Thai Journal of Mathematics
Volume 5(2007) Number 1 : 151–161
www.math.science.cmu.ac.th/thaijournal

Weak and Strong Convergence of Iterates of Non-Lipschitzian Asymptotically Quasi-Nonexpansive Type Mapping

A. Banerjee

Keywords : iteration process, nonlipschitzian asymptotically nonexpansive mapping
2002 Mathematics Subject Classification : 47H10

1 Introduction

Throughout this paper, let K be a nonempty subset of a real normed space E and $T : K \to K$ a nonlinear mappings, and $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$. In this paper throughout we assume that $F(T) \neq \phi$.

T is said to be nonexpansive provided

$$\|Tx - Ty\| \leq \|x - y\|$$

for all x, y in K.

T is said to be asymptotically nonexpansive [4] if there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all x, y in K and $n \in \mathbb{N}$.

T is said to be asymptotically nonexpansive type [7] if

$$\limsup_{n\to\infty} \sup_{x\in K} \{\|T^n x - T^n y\| - \|x - y\|\} \leq 0.$$

for any $y \in K$.

Remark 1.1 *From the above definitions, it follows that a nonexpansive mapping must be asymptotically nonexpansive, and any asymptotically nonexpansive mapping must be asymptotically nonexpansive type (see [7]).*

The concept of quasi-nonexpansiveness was introduced by Diaz and Metcalf [2] in 1967.

The mapping T is said to be quasi-nonexpansive if

$$\|Tx - p\| \leq \|x - p\|$$

for all $x \in K$ and $p \in F(T)$.

Using similar concept asymptotically quasi-nonexpansive mapping have been defined [14]:
The mapping T is said to be asymptotically quasi-nonexpansive if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - p\| \leq k_n \|x - p\|$$

for all $x \in K$ and $p \in F(T)$, $n \in \mathbb{N}$.

Similarly asymptotically quasi-nonexpansive type mappings can be defined as :

The mapping T is said to be asymptotically quasi-nonexpansive type if

$$\limsup_{n\to\infty} \sup_{x\in K} \{\|T^n x - p\| - \|x - p\|\} \leq 0 \tag{1.1}$$

for $p \in F(T)$.

Remark 1.2 *From above definition, it follows that a quasi-nonexpansive mapping must be asymptotically quasi-nonexpansive mapping and asymptotically quasi-nonexpansive mapping must be asymptotically quasi-nonexpansive type.*

The iterative approximation problem for nonexpansive mapping, asymptotically nonexpansive mapping, quasi-nonexpansive mapping, asymptotically quasi-nonexpansive mapping and recently of asymptotically nonexpansive type mappings were studied extensively by ([9, 12, 15] and reference therein).

In the most of the above study authors have mainly studied Mann [10] and Ishikawa [6] type iteration process for iterative approximation of above class of mappings.

In this paper our main interest is to study the problem of approximation of fixed point of the more general class of asymptotically quasi nonexpansive type mappings using generalized iteration process (three-step iteration process) [17], which is more general than the Mann and Ishikawa iteration process and defined as below:

Definition 1.3 ***Generalized Iteration Process :***

For a given $x_o \in K$, define sequence $\{x_k\}$,$\{y_k\}$ and $\{z_k\}$ by

$$\begin{aligned} x_{n+1} =& (1-\alpha_n)x_n + \alpha_n T^n y_n, \\ y_n =& (1-\beta_n)x_n + \beta_n T^n z_n, \\ z_n =& (1-\gamma_n)x_n + \gamma_n T^n x_n \end{aligned} \tag{1.2}$$

$n \geq 0$, *where* $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\}$ *are real sequence in* $[0,1]$.

For $\gamma_n = 0$, the iteration process (1.2) reduces to :

For a given $x_o \in K$, define sequence $\{x_k\}$ and $\{y_k\}$ by

$$\begin{aligned} x_{n+1} =& (1-\alpha_n)x_n + \alpha_n T^n y_n, \\ y_n =& (1-\beta_n)x_n + \beta_n T^n x_n \end{aligned} \tag{1.3}$$

$n \geq 0$, where $\{\alpha_n\}$,$\{\beta_n\}$ are real sequence in $[0,1]$.

Iteration scheme (1.3) is called an Ishikawa type iteration process.

For $\beta_n = 0$ and $\gamma_n = 0$, the iteration process (1.2) reduces to :

For a given $x_o \in K$, define sequence $\{x_k\}$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n \tag{1.4}$$

$n \geq 0$, where $\{\alpha_n\}$ is a real sequence in $[0,1]$.

Iteration scheme (1.4) is called an Mann type iteration process.

For a suitable choice of α_n, β_n and γ_n, one can obtain a number of new and known iteration schemes for solving nonlinear equations in Banach spaces and Hilbert spaces; see [8, 12, 15] and the references there in.

Our result extend and generalize various results obtained quite recently by Deng [1], Liu [9], Schu [12] and Tan and Xu [15] to more general type of space and more general family of mappings as well as for generalized iteration process.

Lemma 1.4 *[15] Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequenes of non-negative numbers such that $a_{n+1} \leq a_n + b_n$, for all $n \geq 1$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists.*

Lemma 1.5 *Let K be a nonempty convex subset of a normed space E and let $T : K \to K$ be a mapping of asymptotically quasi-nonexpansive type. Let sequence $\{x_n\}$ be defined by (1.2). If $p \in F(T)$, then*

$$\|x_{n+1} - p\| \leq \|x_n - p\| + 3 \sup_{x\in K}\{\|T^n x - p\| - \|x - p\|\}; \quad n \in \mathbb{N}.$$

Proof.

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \|(1-\alpha_n)x_n + \alpha_n T^n y_n - p\| \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n[\|T^n y_n - p\| - \|y_n - p\|] \\
&\quad + \alpha_n\|y_n - p\| \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \sup_{x\in K}[\|T^n x - p\| - \|x - p\|] \\
&\quad + \alpha_n[(1-\beta_n)\|x_n - p\| + \beta_n\|T^n z_n - p\|] \\
&\leq [(1-\alpha_n) + \alpha_n(1-\beta_n)]\|x_n - p\| \\
&\quad + \alpha_n \sup_{x\in K}[\|T^n x - p\| - \|x - p\|] + \alpha_n\beta_n\|T^n z_n - p\| \\
&\leq [(1-\alpha_n) + \alpha_n(1-\beta_n)]\|x_n - p\| \\
&\quad + \alpha_n \sup_{x\in K}[\|T^n x - p\| - \|x - p\|] \\
&\quad + \alpha_n\beta_n[\|T^n z_n - p\| - \|z_n - p\|] + \alpha_n\beta_n\|z_n - p\| \\
&\leq [(1-\alpha_n) + \alpha_n(1-\beta_n)]\|x_n - p\| \\
&\quad + \alpha_n(1+\beta_n) \sup_{x\in K}[\|T^n x - p\| - \|x - p\|] \\
&\quad + \alpha_n\beta_n[\gamma_n\|T^n x_n - p\| + (1-\gamma_n)\|x_n - p\|]
\end{aligned}$$

$$
\begin{aligned}
&\le [(1-\alpha_n)+\alpha_n(1-\beta_n)+\alpha_n\beta_n]\|x_n-p\| \\
&\quad + \alpha_n(1+\beta_n)\sup_{x\in K}[\|T^n x-p\|-\|x-p\|] \\
&\quad + \alpha_n\beta_n\gamma_n[\|T^n x_n-p\|-\|x_n-p\|] \\
&\le \|x_n-p\|+\alpha_n[1+\beta_n(1+\gamma_n)]\sup_{x\in K}[\|T^n x-p\|-\|x-p\|] \\
&\le \|x_n-p\|+(2+\gamma_n)\sup_{x\in K}[\|T^n x-p\|-\|x-p\|] \\
&\le \|x_n-p\|+3\sup_{x\in K}[\|T^n x-p\|-\|x-p\|.
\end{aligned}
$$

□

Lemma 1.6 *Let K be a nonempty convex subset of a normed space E and let $T: K \to K$ be a mapping of asymptotically quasi-nonexpansive type. Let sequence $\{x_n\}$ be defined by (1.2). If $p \in F(T)$, then* $\lim_{n\to\infty}\|x_n-p\|$ *exists.*

Proof. Using (1.1) choose n_o such that $n \ge n_o$ imply

$$\sup_{k\ge n}\{\sup_{x\in K}(\|T^k x-p\|-\|x-p\|)\} \le \frac{1}{3n^2} \quad . \tag{1.5}$$

Hence, by lemma 1.5

$$\|x_{n+1}-p\| \le \|x_n-p\|+\frac{1}{n^2} \quad .$$

Therefore for $n, m \ge n_0$, we have

$$\|x_{m+n}-p\| \le \|x_n-p\|+\sum_{i=n}^{n+m-1}\frac{1}{i^2}$$

By lemma 1.4 we get that $\lim_{n\to\infty}\|x_n-p\|$ exists. □

2 Strong Convergence of Iterates of Asymptotically Quasi-Nonexpansive type Mappings

Lemma 2.1 *Let K be a closed convex subset of a normed space E, let $T: K \to K$ be asymptotically quasi-nonexpansive type with $F(T)$ be a* *nonempty set, and the iterative sequence* $\{x_n\}$ *is defined by (1.2). If* $\lim_{n\to\infty} d(x_n, F(T)) = 0$*, then* $\{x_n\}$ *is a Cauchy sequence. Where* $d(x_n, F(T))$ *denotes the distance from the point* x_n *to the set* $F(T)$*.*

Proof. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, then for all $\varepsilon > 0$ there exists $k(\varepsilon) \in \mathbb{N}$ such that for all $n \geq k(\varepsilon)$

$$d(x_n, F(T)) < \frac{\varepsilon}{2} \tag{2.1}$$

this implies that there exist $p \in F(T)$ such that that for all $n \geq k(\varepsilon)$

$$d(x_n, p) < \frac{\varepsilon}{2}. \tag{2.2}$$

Since the sequence $\{\|x_n - p\|\}$ is nonincreasing, we have for $m, n \geq k(\varepsilon)$

$$\begin{aligned} \|x_n - x_m\| &= \|x_n - p\| + \|x_m - p\| \\ &= 2\|x_{k(\varepsilon)} - p\| < \varepsilon \end{aligned} \tag{2.3}$$

which shows that $\{x_n\}$ is a Cauchy sequence. □

Theorem 2.2 *Let* K *be a closed convex subset of a Banach space* E*, let* $T : K \to K$ *be asymptotically quasi-nonexpansive type with* $F(T)$ *be a nonempty set, and the iterative sequence* $\{x_n\}$ *is defined by (1.2). Then*

1. $\lim_{n\to\infty} d(x_n, F(T)) = 0$ *if* $\{x_n\}$ *converges strongly to a fixed point in* $F(T)$*.*

2. $\{x_n\}$ *converges strongly to a point in* $F(T)$ *if* $\lim_{n\to\infty} d(x_n, F(T)) = 0$*.*

Proof. (1) Since $F(T)$ is closed and the map $x \mapsto d(x, F(T))$ is continuous, then

$$\lim_{n\to\infty} d(x_n, F(T)) = d(\lim_{n\to\infty} x_n, F(T)) = 0.$$

(2) From lemma 2.1, $\{x_n\}$ is a Cauchy sequence, so $\{x_n\}$ converges to a point, say p in K. Since $F(T)$ is closed, then $0 = \lim_{n\to\infty} d(x_n, F(T)) = d(\lim_{n\to\infty} x_n, F(T))$ implies that $p \in F(T)$. □

Theorem 2.3 *Let K be a closed convex subset of a Banach space E, let $T : K \to K$ be asymptotically quasi-nonexpansive type with $F(T)$ be a nonempty set, and the iterative sequence $\{x_n\}$ is defined by (1.2). Then $\{x_n\}$ converges strongly to a point in $F(T)$ if and only if* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Proof. The necessity of the condition is obvious. Thus, we will only prove the sufficiency.

From lemma 1.5,we have

$$\|x_{n+1} - p\| \le \|x_n - p\| + 3 \sup_{x\in K}\{\|T^n x - p\| - \|x - p\|\} \tag{2.4}$$

for any $n \in \mathbb{N}$ and $p \in F(T)$.

From (2.4) and (1.5), we have

$$d(x_{n+1}, F(T)) \le d(x_n, F(T)) + \frac{1}{n^2} \ .$$

Therefor for $n, m \ge n_0$, we have

$$d(x_{m+n}, F(T)) \le d(x_n, F(T)) + \sum_{i=n}^{n+m-1} \frac{1}{i^2}$$

By Lemma 1.4 we get that $\lim_{n\to\infty} d(x_n, F(T))$ exists and it follows from

$$\liminf_{n\to\infty} d(x_n, F(T) = 0$$

that

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

From Theorem 2.2 $\{x_n\}$ converges strongly to a point in $F(T)$. □

Remark 2.4 *Threorem 2.2 and Threorem 2.3 partially extend and generalize Theorem 1.1' of Petryshyn and Williamson [16], Theorem 3.1 of Gosh and Debnath [5] for the larger class of mapping and iteration scheme considered here.*

A self mapping T with $F(T) \neq \phi$ on a convex subset K of a Banach space is said to satisfy condition (S, D) if there is a nondecreasing function $f : [0, M] \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, M)$ and $\|x - Tx\| \geq f(d(x, F(T))$ for all $x \in K$.

This condition was introduced by Senter and Dotson [13].

Theorem 2.5 *Let K be a closed convex subset of a Banach space E, let $T : K \to E$ be asymptotically quasi-nonexpansive type with $F(T)$ a nonempty set, and the iterative sequence $\{x_n\}$ be an defined by (1.2). Let $\{x_n\}$ is a approximate fixed point sequence for T, i.e. $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, K satisfy conditon (S,D). Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. For any $p \in F(T)$, the sequence $\{\|x_n - p\|\}$ is nonincreasing, so is $\{d(x_n, F(T))\}$ and then $\lim_{n\to\infty} d(x_n, F(T)) = r$ for some $r \geq 0$.

Now in view of Theorem 2.3, to complete the proof we must show that $r = 0$.

From condition (S,D), we have

$$\|x_n - Tx_n\| \geq f(d(x_n, F(T)) \geq f(r).$$

Since $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, we get that $f(r) = 0$ and so $r = 0$.

This completes the proof. □

Remark 2.6 *Theorem 2.5 generalizes theorem 3 of Liu [9], Theorem 3 of Tan and Xu [15] and Theorem 3 of Zeng [18] for the iteration process (1.2) and the mapping of asymptotically quasi nonexpansive type considered here.*

3 Weak Convergence of Iterates of Asymptotically Quasi-Nonexpansive type Mappings

Theorem 3.1 *Let E be a Banach space which satisfies Opial's condition and let K be a closed compact convex subset of E. Let $T : K \to K$ be asymptotically quasi-nonexpansive type, and the sequence $\{x_n\}$ defined by (1.2) is weakly sequentially compact. Suppose T has a fixed point, $I - T$ is demiclosed with respect to zero, and $\{x_n\}$ is a approximate fixed point sequence for T, i.e. $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Since $\{x_n\} \subset K$ is weakly sequentially compact, and K is closed convex set and thus weakly closed, there exists subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $p \in K$ such that $x_{n_k} \rightharpoonup p$, since $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $I - T$ is demiclosed with respect to zero, $p \in F(T)$. If $\{x_n\}$ does't converge weakly to p, then there are subsequence $\{x_{n_j}\}$ which converges to $p \in K$ and $q \in F(T)$, $q \neq p$. Lemma 1.6 provide the existence of $a = \lim_{n\to\infty} \|x_n - p\|$ and $b = \lim_{n\to\infty} \|x_n - q\|$.

By Opial's condition we have

$$a = \liminf_{n\to\infty} \|x_{n_k} - p\| < \liminf_{n\to\infty} \|x_{n_k} - q\| = b = \liminf_{n\to\infty} \|x_{n_j} - q\| < \liminf_{n\to\infty} \|x_{n_j} - p\| = a$$

a contradiction.

This completes the proof.

□

Remark 3.2 *Theorem 3.1 generalizes theorem 2 of Deng [1] (which itself generalizes a result of Emmanuele [3]), Theorem 2.1 of Schu [12] for the iteration process (1.2) and the mapping of asymptotically quasi nonexpansive type considered here.*

References

[1] L.Deng, *Convergence of the Ishikawa iteration process for nonexpansive mappings, J.Math.Anal.Appl. 199(1996),769-775.*

[2] J.B.Diaz and F.T.Metcalf, *On the structure of the set of subsequential limit points of successive approximations, Bull.Amer.Math.Soc. 73(1967), 516-519.*

[3] G.Emmanuele, *Convergence of the Mann-Ishikawa iterative process for nonexpansive mappings., Nonlinear Anal. 6(1982),1135-1141.*

[4] K.Goebel and W.A.Kirk, *A fixed point theorem for asymptotically nonexpansive mappings, Proc.Amer.Math.Soc. 35(1972),171-174.*

[5] M.K.Gosh and L.Debnath,*Convergence of Ishikawa iterates of quasi-nonexpansive mappings, J.Math.Anal.Appl. 207(1997),96-103.*

[6] S.Ishikawa, *Fixed point by a new iteration, Proc.Amer.Math.Sco. 44(1974),147-150.*

[7] W.A.Kirk, *Fixed point theorems from non-Lipschitzian mappings of asymptotically nonexpansive type. Israel J.Math. 17(1974), 339-346.*

[8] Q.Liu,*Convergence theorems of the sequence of iterates for asymptotically demicontractive and hemicontractive mappings, Nonlinear Anal. 26(1996),1835-1842.*

[9] Q.Liu, *Iterative sequences for asymptotically quasi-nonexpansive mappings, J.Math.Anal.Appl. 259(2001),1-7*

[10] W.R.Mann, *Mean value methods in iteration, Proc.Amer.Math.Soc. 4(1953),506-510.*

[11] Z.Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mapping. Bull.Amer.Math.Soc. 73(1967), 591-597.*

[12] J.Schu, *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull.Austral.Math.Soc. 43(1991),153-159.*

[13] H.F.Senter and W.G.Dotson,Jr., *Approximating fixed points of nonexpansive mappings, Proc.Amer.Math.Soc. 44(1974),375-380.*

[14] P.Srivastava and S.C.Srivastava, *On asymptotically quasinonexpansive families of mappings. In : Nonlinear analysis and applications (St. Johns, 1981), Lecture Notes in Pure and Appl. Math. 80, pp. 265-270, Dekker (1982).*

[15] K.K.Tan and H.K.Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J.Math.Anal.Appl. 178(1993), 301-308.*

[16] W.V.Petryshyn and T.E.Williamson, *Strong and weak convergence of the sequence of successive approximations for quasi-nonexpansive mappings, J.Math.Anal.Appl. 43(1973),459-497.*

[17] B.Xu, and M.A.Noor,*Fixed-point iterations for asymptotically nonexpansive mappings in Banach spaces. J.Math.Anal.Appl. 267(2002),444-453.*

[18] Zeng, L.C.,*A note on approximating fixed points of nonexpansive mappings by the Ishikawa iteration process. J.Math.Anal.Appl. 226(1998),245-250.*

(Received 19 May 2007)

Amitabh Banerjee,
Govt. D.B. Girls' P.G. (Autonomous),
College, Raipur - 492010 (C.G.),
India
E-mail: amitabh61@yahoo.com