, gfu, gfu), G(gu, ffu, gfu), G(gu, gfu, ffu)\} \\ &= \alpha \max\{G(fu, ffu, ffu), G(fu, ffu, ffu), G(fu, ffu, ffu)\} \\ &= \alpha G(fu, ffu, ffu) \end{aligned}$$

by (3.2). This shows that $ffu = fu$ and $ffu = fgu = gfu = fu$. So fu is a common fixed point of f and g.

To show the uniqueness, assume that u, v in X such that $fu = gu = u$, $fv = gv = v$ and $u \neq v$. Then by (3.2), we have

$$\begin{aligned} G(u, v, v) = G(fu, fv, fv) &\precsim \alpha \max\{G(fu, gv, gv), G(gu, fv, gv), G(gu, gv, fv)\} \\ &= \alpha \max\{G(u, v, v), G(u, v, v), G(u, v, v)\} \\ &= \alpha G(u, v, v) \end{aligned}$$

which is a contradiction. Therefore, $u = v$ and the common fixed point of f and g is unique. □

Theorem 3.9. *Let (X, G) be a complete complex valued G_b-metric space with coefficient $s > 1$ and $T : X \to X$ be a mapping satisfying the following:*

$$G(Tx, Ty, Tz) \precsim \beta \max\{G(x, y, z), G(x, x, Tx), G(y, y, Ty), G(z, z, Tz)\} \quad (3.1)$$

$\forall x, y, z \in X$. where $\beta \in [0, 1)$. Then T has a unique fixed point $w \in X$ and $G(w, w, w) = 0$.

Proof. Let us prove that if a fixed point of T exists, then it is unique. Let $u, v \in X$ be two fixed points of T, $u \neq v$, that is $Tu = u$ and $Tv = v$. It follows from (3.1):

$$\begin{aligned} G(u, u, v) &= G(Tu, Tu, Tv) \\ &\precsim \beta \max\{G(u, u, v), G(u, u, Tu), G(u, u, Tu), G(v, v, Tv)\} \\ &= \beta \max\{G(u, u, v), G(u, u, u), G(v, v, v)\} \\ &= \beta G(u, u, v) \\ &\prec G(u, u, v) \end{aligned}$$

since $\beta < 1$. We obtain $G(u,u,v) < G(u,u,v)$ which gives $G(u,u,v) = 0$, then $u = v$. Therefore, if a fixed point of T exists, then it is unique.

Let $x_0 \in X$ and define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. For any n, we obtain from (3.1),

$$\begin{aligned}G(x_{n+1},x_{n+1},x_n) &= G(Tx_n,Tx_n,Tx_{n-1})\\ &\precsim \beta\max\{G(x_n,x_n,x_{n-1}),G(x_n,x_n,Tx_n),G(x_n,x_n,Tx_n),\\ &\quad G(x_{n-1},x_{n-1},Tx_{n-1})\}\\ &= \beta\max\{G(x_n,x_n,x_{n-1}),G(x_n,x_n,x_{n+1}),G(x_{n-1},x_{n-1},x_n)\}.\end{aligned}$$

By the symmetry, we have

$$G(x_{n-1},x_{n-1},x_n) = G(x_n,x_n,x_{n-1}),$$

and so

$$G(x_{n+1},x_{n+1},x_n) \precsim \beta\max\{G(x_n,x_n,x_{n-1}),G(x_n,x_n,x_{n+1})\}.$$

If $\max\{G(x_n,x_n,x_{n-1}),G(x_n,x_n,x_{n+1})\} = G(x_n,x_n,x_{n+1})$, then we get a contradiction such that:

$$\begin{aligned}G(x_{n+1},x_{n+1},x_n) &\precsim \beta G(x_n,x_n,x_{n+1})\\ &= \beta G(x_{n+1},x_{n+1},x_n)\\ &< G(x_{n+1},x_{n+1},x_n).\end{aligned}$$

Therefore, $\max\{G(x_n,x_n,x_{n-1}),G(x_n,x_n,x_{n+1})\} = G(x_n,x_n,x_{n-1})$ and

$$G(x_{n+1},x_{n+1},x_n) \precsim \beta G(x_n,x_n,x_{n-1}) \tag{3.2}$$

that is

$$G(Tx_n,Tx_n,Tx_{n-1}) \precsim \beta G(x_n,x_n,x_{n-1}). \tag{3.3}$$

If we continue to the same procedure, we get

$$G(x_{n+1},x_{n+1},x_n) \precsim \beta^n G(x_1,x_1,x_0). \tag{3.4}$$

For $n,m \in \mathbb{N}$, $m > n$, we have

$$\begin{aligned}G(x_n,x_n,x_m) &\precsim s[G(x_n,x_{n+1},x_{n+1}) + G(x_{n+1},x_n,x_m)]\\ &\precsim sG(x_n,x_{n+1},x_{n+1})+s^2[G(x_{n+1},x_{n+2},x_{n+2})+G(x_{n+2},x_n,x_m)]\\ &\precsim sG(x_n,x_{n+1},x_{n+1}) + s^2G(x_{n+1},x_{n+2},x_{n+2})\\ &\quad + s^3[G(x_{n+2},x_{n+3},x_{n+3}) + G(x_{n+3},x_n,x_m)]\\ &\vdots\\ &\precsim sG(x_n,x_n,x_{n+1})+s^2G(x_{n+1},x_{n+1},x_{n+2})+s^3G(x_{n+2},x_{n+2},x_{n+3})\\ &\quad + \ldots + s^{m-n}G(x_{m-1},x_{m-1},x_m) + s^{m-n}G(x_n,x_n,x_m)\end{aligned}$$

and using the inequality (3.4), we obtain

$$\begin{aligned}(1-s^{m-n})G(x_n,x_n,x_m) &\precsim s\beta^n G(x_1,x_1,x_0)+s^2\beta^{n+1}G(x_1,x_1,x_0)\\ &\quad + s^3\beta^{n+2}G(x_1,x_1,x_0)+\ldots+s^{m-n}\beta^{m-1}G(x_1,x_1,x_0)\\ &\precsim s\beta^n[1+s\beta+(s\beta)^2+\ldots+(s\beta)^{m-n-1}]G(x_1,x_1,x_0)\\ G(x_n,x_n,x_m) &\precsim \frac{s\beta^n}{1-s^{m-n}}[\frac{1-(s\beta)^{m-1}}{1-s\beta}]G(x_1,x_1,x_0).\end{aligned}$$

Taking limit $n\to\infty$, we obtain $G(x_n,x_n,x_m)\to 0$. Hence, $=0$. Thus, $\{x_n\}$ is a complex valued G_b-Cauchy sequence in X. Since X is a complex valued complete G_b-metric space, then there exists $u\in X$ such that

$$\lim_{n\to\infty} G(x_n,x_n,u)=\lim_{n\to\infty} G(x_n,x_n,x_m)=G(u,u,u)=0. \tag{3.5}$$

We now prove that u is a fixed point of T. For all $n\in\mathbb{N}$, we obtain

$$\begin{aligned}G(u,u,Tu) &\precsim s[G(u,x_{n+1},x_{n+1})+G(x_{n+1},u,Tu)]\\ &= s[G(u,u,x_{n+1})+G(u,Tu,x_{n+1})]\\ &\precsim sG(u,u,x_{n+1})+s^2[G(u,x_{n+1},x_{n+1})+G(x_{n+1},Tu,x_{n+1})]\\ &\precsim sG(u,u,x_{n+1})+s^2G(u,x_{n+1},x_{n+1})+s^2G(x_{n+1},Tu,x_{n+1})\\ &= (s+s^2)G(u,u,x_{n+1})+s^2G(Tx_n,Tu,Tx_n).\end{aligned}$$

If we use (3.3), we obtain $G(Tx_n,Tu,Tx_n)\precsim \beta G(x_n,u,x_n)$. Thus we have the following:

$$\begin{aligned}G(u,u,Tu) &\precsim (s+s^2)G(u,u,x_{n+1})+\beta s^2G(x_n,u,x_n)\\ &\precsim (s+s^2)G(u,u,u)+\beta s^2G(u,u,u)\end{aligned}$$

as $n\to\infty$. By (3.5), we have $G(u,u,Tu)=0$, then $Tu=u$. Therefore, u is a fixed point of T and it is unique. □

Theorem 3.10. *Let (X,G) be a complex valued G_b-complete metric space. If $T:X\to X$ is a continuous map satisfying*

$$G(Tx,Ty,Tz)\precsim \phi(G(x,y,z)) \quad \textit{for all } x,y,z\in X \tag{3.6}$$

where $\phi:[0,+\infty)\to[0,+\infty)$ is an increasing function such that

$$\lim_{n\to\infty}\phi^n(t)=0 \quad \textit{for all } t>0,$$

then T has a unique fixed point in X.

Proof. Let $x\in X$ and $\epsilon\succ 0$. Consider a natural number n such that $\phi^n(\epsilon)\prec\frac{\epsilon}{4s^2}$. If we assume that $F=T^n$ and $x_k=F^k(x)$ for $k\in\mathbb{N}$, then we have the following

$$G(Fx,Fx,Fy)\precsim \phi^n(G(x,x,y))=\alpha(G(x,x,y))$$

for $x, y \in X$ and $\alpha = \phi^n$. So taking limit as $k \to \infty$, we get $|G(x_{k+1}, x_{k+1}, x_k)| \to 0$. Thus, let k be a number such that

$$G(x_{k+1}, x_{k+1}, x_k) \prec \frac{\epsilon}{4s^2}.$$

We can define the ball $B(x_k, \epsilon)$ such that for each

$$z \in B(x_k, \epsilon) = \{y \in X : G(x_k, x_k, y) \precsim \epsilon\}.$$

By the fact that $x_k \in B(x_k, \epsilon)$, $B(x_k, \epsilon) \neq \emptyset$. For all $z \in B(x_k, \epsilon)$, we obtain the following inequalities:

$$\begin{aligned} G(Fz, Fz, Fx_k) &\precsim \alpha(G(z, z, x_k)) \quad = \alpha(G(x_k, x_k, z)) \\ &\precsim \alpha(\epsilon) \\ &= \phi^n(\epsilon) \\ &\prec \frac{\epsilon}{4s^2}. \end{aligned}$$

From the fact that $G(Fx_k, Fx_k, x_k) = G(x_{k+1}, x_{k+1}, x_k) \prec \frac{\epsilon}{s^2}$, we obtain the following:

$$\begin{aligned} G(x_k, x_k, Fz) &\precsim s[G(x_k, x_{k+1}, x_{k+1}) + G(x_{k+1}, x_k, Fz)] \\ &= s[G(x_{k+1}, x_{k+1}, x_k) + G(x_{k+1}, x_k, Fz)] \\ &\precsim s[G(x_{k+1}, x_k, Fz) + G(x_{k+1}, x_k, Fz)] \\ &\precsim 2sG(x_{k+1}, x_k, Fz) \\ &= 2sG(x_k, Fz, x_{k+1}) \\ &\precsim 2s^2[G(x_k, x_{k+1}, x_{k+1}) + G(x_{k+1}, Fz, x_{k+1})] \\ &= 2s^2[G(x_{k+1}, x_{k+1}, x_k) + G(x_{k+1}, x_{k+1}, Fz)] \\ &= 2s^2[G(x_k, x_k, x_{k+1}) + G(Fz, Fz, x_{k+1})] \\ &\precsim 2s^2[\frac{\epsilon}{4s^2} + \frac{\epsilon}{4s^2}] = \epsilon. \end{aligned}$$

Thus, F maps $B(x_k, \epsilon)$ to itself. Since $x_k \in B(x_k, \epsilon)$, we get $Fx_k \in B(x_k, \epsilon)$. Continuing the same procedure, it is obtained that

$$F^m_{x_k} \in B(x_k, \epsilon) \quad \text{for all } m \in \mathbb{N},$$

that is $x_l \in B(x_k, \epsilon)$ for each $l \geq k$. As a result,

$$G(x_m, x_m, x_l) \prec \epsilon \quad \text{for all } m, l > k$$

and $\{x_k\}$ is a complex valued G_b-Cauchy sequence. By the completeness of (X, G), there exists $u \in X$ such that $x_k \to u$ as $k \to \infty$. Since

$$u = \lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty} x_k = F(u),$$

u is a fixed point of F.

We need to prove the uniqueness of this fixed point. Let u and u_1 be two fixed points of F. Considering $\alpha(t) = \phi^n(t) < t$ for any $t > 0$, the following inequalities

$$\begin{aligned} G(u, u, u_1) &= G(Fu, Fu, Fu_1) \\ &\precsim \phi^n(u, u, u_1) \\ &= \alpha(G(u, u, u_1)) \\ &\precsim G(u, u, u_1) \end{aligned}$$

show that $G(u, u, u_1) = 0$, i.e., $u = u = 1$. So F has a unique fixed point in X.

If we take limit as $k \to \infty$, we get $T^{nk+r}(x) = F^k(T^r(x)) \to u$. Thus $T^m x \to u$ as $m \to \infty$ for every x, that is,

$$u = \lim_{m\to\infty} Tx_m = T(u).$$

This shows that T has a fixed point and the proof is completed. □

Acknowledgement : The authors sincerely thank the referee for valuable suggestions of this paper.

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(Received 10 January 2018)
(Accepted 26 December 2018)