



On Modular Difference Sequence Spaces

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Abstract : In this paper, using m -th order difference operator $\Delta^{(m)}$ and a sequence $\{\alpha_n\}_{n=0}^{\infty}$ of strictly positive real numbers, sequence spaces $\Delta^{(m)}l_{\alpha}\{f_n\} = \{x \in s : (\Delta^{(m)}x)_j \in l_{\alpha}\{f_n\}\}$ and $\Delta^{(m)}l^{\alpha}\{g_n\} = \{x \in s : (\Delta^{(m)}x)_j \in l^{\alpha}\{g_n\}\}$ are introduced, where $x = \{\xi_j\}_{j=0}^{\infty} \in s$ and $\{f_n\}_{n=0}^{\infty}, \{g_n\}_{n=0}^{\infty}$ are sequence of Orlicz functions. It is shown that these are separable Banach spaces and dense F_{σ} -set of the first Baire category in s , the space of all real sequences with the Frèchet metric. Some earlier results related to Baire category are obtained when the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is chosen specifically.

Keywords : modular sequence spaces; modular spaces; difference sequence; Baire category.

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1 Introduction

The geometric properties of Banach space such as H -property, uniform Opial property, rotundity, uniform rotundity etc. are recent interest of study in modular function spaces or modular sequence spaces by many mathematicians and researchers. For example, geometric properties like uniform rotundity, uniform Opial property, H -property were discussed by Kamińska [1], Manna and Srivastava [2], Mongkolkeha and Kumam [3, 4], Cui and Hudzik [5] and many others. Similarly, the Baire category results on modular sequence spaces have received attention in some of the recent papers. In the year of 1980, Šalát [6] shown that

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the sequence space l_q is a set of the first Baire category in l_r , where $1 \leq q < r$. In 1973, Woo [7] introduced and studied the modular sequence spaces. Šalát & Ewert [8] further extended the result of Šalát [6] to modular sequence spaces. The theory of difference sequence spaces was first introduced and studied by Kizmaz [9]. The results of Kizmaz were generalized to m -th order difference sequence spaces by Malkowsky & Parashar [10]. In the present paper, an attempt has been made to introduce and study the modular difference sequence spaces $\Delta^{(m)}l_\alpha\{f_n\}$ and $\Delta^{(m)}l_\alpha\{g_n\}$ defined by using m -th order difference operator and a sequence $\{\alpha_n\}_{n=0}^\infty$ of strictly positive real numbers.

2 Preliminaries

Let X be a real vector space. A functional $\varrho : X \rightarrow [0, \infty]$ is said to be a convex modular if for arbitrary $x, y \in X$, the following conditions hold:

- (i) $\varrho(x) = 0$ if and only if $x = 0$,
- (ii) $\varrho(-x) = \varrho(x)$,
- (iii) $\varrho(\alpha x + \beta y) \leq \alpha\varrho(x) + \beta\varrho(y)$ for $x, y \in X, \alpha, \beta \geq 0, \alpha + \beta = 1$.

The set

$$X_\varrho = \{x \in X : \varrho(\lambda x) < \infty, \text{ for some } \lambda > 0\}$$

is a linear subspace of a real vector space X and it is called modular space determined by ϱ . The relation

$$\|x\| = \inf \left\{ \lambda > 0 : \varrho\left(\frac{x}{\lambda}\right) \leq 1 \right\} \text{ defines a norm on } X_\varrho \text{ [11, 12].}$$

A function $f : [0, \infty) \rightarrow [0, \infty)$ is said to be an Orlicz function if it is continuous, non decreasing, convex and $f(0) = 0$ and $\lim_{x \rightarrow \infty} f(x) = \infty$. If $f(x) = 0$ for some $x > 0$, then f is said to be a degenerate Orlicz function. Orlicz functions, which are not degenerate called non-degenerate.

Remark 2.1. An Orlicz function can be represented as $f(x) = \int_0^x p(t)dt$, where $p(t)$ is called the kernel of f having the properties $p(0) = 0, p(t) \rightarrow \infty$ as $t \rightarrow \infty$ & is right differentiable for $t \geq 0$.

A sequence space λ with a linear topology is called a K -space if each of the projection maps $P_i : \lambda \rightarrow \mathbb{K}$, given by $P_i(x) = x_i, i \geq 1$ is continuous where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} [13].

Let w be the class of all real sequences. For a sequence $\{f_n\}_{n=0}^\infty$ of Orlicz functions, the modular sequence space is defined as below:

$$l\{f_n\} = \left\{ x \in w : \sum_{n=0}^{\infty} f_n\left(\frac{|x_n|}{\lambda}\right) < \infty, \text{ for some } \lambda > 0 \right\}.$$

This is a Banach space with respect to the norm $\|\cdot\|_{l\{f_n\}}$ defined as follows:

$$\|x\|_{l\{f_n\}} = \inf \left\{ \lambda > 0 : \sum_{n=0}^{\infty} f_n \left(\frac{|x_n|}{\lambda} \right) \leq 1 \right\}.$$

These spaces were introduced and studied by Woo [7]. For an arbitrary positive integer m , the operators $\Delta^{(m)}, \Sigma^{(m)} : s \rightarrow s$ are continuous, linear and defined by

$$(\Delta^{(1)}x)_k = \Delta^{(1)}\xi_k = \xi_k - \xi_{k-1}, (\Sigma^{(1)}x)_k = \sum_{j=0}^k \xi_j, \quad k = 0, 1, 2, \dots$$

$$\Delta^{(m)} = \Delta^{(1)} \circ \Delta^{(m-1)}, \Sigma^{(m)} = \Sigma^{(1)} \circ \Sigma^{(m-1)}.$$

Also $\Sigma^{(m)} \circ \Delta^{(m)} = \Delta^{(m)} \circ \Sigma^{(m)} = id$, the identity on s .

In general, the m -th order difference sequence space $(\Delta^{(m)}x)_k$ & its inverse $(\Sigma^{(m)}x)_k$ is defined by the finite sum given below:

$$(\Delta^{(m)}x)_k = \sum_{i=0}^m (-1)^i \binom{m}{i} \xi_{k-i} \quad \& \quad (\Sigma^{(m)}x)_k = \sum_{i=0}^k \binom{m+k-i-1}{k-i} \xi_i$$

respectively, with all negative indices are assumed to be zero [10]. Malkowsky and Parashar [10] defined and studied the sequence spaces

$$\Delta^{(m)}(X) = \{x = (\xi_k) : (\Delta^{(m)}x)_k \in X\} \text{ for sequence spaces } X = l_{\infty}, c_0, c.$$

Let s be the linear space of all real sequences with the Frèchet metric d defined by

$$d(x, y) = \sum_{j=0}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|}$$

where $x = \{\xi_j\}_{j=0}^{\infty} \in s$ and $y = \{\eta_j\}_{j=0}^{\infty} \in s$.

Definition 2.2. [14] A sequence $\{f_n\}_{n=0}^{\infty}$ of Orlicz function is said to satisfy *uniform Δ_2 -condition* if there exists $K > 0$ and an integer n_0 such that, for all $n \geq n_0$ we have $\frac{f_n(2t)}{f_n(t)} \leq K$ for each $t \in (0, \frac{1}{2}]$.

Lemma 2.3. [7, Corollary 3.3.] *If $\{f_n\}_{n=0}^{\infty}$ satisfies uniform Δ_2 -condition, then*

$$\sum_{n=0}^{\infty} f_n(|x_n|) < \infty \text{ implies } \lim_{n \rightarrow \infty} x_n = 0.$$

3 Modular Difference Sequence Spaces $\Delta^{(m)}l_{\alpha}\{f_n\}$ and $\Delta^{(m)}l^{\alpha}\{g_n\}$

Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of Orlicz functions and $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence of strictly positive real numbers. The following sequence spaces $l_{\alpha}\{f_n\}$ and $l^{\alpha}\{g_n\}$ are defined as

$$l_\alpha\{f_n\} = \left\{ x = \{\xi_k\}_{k=0}^\infty \in s : \sum_{k=0}^\infty f_k \left(\frac{|\xi_k|}{\lambda \alpha_k} \right) < \infty, \text{ for some } \lambda > 0 \right\} \text{ and}$$

$$l^\alpha\{g_n\} = \left\{ x = \{\xi_k\}_{k=0}^\infty \in s : \sum_{k=0}^\infty g_k \left(\frac{\alpha_k |\xi_k|}{\lambda} \right) < \infty, \text{ for some } \lambda > 0 \right\}, \text{ respectively.}$$

Construct a functional $\varrho_{\alpha, \Delta^{(m)}}$ defined on s as $\varrho_{\alpha, \Delta^{(m)}}(x) = \sum_{k=0}^\infty f_k \left(\frac{|\Delta^{(m)} \xi_k|}{\alpha_k} \right)$.

Then it is easy to verify that $\varrho_{\alpha, \Delta^{(m)}}$ satisfies all the properties of a convex modular. Now for each $x = \{\xi_k\}_{k=0}^\infty \in s$ the sequence spaces $\Delta^{(m)}l_\alpha\{f_n\}$ & $\Delta^{(m)}h_\alpha\{f_n\}$ are defined as follows:

$$\Delta^{(m)}l_\alpha\{f_n\} = \{x \in s : (\Delta^{(m)}x)_k \in l_\alpha\{f_n\}\} \text{ or equivalently}$$

$$\Delta^{(m)}l_\alpha\{f_n\} = \{x \in s : \varrho_{\alpha, \Delta^{(m)}}\left(\frac{x}{\lambda}\right) < \infty \text{ for some } \lambda > 0\}$$

and

$$\Delta^{(m)}h_\alpha\{f_n\} = \{x \in s : \varrho_{\alpha, \Delta^{(m)}}\left(\frac{x}{\lambda}\right) < \infty \text{ for all } \lambda > 0\}.$$

Then $\Delta^{(m)}l_\alpha\{f_n\}$ is a modular space determined by $\varrho_{\alpha, \Delta^{(m)}}$ and it will be called as modular difference sequence space. For $x \in \Delta^{(m)}l_\alpha\{f_n\}$, the corresponding norm is defined as

$$\|x\|_{\alpha, \Delta^{(m)}} = \inf \left\{ \lambda > 0 : \varrho_{\alpha, \Delta^{(m)}}\left(\frac{x}{\lambda}\right) = \sum_{k=0}^\infty f_k \left(\frac{|\Delta^{(m)} \xi_k|}{\lambda \alpha_k} \right) \leq 1 \right\}.$$

Note: Since $(\Delta^{(m)}x)_k = \Delta^{(m)}\xi_k$, so the norm $\|x\|_{\alpha, \Delta^{(m)}}$ can also be denoted by $\|(\Delta^{(m)}x)_k\|_{l_\alpha\{f_n\}}$.

In a similar way, the modular difference sequence spaces $\Delta^{(m)}l^\alpha\{g_n\}$ are defined as

$$\Delta^{(m)}l^\alpha\{g_n\} = \{x \in s : (\Delta^{(m)}x)_k \in l^\alpha\{g_n\}\}$$

equipped with the norm $\|x\|_{\Delta^{(m)}}^\alpha = \|(\Delta^{(m)}x)_k\|_{l^\alpha\{g_n\}}$.

4 Main Results

In the following theorem, a topological structure of the sequence space $\Delta^{(m)}l_\alpha\{f_n\}$ has been given.

Theorem 4.1. *Let $\{f_n\}_{n=0}^\infty$ be a sequence of Orlicz functions. Then the followings hold:*

- $\Delta^{(m)}l_\alpha\{f_n\}$ is a Banach K -space endowed with the norm $\|\cdot\|_{\alpha, \Delta^{(m)}}$.
- $\Delta^{(m)}h_\alpha\{f_n\}$ is a closed subspace of $\Delta^{(m)}l_\alpha\{f_n\}$.
- If $\{f_n\}_{n=0}^\infty$ satisfies uniform Δ_2 -condition, then $\Delta^{(m)}l_\alpha\{f_n\} = \Delta^{(m)}h_\alpha\{f_n\}$.

Proof. (a) For $x \in \Delta^{(m)}l_\alpha\{f_n\}$, the following norm is defined in Section 3:

$$\|x\|_{\alpha, \Delta^{(m)}} = \inf \left\{ \lambda > 0 : \varrho_{\alpha, \Delta^{(m)}}\left(\frac{x}{\lambda}\right) \leq 1 \right\}.$$

To show the completeness of the space, let $x^{(q)} = \{\xi_k^{(q)}\}_{k=0}^\infty$, $q = 0, 1, 2, \dots$ be a Cauchy sequence in $\Delta^{(m)}l_\alpha\{f_n\}$ and $\epsilon > 0$. Then there exists an $N > 0$ such that for every $\epsilon > 0$ there is an λ_ϵ with $\lambda_\epsilon < \epsilon$, one gets

$$\varrho_{\alpha, \Delta^{(m)}}\left(\frac{x^{(q)} - x^{(r)}}{\lambda_\epsilon}\right) \leq 1 \quad \text{for } q, r \geq N.$$

So, by the definition of $\varrho_{\alpha, \Delta^{(m)}}$, one obtains

$$\sum_{k=0}^\infty f_k \left(\frac{|\Delta^{(m)}\xi_k^{(q)} - \Delta^{(m)}\xi_k^{(r)}|}{\lambda_\epsilon \alpha_k} \right) \leq 1 \quad \text{for } q, r \geq N,$$

which implies that

$$f_k \left(\frac{|\Delta^{(m)}\xi_k^{(q)} - \Delta^{(m)}\xi_k^{(r)}|}{\lambda_\epsilon \alpha_k} \right) \leq 1 \quad \text{for each } k = 0, 1, 2, \dots \text{ and for } q, r \geq N.$$

Let p_k 's be the corresponding kernel of the Orlicz functions f_k 's. Now one can choose a constant $s_0 > 0$ and $\gamma > 1$ such that $\gamma \frac{s_0}{2} p_k(\frac{s_0}{2}) \geq 1$ (easily follows from $f(\frac{s_0}{2}) = \int_0^{\frac{s_0}{2}} p(t)dt$ and $s_0 > 0$) for each $k \in \mathbb{N}_0$, the set of natural numbers including 0. The notation \mathbb{N}_0 will be used wherever it will appear in the text.

Therefore by using integral representation of Orlicz function, above inequality reduces to

$$|\Delta^{(m)}\xi_k^{(q)} - \Delta^{(m)}\xi_k^{(r)}| \leq \lambda_\epsilon \alpha_k \gamma s_0 \quad \text{for each } k \in \mathbb{N}_0 \text{ and for } q, r \geq N.$$

Otherwise, one can find a k with $\frac{|\Delta^{(m)}\xi_k^{(q)} - \Delta^{(m)}\xi_k^{(r)}|}{\lambda_\epsilon \alpha_k} > \gamma s_0$, so that the following holds:

$$f_k \left(\frac{|\Delta^{(m)}\xi_k^{(q)} - \Delta^{(m)}\xi_k^{(r)}|}{\lambda_\epsilon \alpha_k} \right) \geq \int_{\frac{\gamma s_0}{2}}^{\frac{|\Delta^{(m)}\xi_k^{(q)} - \Delta^{(m)}\xi_k^{(r)}|}{\lambda_\epsilon \alpha_k}} p_k(t)dt > \frac{\gamma s_0}{2} p_k\left(\frac{\gamma s_0}{2}\right) > \frac{\gamma s_0}{2} p_k\left(\frac{s_0}{2}\right)$$

and a contradiction is attained. Therefore for each $k \in \mathbb{N}_0$, $\{\Delta^{(m)}\xi_k^{(r)}\}_{r=0}^\infty$ forms a Cauchy sequence of real numbers and hence converges.

Let $\lim_{r \rightarrow \infty} \Delta^{(m)}\xi_k^{(r)} = \Delta^{(m)}\xi_k$. Since f_k 's & the operator $\Delta^{(m)}$ are continuous, so from the above inequations, one obtains

$$\sum_{k=0}^\infty f_k \left(\frac{|\Delta^{(m)}\xi_k^{(q)} - \Delta^{(m)}\xi_k|}{\lambda_\epsilon \alpha_k} \right) \leq 1 \quad \text{for } q \geq N \text{ as } r \rightarrow \infty \text{ (keeping } m \text{ fixed)}$$

which implies $\|x^{(q)} - x\|_{\alpha, \Delta^{(m)}} \leq \lambda_\epsilon < \epsilon$ for $q \geq N$. To show that $x \in \Delta^{(m)}l_\alpha\{f_n\}$ for some $\lambda = 2\lambda_0 > 0$, consider the following expression:

$$\begin{aligned} & \sum_{k=0}^i f_k \left(\frac{|\Delta^{(m)}\xi_k|}{2\lambda_0\alpha_k} \right) \\ &= \sum_{k=0}^i f_k \left(\frac{|\Delta^{(m)}\{(\xi_k - \xi_k^{(q)}) + \xi_k^{(q)}\}|}{2\lambda_0\alpha_k} \right) \\ &\leq \frac{1}{2} \sum_{k=0}^i f_k \left(\frac{|\Delta^{(m)}(\xi_k - \xi_k^{(q)})|}{\lambda_0\alpha_k} \right) + \frac{1}{2} \sum_{k=0}^i f_k \left(\frac{|\Delta^{(m)}\xi_k^{(q)}|}{\lambda_0\alpha_k} \right). \end{aligned} \tag{4.1}$$

Using the continuity of the operator $\Delta^{(m)}$ one may choose a $\delta > 0$ with $|\xi_k^{(q)} - \xi_k| < \delta$ such that for each $k = 0, 1, 2, \dots, i$ and for a given $\epsilon > 0$ the following holds:

$$|\Delta^{(m)}\xi_k^{(q)} - \Delta^{(m)}\xi_k| < \frac{\epsilon\alpha_k\lambda_0 f_k^{-1}(1)}{2^{(k+1)}}.$$

Applying the nondecreasing property of f_k 's, one gets

$$\sum_{k=0}^i f_k \left(\frac{|\Delta^{(m)}\xi_k^{(q)} - \Delta^{(m)}\xi_k|}{\lambda_0\alpha_k} \right) < \epsilon \sum_{k=0}^i \frac{1}{2^{(k+1)}}. \tag{4.2}$$

Since $x^{(q)} \in \Delta^{(m)}l_\alpha\{f_n\}$ choosing $i \rightarrow \infty$ in Eqn. (4.1) & Eqn. (4.2), it follows that $\sum_{k=0}^\infty f_k \left(\frac{|\Delta^{(m)}\xi_k|}{2\lambda_0\alpha_k} \right) < \infty$, i.e., $x \in \Delta^{(m)}l_\alpha\{f_n\}$. Hence $\Delta^{(m)}l_\alpha\{f_n\}$ is a Banach space.

Now the space $\Delta^{(m)}l_\alpha\{f_n\}$ is a K space will be presented next. Choose $x^{(q)} \rightarrow x$ in the norm $\|\cdot\|_{\alpha, \Delta^{(m)}}$ for large q . Then by definition of $\|\cdot\|_{\alpha, \Delta^{(m)}}$, for every $\epsilon > 0$ there exists a natural number q_0 such that

$$\|x^{(q)} - x\|_{\alpha, \Delta^{(m)}} = \inf \left\{ \lambda > 0 : \sum_{k=0}^\infty f_k \left(\frac{|\Delta^{(m)}(\xi_k^{(q)} - \xi_k)|}{\lambda\alpha_k} \right) \leq 1 \right\} < \epsilon \text{ for } q \geq q_0,$$

i.e., $x^{(q)} - x \rightarrow 0$ as $q \rightarrow \infty$ in the norm $\|\cdot\|_{\alpha, \Delta^{(m)}}$, i.e., $\epsilon \rightarrow 0$ as $q \rightarrow \infty$. Using the similar techniques used as above, there exists a λ_ϵ with $\lambda_\epsilon < \epsilon$ such that

$$|\Delta^{(m)}(\xi_k^{(q)} - \xi_k)| \leq \lambda_\epsilon\alpha_k\gamma s_0 \text{ for } q \geq q_0 \text{ and for each } k \in \mathbb{N}_0,$$

which gives $|\Delta^{(m)}(\xi_k^{(q)} - \xi_k)| \rightarrow 0$ as $q \rightarrow \infty$, which in turn implies that $(\xi_k^{(q)} - \xi_k) \in \Delta^{(m)}c_0$ for each $k \in \mathbb{N}_0$.

It is known that $\Delta^{(m)}c_0$ is isometrically isomorphic to c_0 , which can be viewed by the mapping $T : \Delta^{(m)}c_0 \rightarrow c_0$ defined by $Tx = \Delta^{(m)}x$ with the norm defined by $\|x\|_{\Delta^{(m)}c_0} = \sup_k |\Delta^{(m)}\xi_k|$, where $x = \{\xi_k\}_{k=0}^\infty$. Hence $\Delta^{(m)}l_\alpha\{f_n\}$ is a K -space.

(b) It is clear that $\Delta^{(m)}h_\alpha\{f_n\}$ is a subspace of $\Delta^{(m)}l_\alpha\{f_n\}$. Let for each $i \in \mathbb{N}_0$, $x^{(i)} = \{\xi_k^{(i)}\}_{k=0}^\infty \in \Delta^{(m)}h_\alpha\{f_n\}$ be a sequence such that $\lim_{i \rightarrow \infty} \|x^{(i)} - x\|_{\alpha, \Delta^{(m)}} = 0$,

where $x \in \Delta^{(m)}l_\alpha\{f_n\}$. Let $\lambda > 0$ be arbitrary. Corresponding to this λ , one can choose a positive integer i_0 such that $\|x^{(i_0)} - x\|_{\alpha, \Delta^{(m)}} < \frac{\lambda}{2}$. Now it follows that

$$\begin{aligned} \sum_{k=0}^{\infty} f_k\left(\frac{|\Delta^{(m)}\xi_k|}{\lambda\alpha_k}\right) &= \sum_{k=0}^{\infty} f_k\left(\frac{|\Delta^{(m)}(\xi_k^{(i_0)} + \xi_k - \xi_k^{(i_0)})|}{\lambda\alpha_k}\right) \\ &\leq \frac{1}{2} \sum_{k=0}^{\infty} f_k\left(\frac{2|\Delta^{(m)}\xi_k^{(i_0)}|}{\lambda\alpha_k}\right) + \frac{1}{2} \sum_{k=0}^{\infty} f_k\left(\frac{|\Delta^{(m)}(\xi_k^{(i_0)} - \xi_k)|}{\alpha_k\|x^{(i_0)} - x\|_{\alpha, \Delta^{(m)}}}\right) \\ &< \infty. \end{aligned}$$

Thus $x \in \Delta^{(m)}h_\alpha\{f_n\}$.

(c) Let $x \in \Delta^{(m)}l_\alpha\{f_n\}$ be a sequence. Then there exists an $r > 0$ such that $\varrho_{\alpha, \Delta^{(m)}}\left(\frac{x}{r}\right)$ is finite. For an arbitrary λ such that $\lambda \geq r > 0$, one gets $\varrho_{\alpha, \Delta^{(m)}}\left(\frac{x}{\lambda}\right) \leq \varrho_{\alpha, \Delta^{(m)}}\left(\frac{x}{r}\right)$, which is finite. If $\lambda < r$, i.e., when $\frac{r}{\lambda} > 1$ then one can find an integer $l \geq 1$ such that $r \leq 2^l\lambda$. Now by Lemma 2.3, it follows that $\lim_{k \rightarrow \infty} \frac{\Delta^{(m)}\xi_k}{r\alpha_k} = 0$. Hence there exist natural numbers k_1, k_2 and a constant $K > 0$ such that

$$\begin{aligned} \frac{|\Delta^{(m)}\xi_k|}{r\alpha_k} &\leq \frac{1}{2} \quad \text{for all } k \geq k_1 \\ \text{and } f_k(2t) &\leq Kf_k(t) \quad \text{for all } k \geq k_2, t \in (0, \frac{1}{2}]. \end{aligned}$$

Let $k_0 = \max\{k_1, k_2\}$. Then for $k \geq k_0$, one gets

$$\begin{aligned} \sum_{k=0}^{\infty} f_k\left(\frac{|\Delta^{(m)}\xi_k|}{\lambda\alpha_k}\right) &= \sum_{k=0}^{k_0-1} f_k\left(\frac{|\Delta^{(m)}\xi_k|}{\lambda\alpha_k}\right) + \sum_{k=k_0}^{\infty} f_k\left(\frac{r}{\lambda} \frac{|\Delta^{(m)}\xi_k|}{r\alpha_k}\right) \\ &\leq \sum_{k=0}^{k_0-1} f_k\left(\frac{|\Delta^{(m)}\xi_k|}{\lambda\alpha_k}\right) + \sum_{k=k_0}^{\infty} f_k\left(2^l \frac{|\Delta^{(m)}\xi_k|}{r\alpha_k}\right) \\ &\leq \sum_{k=0}^{k_0-1} f_k\left(\frac{|\Delta^{(m)}\xi_k|}{\lambda\alpha_k}\right) + K^l \sum_{k=k_0}^{\infty} f_k\left(\frac{|\Delta^{(m)}\xi_k|}{r\alpha_k}\right) \\ &< \infty \end{aligned}$$

that is, $\varrho_{\alpha, \Delta^{(m)}}\left(\frac{x}{\lambda}\right) < \infty$ and hence $x \in \Delta^{(m)}h_\alpha\{f_n\}$. This completes the proof of the theorem. \square

Theorem 4.2. *The modular difference sequence space $\Delta^{(m)}l_\alpha\{g_n\}$ is a BK-space equipped with the norm $\|\cdot\|_{\Delta^{(m)}}^\alpha$ defined by*

$$\|x\|_{\Delta^{(m)}}^\alpha = \inf \left\{ \lambda > 0 : \sum_{k=0}^{\infty} g_k\left(\frac{\alpha_k |\Delta^{(m)}\xi_k|}{\lambda}\right) \leq 1 \right\}.$$

Proof. The proof runs on the parallel lines of Theorem 4.1 and hence omitted. \square

Proposition 4.3. Let $\{f_k\}_{k=0}^\infty$ be a sequence of non degenerate Orlicz functions.

(i) If $\alpha = \{\alpha_k\}_{k=0}^\infty \in l_\infty$ and $\sum_{k=0}^\infty f_k\left(\frac{L}{\lambda\alpha_k}\right) < \infty$, $L = \sup_k |\Delta^{(m)}\xi_k|$ for some $\lambda > 0$,

then $\Delta^{(m)}l_\alpha\{f_n\} = \Delta^{(m)}l_\infty$.

(ii) If $\alpha \in l_p, p \geq 1$ then $\Delta^{(m)}l_\alpha\{f_n\} \subset \Delta^{(m)}l_p$. Similarly, if $\frac{1}{\alpha} \in l_p$ then $\Delta^{(m)}l^\alpha\{f_n\} \subset \Delta^{(m)}l_p$.

Proof. (i) Let $x \in \Delta^{(m)}l_\alpha\{f_n\}$ be a sequence. Then by the definition of norm, it follows that

$$f_k\left(\frac{|\Delta^{(m)}\xi_k|}{\alpha_k\|x\|_{\alpha,\Delta^{(m)}}}\right) \leq 1 \quad \text{for each } k \in \mathbb{N}_0.$$

It is known that for each $k \in \mathbb{N}_0$, there is a constant $s_0 > 0$ and $\gamma > 1$ such that $\gamma\frac{s_0}{2}p_k(\frac{s_0}{2}) \geq 1$ holds. Using it and the integral representation of Orlicz function, one gets

$$|\Delta^{(m)}\xi_k| \leq \alpha_k\gamma s_0\|x\|_{\alpha,\Delta^{(m)}}.$$

Since $\alpha \in l_\infty$, so the above implies that $(\Delta^{(m)}\xi_k) \in l_\infty$ which gives $\{\xi_k\}_{k=0}^\infty \in \Delta^{(m)}l_\infty$.

To establish the converse inclusion, suppose that $x \in \Delta^{(m)}l_\infty$. Then there exists a constant $L > 0$ such that $|\Delta^{(m)}\xi_k| \leq L$. The nondecreasing property of f_k 's implies that

$$\sum_{k=0}^\infty f_k\left(\frac{|\Delta^{(m)}\xi_k|}{\lambda\alpha_k}\right) \leq \sum_{k=0}^\infty f_k\left(\frac{L}{\lambda\alpha_k}\right) < \infty \quad \text{for some } \lambda > 0,$$

which gives $x \in \Delta^{(m)}l_\alpha\{f_n\}$, which in turn implies that $\Delta^{(m)}l_\alpha\{f_n\} = \Delta^{(m)}l_\infty$.

(ii) Using the similar characteristics of Orlicz functions one can easily obtains the inclusions. Therefore it is omitted. □

Remark 4.4. The inclusions in (ii) are strict. Indeed, for the first inclusion, choose $\alpha = \{\alpha_k\}_{k=0}^\infty = \left\{\frac{1}{(k+1)^3}\right\}_{k=0}^\infty$ and $x = \{\xi_k\}_{k=0}^\infty = \left\{\Sigma^{(m)}\frac{1}{(k+1)^2}\right\}_{k=0}^\infty$, then $x \in \Delta^{(m)}l_p$ but $x \notin \Delta^{(m)}l_\alpha\{f_n\}$. For the second inclusion, choose $\alpha = \{\alpha_k\}_{k=0}^\infty = \{(k+1)^3\}_{k=0}^\infty$ and x is same as above then it follows that $x \in \Delta^{(m)}l_p$ but $x \notin \Delta^{(m)}l^\alpha\{f_n\}$.

In the following, the sequence of vectors whose $(n + 1)$ -th coordinate is 1 and others are zero will be denoted by $\{e_n\}_{n=0}^\infty$.

Theorem 4.5. (Schauder Basis) Let $\{f_n\}_{n=0}^\infty$ be a sequence of Orlicz functions satisfying the uniform Δ_2 -condition. Then

(a) $\Delta^{(m)}l_\alpha\{f_n\}$ has Schauder basis $\{\Sigma^{(m)}e_k\}_{k=0}^\infty$ and for each $x = \{\xi_k\}_{k=0}^\infty \in \Delta^{(m)}l_\alpha\{f_n\}$, the expression for $x = \sum_{k=0}^\infty (\Delta^{(m)}x)_k(\Sigma^{(m)}e_k)$ is unique,

(b) $\Delta^{(m)}l_\alpha\{f_n\}$ is separable.

Proof. (a) It is evident that for each $k \in \mathbb{N}_0$ one gets $e_k \in l_\alpha\{f_n\}$ which implies that $\{\Sigma^{(m)}e_k\}_{k=0}^\infty$ is a sequence of elements of $\Delta^{(m)}l_\alpha\{f_n\}$. The authors claim that $\{\Sigma^{(m)}e_k\}_{k=0}^\infty$ is a Schauder basis of $\Delta^{(m)}l_\alpha\{f_n\}$. To prove our claim, choosing $x \in$

$$\Delta^{(m)}l_\alpha\{f_n\} \text{ and putting } x^{[j]} = \sum_{k=0}^j (\Delta^{(m)}x)_k (\Sigma^{(m)}e_k) = \sum_{k=0}^j \Sigma^{(m)}((\Delta^{(m)}x)_k e_k).$$

Since the sequence $\{f_n\}_{n=0}^\infty$ of Orlicz functions satisfies the uniform Δ_2 -condition,

so by Theorem 4.1(c), one obtains $\sum_{k=0}^\infty f_k\left(\frac{|\Delta^{(m)}\xi_k|}{\lambda\alpha_k}\right) < \infty$ for every $\lambda > 0$.

Therefore for each $\epsilon \in (0, 1)$ there exists a $j_0 \in \mathbb{N}$ such that

$$\sum_{k \geq j_0} f_k\left(\frac{|\Delta^{(m)}\xi_k|}{\epsilon\alpha_k}\right) \leq 1 \text{ holds.}$$

Now for $j \geq j_0$, the following expression is obtained:

$$\begin{aligned} \|x - x^{[j]}\|_{\alpha, \Delta^{(m)}} &= \|\Delta^{(m)}x - \Delta^{(m)}x^{[j]}\|_{l_\alpha\{f_n\}} \\ &= \inf \left\{ \lambda > 0 : \sum_{k=j+1}^\infty f_k\left(\frac{|\Delta^{(m)}\xi_k|}{\lambda\alpha_k}\right) \leq 1 \right\} \\ &\leq \inf \left\{ \lambda > 0 : \sum_{k=j_0}^\infty f_k\left(\frac{|\Delta^{(m)}\xi_k|}{\lambda\alpha_k}\right) \leq 1 \right\} \leq \epsilon, \end{aligned}$$

which gives $x = \sum_{k=0}^\infty (\Delta^{(m)}x)_k (\Sigma^{(m)}e_k) = \sum_{k=0}^\infty \Sigma^{(m)}((\Delta^{(m)}x)_k e_k)$.

To show uniqueness of x , if possible let $x \in \Delta^{(m)}l_\alpha\{f_n\}$ has another representation

as $x = \sum_{k=0}^\infty \sigma_k (\Sigma^{(m)}e_k) = \sum_{k=0}^\infty \Sigma^{(m)}(\sigma_k e_k)$. Then $\Delta^{(m)}x = \sum_{k=0}^\infty \sigma_k e_k$. On the other

hand, $\Delta^{(m)}x = \sum_{k=0}^\infty (\Delta^{(m)}x)_k e_k$ which gives $\sigma_k = (\Delta^{(m)}x)_k$ and therefore the expression for x is unique.

(b) Let us define the set $S = \left\{ \sum_{k=0}^{k_0} t_k (\Sigma^{(m)}e_k) : t_0, t_1, \dots, t_{k_0} \in \mathbb{Q} \right\}$. It is required

to show that S is a countable dense subset of $\Delta^{(m)}l_\alpha\{f_n\}$. Clearly S is countable as the coefficients $t_k \in \mathbb{Q}$ for each $k = 0, 1, \dots, k_0$.

Let $x = \{\xi_k\}_{k=0}^\infty \in \Delta^{(m)}l_\alpha\{f_n\}$. So for every $\epsilon > 0$, $\varrho_{\alpha, \Delta^{(m)}}\left(\frac{x}{\epsilon}\right) < \infty$, i.e.,

$$\sum_{k=0}^\infty f_k\left(\frac{|\Delta^{(m)}\xi_k|}{\epsilon\alpha_k}\right) < \infty, \text{ which implies that there exists a } k_0 \in \mathbb{N} \text{ such that the}$$

following holds:

$$\sum_{k=k_0+1}^\infty f_k\left(\frac{|\Delta^{(m)}\xi_k|}{\epsilon\alpha_k}\right) < \frac{1}{2}. \tag{4.3}$$

Since the operator $\Delta^{(m)}$ is continuous, ξ_k 's are real numbers, t_k 's are rational numbers for each $k = 0, 1, \dots, k_0$ and \mathbb{Q} is a dense subset of \mathbb{R} , so one gets

$$|\Delta^{(m)}\xi_k - \Delta^{(m)}t_k| < \frac{\epsilon \alpha_k f_k^{-1}(1)}{2(k_0+1)} \quad \text{for } k = 0, 1, 2, \dots, k_0.$$

Using the nondecreasing property of f_k 's, one obtains

$$\sum_{k=0}^{k_0} f_k \left(\frac{|\Delta^{(m)}\xi_k - \Delta^{(m)}t_k|}{\epsilon \alpha_k} \right) < \frac{1}{2}.$$

Note that the element $u = \Delta^{(m)}t_0(\Sigma^{(m)}e_0) + \Delta^{(m)}t_1(\Sigma^{(m)}e_1) + \Delta^{(m)}t_2(\Sigma^{(m)}e_2) + \dots + \Delta^{(m)}t_{k_0}(\Sigma^{(m)}e_{k_0}) \in S$.

Now using Eqn. (4.3), one gets

$$\varrho_{\alpha, \Delta^{(m)}}\left(\frac{x-u}{\epsilon}\right) = \sum_{k=0}^{k_0} f_k \left(\frac{|\Delta^{(m)}\xi_k - \Delta^{(m)}t_k|}{\epsilon \alpha_k} \right) + \sum_{k=k_0+1}^{\infty} f_k \left(\frac{|\Delta^{(m)}\xi_k|}{\epsilon \alpha_k} \right) < 1.$$

Therefore $\|x - u\|_{\alpha, \Delta^{(m)}} < \epsilon$, which says that the space $\Delta^{(m)}l_\alpha\{f_n\}$ is separable. □

Corollary 4.6. *Let $\{g_n\}_{n=0}^\infty$ be a sequence of Orlicz functions satisfying the uniform Δ_2 -condition. Then the space $\Delta^{(m)}l_\alpha\{g_n\}$ is separable and has Schauder basis $\{\Sigma^{(m)}e_k\}_{k=0}^\infty$. Also for each $x = \{\xi_k\}_{k=0}^\infty \in \Delta^{(m)}l_\alpha\{g_n\}$ one gets an unique expression for $x = \sum_{k=0}^\infty (\Delta^{(m)}x)_k(\Sigma^{(m)}e_k)$.*

Proof. Applying the similar approach presented in Theorem 4.5, one can achieve this result. So the proof is omitted. □

5 Category Results on $\Delta^{(m)}l_\alpha\{f_n\}$ & $\Delta^{(m)}l_\alpha\{g_n\}$

In this section, it is contemplated to establish certain results related to Baire category for the sequence spaces $\Delta^{(m)}l_\alpha\{f_n\}$ and $\Delta^{(m)}l_\alpha\{g_n\}$. Firstly, the following theorem will be presented:

Theorem 5.1. *For a sequence $\{f_n\}_{n=0}^\infty$ of non degenerate Orlicz functions, the sequence space $\Delta^{(m)}l_\alpha\{f_n\}$ is a dense F_σ -set of the first Baire category in s .*

Proof. In Section 4, it is proved that $\Delta^{(m)}l_\alpha\{f_n\}$ is a Banach space equipped with the norm $\|\cdot\|_{\alpha, \Delta^{(m)}}$ defined below:

$$\|x\|_{\alpha, \Delta^{(m)}} = \inf \left\{ \lambda > 0 : \sum_{k=0}^{\infty} f_k \left(\frac{|\Delta^{(m)}\xi_k|}{\lambda \alpha_k} \right) \leq 1 \right\}.$$

Let us consider the set A_t as

$$A_t = \{x = \{\xi_k\}_{k=0}^\infty \in \Delta^{(m)}l_\alpha\{f_n\} : \|x\|_{\alpha, \Delta^{(m)}} \leq t\}, \quad t = 1, 2, 3, \dots$$

Since each element of $\Delta^{(m)}l_\alpha\{f_n\}$ is in some A_t , so clearly $\Delta^{(m)}l_\alpha\{f_n\} = \bigcup_{t=1}^\infty A_t$.

Now, it will be shown that for each $t \in \mathbb{N}$, A_t 's are nowhere dense sets in s , the space of all real sequences with the Frèchet metric d . For this, A_t 's for each $t \in \mathbb{N}$ are closed sets in s will be proved first. Let $a = \{a^{(i)}\}_{i=0}^\infty \in \bar{A}_t$, the closure of A_t and $a_k = \{a_k^{(i)}\}_{i=0}^\infty \in A_t$ with $a_k \rightarrow a$ as $k \rightarrow \infty$ in s . Then, one gets

$$d(a_k, a) \rightarrow 0 \text{ as } k \rightarrow \infty \text{ which implies } |a_k^{(i)} - a^{(i)}| \rightarrow 0 \text{ as } k \rightarrow \infty, \text{ for each } i \in \mathbb{N}_0.$$

Denote $\lim_{k \rightarrow \infty} a_k^{(i)} = a^{(i)}$ for each $i \in \mathbb{N}_0$. Since $a_k \in A_t$, i.e., $\{a_k^{(i)}\}_{i=0}^\infty \in \Delta^{(m)}l_\alpha\{f_n\}$, so for each $k \in \mathbb{N}_0$, one obtains

$$\sum_{i=0}^\infty f_i \left(\frac{|\Delta^{(m)}a_k^{(i)}|}{\lambda\alpha_i} \right) < \infty \text{ for some } \lambda > 0 \text{ and } \|\{a_k^{(i)}\}_{i=0}^\infty\|_{\alpha, \Delta^{(m)}} \leq t, t = 1, 2, \dots$$

Let $\|\{a_k^{(i)}\}_{i=0}^\infty\|_{\alpha, \Delta^{(m)}} = \lambda_0$ (say), then for each $k \in \mathbb{N}_0$, the followings hold:

$$\sum_{i=0}^\infty f_i \left(\frac{|\Delta^{(m)}a_k^{(i)}|}{\lambda_0\alpha_i} \right) \leq 1 \text{ and } \lambda_0 \leq t.$$

The continuity of f_i 's and $\Delta^{(m)}$'s implies that

$$\sum_{i=0}^\infty f_i \left(\frac{|\Delta^{(m)}a^{(i)}|}{\lambda\alpha_i} \right) < \infty, \text{ for some } \lambda > 0 \text{ and } \sum_{i=0}^\infty f_i \left(\frac{|\Delta^{(m)}a^{(i)}|}{\lambda_0\alpha_i} \right) \leq 1,$$

because $a_k^{(i)} \rightarrow a^{(i)}$ as $k \rightarrow \infty$, i.e., $a = \{a^{(i)}\}_{i=0}^\infty \in \Delta^{(m)}l_\alpha\{f_n\}$ and

$$\|\{a^{(i)}\}_{i=0}^\infty\|_{\alpha, \Delta^{(m)}} \leq \lambda_0 \leq t.$$

Hence $a \in A_t$ and so A_t 's are closed sets in s . Therefore $\Delta^{(m)}l_\alpha\{f_n\}$ is an F_σ -set in s .

Now A_t is a nowhere dense set in s will be proved. A well known result is that a closed set in a metric space is nowhere dense if and only if its complement is everywhere dense (see [15, p. 75]). Denote the complement of A_t by B_t , where B_t has the following expression

$$B_t = \{x = \{\xi_k\}_{k=0}^\infty \in \Delta^{(m)}l_\alpha\{f_n\} : \|x\|_{\alpha, \Delta^{(m)}} > t\}, t = 1, 2, 3, \dots$$

Indeed, the set B_t is everywhere dense in s will be proved now.

Firstly, B_t is nonempty as for fixed t as the element $v = \{\zeta_k\}_{k=0}^\infty$ is in B_t , where $\zeta_k = \Sigma^{(m)}[\lambda\alpha_k f_k^{-1} \{ \frac{6(t+1)}{\pi^2 \lambda (k+1)^2} \}]$ for some $\lambda > 0$. To show that $\bar{B}_t = s$, first it is to be noted that $B_t \neq \phi$ and hence one can choose a sequence $y = \{\eta_k\}_{k=0}^\infty \in B_t$.

For a given $\epsilon > 0$ there exists $p \in \mathbb{N}$ such that $\sum_{k=p}^{\infty} 2^{-k} < \epsilon$.

Let us construct the sequence $z = \{\tau_k\}_{k=0}^{\infty}$ as

$$\begin{aligned} \tau_k &= \xi_k & 0 \leq k \leq p-1 \\ &= \tau_p & k = p \\ &= \tau_{p+r}, & k = p+r, r = 1, 2, 3, \dots, \end{aligned}$$

where τ_p will be determined from the relation $\frac{\Delta^{(m)}\tau_p}{\lambda\alpha_p} = f_p^{-1}\left\{\sum_{k=0}^p f_k\left(\frac{|\Delta^{(m)}\eta_k|}{\lambda\alpha_k}\right)\right\}$

and τ_{p+r} has the following expression

$$\tau_{p+r} = \eta_{p+r} - \binom{m}{1}(\eta_{p+r-1} - \tau_{p+r-1}) + \binom{m}{2}(\eta_{p+r-2} - \tau_{p+r-2}) + \dots + (-1)^m(\eta_{p+r-m} - \tau_{p+r-m}).$$

Then one gets

$$\begin{aligned} \sum_{k=0}^{\infty} f_k\left(\frac{|\Delta^{(m)}\tau_k|}{\lambda\alpha_k}\right) &= \sum_{k=0}^{p-1} f_k\left(\frac{|\Delta^{(m)}\xi_k|}{\lambda\alpha_k}\right) + f_p\left(f_p^{-1}\left\{\sum_{k=0}^p f_k\left(\frac{|\Delta^{(m)}\eta_k|}{\lambda\alpha_k}\right)\right\}\right) \\ &+ \sum_{k=p+1}^{\infty} f_k\left(\frac{|\Delta^{(m)}\eta_k|}{\lambda\alpha_k}\right) \\ &= \sum_{k=0}^{p-1} f_k\left(\frac{|\Delta^{(m)}\xi_k|}{\lambda\alpha_k}\right) + \sum_{k=0}^{\infty} f_k\left(\frac{|\Delta^{(m)}\eta_k|}{\lambda\alpha_k}\right) < \infty \end{aligned}$$

and

$$\begin{aligned} \|z\|_{\alpha, \Delta^{(m)}} &= \inf \left\{ \lambda > 0 : \sum_{k=0}^{\infty} f_k\left(\frac{|\Delta^{(m)}\tau_k|}{\lambda\alpha_k}\right) \leq 1 \right\} \\ &= \inf \left\{ \lambda > 0 : \sum_{k=0}^{p-1} f_k\left(\frac{|\Delta^{(m)}\xi_k|}{\lambda\alpha_k}\right) + \sum_{k=0}^{\infty} f_k\left(\frac{|\Delta^{(m)}\eta_k|}{\lambda\alpha_k}\right) \leq 1 \right\} \\ &\geq \inf \left\{ \lambda > 0 : \sum_{k=0}^{\infty} f_k\left(\frac{|\Delta^{(m)}\eta_k|}{\lambda\alpha_k}\right) \leq 1 \right\} = \|y\|_{\alpha, \Delta^{(m)}} > t. \end{aligned}$$

So, the sequence $z = \{\tau_k\}_{k=0}^{\infty} \in B_t$. Therefore for arbitrary $x \in s$, one obtains

$$d(x, z) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{|\xi_k - \tau_k|}{1 + |\xi_k - \tau_k|} \leq \sum_{k=p}^{\infty} 2^{-k} < \epsilon.$$

Hence for every ϵ -ball in s contains an element from B_t . Therefore B_t is everywhere dense in s and hence its complement A_t is nowhere dense in s . So, $\Delta^{(m)}l_{\alpha}\{f_n\}$ is a set of the first Baire category in s .

It remains to be showed that $\Delta^{(m)}l_{\alpha}\{f_n\}$ is dense in s . To show this, suppose that

$x = \{\xi_k\}_{k=0}^\infty \in s$ is arbitrary. For a given $\epsilon > 0$, there exists a $p_0 \in \mathbb{N}$ such that $\sum_{k=p_0+1}^\infty \frac{1}{2^k} < \epsilon$. Define the sequence $w = \{w_k\}_{k=0}^\infty$ as $w_k = \xi_k$, for $k = 0, 1, 2, \dots, p_0$ and $w_{p_0+r} = \binom{m}{1}w_{p_0+r-1} - \binom{m}{2}w_{p_0+r-2} + \dots + (-1)^{m+1}w_{p_0+r-m}$, $r=1, 2, 3, \dots$. Then it is easy to verify that $w \in \Delta^{(m)}l_\alpha\{f_n\}$ and

$$d(x, w) = \sum_{k=p_0+1}^\infty \frac{1}{2^k} \frac{|\xi_k - w_k|}{1 + |\xi_k - w_k|} \leq \sum_{k=p_0+1}^\infty \frac{1}{2^k} < \epsilon.$$

Therefore $\Delta^{(m)}l_\alpha\{f_n\}$ is a dense set in s . Combining all the results discussed above, it is concluded that $\Delta^{(m)}l_\alpha\{f_n\}$ is a dense F_σ -set of the first Baire category in s . □

Corollary 5.2. (i) *The sequence space $\Delta^{(m)}l^\alpha\{g_n\}$ is a dense F_σ -set of the first Baire category in s .*

(ii) *If $\{f_n\}_{n=0}^\infty$ satisfies uniform Δ_2 -condition, then by Theorem 4.1(c), $\Delta^{(m)}h_\alpha\{f_n\}$ is a dense F_σ -set of the first Baire category in s .*

(iii) *If $\alpha = \{\alpha_n\}_{n=0}^\infty = \{1, 1, \dots, 1, \dots\}$ and $f_n(x) = x^p, 1 \leq p < \infty$ for each n , then one gets that the sequence space $l_p(\Delta^{(m)})$ [16], which has this property.*

(iv) *The union $\Delta^{(m)}l_\alpha\{f_n\} \cup \Delta^{(m)}l^\alpha\{g_n\}$ is a dense F_σ -set of the first Baire category in s . Also, since the symmetric difference $\Delta^{(m)}l_\alpha\{f_n\} \ominus \Delta^{(m)}l^\alpha\{g_n\} \subset \Delta^{(m)}l_\alpha\{f_n\} \cup \Delta^{(m)}l^\alpha\{g_n\}$, so $\Delta^{(m)}l_\alpha\{f_n\} \ominus \Delta^{(m)}l^\alpha\{g_n\}$ is a set of the first Baire category in s .*

Remark 5.3. Using the Cauchy criterion of convergence of a series of real numbers, for some $\lambda = \lambda_0$ one can write

$$\Delta^{(m)}l_\alpha\{f_n\} = \bigcap_{p=1}^\infty \bigcup_{N \geq 1} \bigcap_{N \leq i < j} \left\{ x \in s : \left| \sum_{k=i}^j f_k \left(\frac{|\Delta^{(m)}\xi_k|}{\lambda_0 \alpha_k} \right) \right| \leq \frac{1}{p} \right\}.$$

Then it is easy to show that the set in second bracket is a closed set in s and hence $\Delta^{(m)}l_\alpha\{f_n\}$ is an $F_{\sigma\delta}$ -set (countable intersection of F_σ -sets) in s .

In the next theorem, intersection of two sequence spaces $\Delta^{(m)}l_\alpha\{f_n\}$ and $\Delta^{(m)}l^\alpha\{g_n\}$ in $\Delta^{(m)}l_\alpha\{f_n\}$ is characterized in respect of Baire category. The following additional assumptions on sequences $\{f_n\}_{n=0}^\infty$ and $\{g_n\}_{n=0}^\infty$ are considered to establish the theorem:

The sequence of non degenerate Orlicz functions $\{f_n\}_{n=0}^\infty$ and $\{g_n\}_{n=0}^\infty$ are said to satisfy condition (T), if the following two conditions hold:

(i) both functions satisfy the uniform Δ_2 -condition and

(ii) $\sum_{k=0}^\infty g_k(\alpha_k t) = \infty$ & $\sum_{k=0}^\infty f_k\left(\frac{t}{\alpha_k}\right) = \infty$ for some $t > 0$, where $\{\alpha_n\}_{n=0}^\infty$ is a sequence of strictly positive real numbers.

Theorem 5.4. Let $\{f_n\}_{n=0}^\infty$ and $\{g_n\}_{n=0}^\infty$ be two sequences of non degenerate Orlicz functions satisfying the condition (T). If $\Delta^{(m)}l_\alpha\{f_n\} \cap \Delta^{(m)}l^\alpha\{g_n\} \neq \Delta^{(m)}l_\alpha\{f_n\}$, then the set $\Delta^{(m)}l_\alpha\{f_n\} \cap \Delta^{(m)}l^\alpha\{g_n\}$ is a dense F_σ -set of the first Baire category in $\Delta^{(m)}l_\alpha\{f_n\}$.

Proof. Since $\{f_n\}_{n=0}^\infty$ satisfies uniform Δ_2 -condition, so for every $\epsilon \in (0, 1)$ there exists $j \in \mathbb{N}$ such that

$$\sum_{k=j+1}^\infty f_k\left(\frac{|\Delta^{(m)}\xi_k|}{\epsilon\alpha_k}\right) \leq 1.$$

Note that the element $u^{[j]} = \sum_{k=0}^j (\Delta^{(m)}x)_k (\Sigma^{(m)}e_k) \in \Delta^{(m)}l_\alpha\{f_n\} \cap \Delta^{(m)}l^\alpha\{g_n\}$

as the sequence $\Delta^{(m)}u^{[j]} \in l_\alpha\{f_n\} \cap l^\alpha\{g_n\}$.

Let $x \in \Delta^{(m)}l_\alpha\{f_n\}$ be an arbitrary sequence. Then by the definition of norm on the space $\Delta^{(m)}l_\alpha\{f_n\}$, one gets

$$\varrho_{\alpha, \Delta^{(m)}}\left(\frac{x-u^{[j]}}{\epsilon}\right) = \sum_{k \geq j+1} f_k\left(\frac{|\Delta^{(m)}\xi_k|}{\epsilon\alpha_k}\right) \leq 1,$$

which implies that $\|x - u^{[j]}\|_{\alpha, \Delta^{(m)}} \leq \epsilon$.

Therefore the set $\Delta^{(m)}l_\alpha\{f_n\} \cap \Delta^{(m)}l^\alpha\{g_n\}$ is dense in $\Delta^{(m)}l_\alpha\{f_n\}$.

An equivalent expression for the set $\Delta^{(m)}l_\alpha\{f_n\} \cap \Delta^{(m)}l^\alpha\{g_n\}$ is given below:

$$\begin{aligned} & \Delta^{(m)}l_\alpha\{f_n\} \cap \Delta^{(m)}l^\alpha\{g_n\} \\ &= \left\{ x \in \Delta^{(m)}l_\alpha\{f_n\} : \sum_{k=0}^\infty g_k\left(\alpha_k|\Delta^{(m)}\xi_k|\right) < \infty \right\} \\ &= \bigcup_{i=1}^\infty \bigcap_{j=0}^\infty \left\{ x \in \Delta^{(m)}l_\alpha\{f_n\} : \sum_{k=0}^j g_k\left(\alpha_k|\Delta^{(m)}\xi_k|\right) \leq i \right\} \\ &= \bigcup_{i=1}^\infty M(i), \end{aligned}$$

where $M(i) = \bigcap_{j=0}^\infty \left\{ x \in \Delta^{(m)}l_\alpha\{f_n\} : \sum_{k=0}^j g_k\left(\alpha_k|\Delta^{(m)}\xi_k|\right) \leq i \right\}$.

Our aim is to establish that $M(i)$ is a nowhere dense set in $\Delta^{(m)}l_\alpha\{f_n\}$. To establish this, first $M(i)$ is a closed set in $\Delta^{(m)}l_\alpha\{f_n\}$ will be proved and later applying ‘a closed set in a metric space is nowhere dense if and only if its complement is everywhere dense’ (see [15, p. 75]) the result will be concluded.

Let $x^{(l)} = \{\xi_k^{(l)}\}_{k=0}^\infty \in M(i)$ be a sequence with $x^{(l)} \rightarrow x = \{\xi_k\}_{k=0}^\infty$ as $l \rightarrow \infty$, where $x \in \overline{M(i)}$. Since $\Delta^{(m)}l_\alpha\{f_n\}$ is a K -space (see Theorem 4.1(a)), so the convergence in this space is equivalent to the coordinate-wise convergence. Using the continuity of g_n for each $n \in \mathbb{N}_0$ and operator $\Delta^{(m)}$, one obtains

$$\sum_{k=0}^j g_k \left(\alpha_k |\Delta^{(m)} \xi_k| \right) = \sum_{k=0}^j g_k \left(\alpha_k |\Delta^{(m)} \lim_{l \rightarrow \infty} \xi_k^{(l)}| \right) = \lim_{l \rightarrow \infty} \sum_{k=0}^j g_k \left(\alpha_k |\Delta^{(m)} \xi_k^{(l)}| \right) \leq i,$$

whence $x \in M(i)$. Therefore $\overline{M(i)} = M(i)$, i.e., $M(i)$ is closed and hence $\Delta^{(m)} l_\alpha \{f_n\} \cap \Delta^{(m)} l_\alpha \{g_n\}$ is an F_σ set.

Now it will be shown that the set $N(i) = \Delta^{(m)} l_\alpha \{f_n\} \setminus \overline{M(i)} = \Delta^{(m)} l_\alpha \{f_n\} \setminus M(i)$ (say and \setminus denotes the difference between two sets) is everywhere dense in $\Delta^{(m)} l_\alpha \{f_n\}$, where $N(i)$ is defined as follows

$$N(i) = \bigcup_{j=0}^{\infty} \left\{ x \in \Delta^{(m)} l_\alpha \{f_n\} : \sum_{k=0}^j g_k \left(\alpha_k |\Delta^{(m)} \xi_k| \right) > i \right\}.$$

By assumption, $\Delta^{(m)} l_\alpha \{f_n\} \cap \Delta^{(m)} l_\alpha \{g_n\} \neq \Delta^{(m)} l_\alpha \{f_n\}$. Hence there exists some $x \in \Delta^{(m)} l_\alpha \{f_n\}$ for which $x \notin \Delta^{(m)} l_\alpha \{g_n\}$. Therefore by definition

$$\sum_{k=0}^{\infty} g_k \left(\alpha_k |\Delta^{(m)} \xi_k| \right) \text{ is diverges,}$$

which in turn gives $N(i) \neq \emptyset$, for each $i \in \mathbb{N}$.

Let $y = \{\eta_k\}_{k=0}^{\infty} \in N(i)$ be a sequence. Then there exists $p_0 (< j) \in \mathbb{N}$, one gets

$$\lim_{j \rightarrow \infty} \sum_{k=0}^j g_k \left(\alpha_k |\Delta^{(m)} \eta_k| \right) = \infty \text{ and } \sum_{k=p_0+1}^{\infty} f_k \left(\frac{2|\Delta^{(m)} \eta_k|}{\epsilon \alpha_k} \right) \leq 1.$$

Further since $x = \{\xi_k\}_{k=0}^{\infty} \in \Delta^{(m)} l_\alpha \{f_n\}$, so for every $\epsilon > 0$ there exists $p_0 \in \mathbb{N}$, one obtains

$$\sum_{k=p_0+1}^{\infty} f_k \left(\frac{2|\Delta^{(m)} \xi_k|}{\epsilon \alpha_k} \right) \leq 1.$$

Now the following sequence $u = \{u_k\}_{k=0}^{\infty}$ is constructed:

$$\begin{aligned} u_k &= \xi_k, & k = 0, 1, 2, \dots, p_0 \\ &= u_{p_0+r}, & k = p_0 + r, r = 1, 2, 3, \dots, \end{aligned}$$

where

$$u_{p_0+r} = \eta_{p_0+r} - \binom{m}{1} (\eta_{p_0+r-1} - u_{p_0+r-1}) + \binom{m}{2} (\eta_{p_0+r-2} - u_{p_0+r-2}) + \dots + (-1)^m (\eta_{p_0+r-m} - u_{p_0+r-m}).$$

Then for $j > p_0$, one gets

$$\begin{aligned} \sum_{k=0}^j g_k \left(\alpha_k |\Delta^{(m)} u_k| \right) &= \sum_{k=0}^{p_0} g_k \left(\alpha_k |\Delta^{(m)} \xi_k| \right) + \sum_{k=p_0+1}^j g_k \left(\alpha_k |\Delta^{(m)} \eta_k| \right) \\ &> i. \end{aligned}$$

That is the sequence $u = \{u_k\}_{k=0}^\infty \in N(i)$ and

$$\begin{aligned} \varrho_{\alpha, \Delta^{(m)}}\left(\frac{x-u}{\epsilon}\right) &= \sum_{k=0}^\infty f_k\left(\frac{|\Delta^{(m)}(\xi_k - u_k)|}{\epsilon\alpha_k}\right) \\ &= \sum_{k=p_0+1}^\infty f_k\left(\frac{|\Delta^{(m)}\xi_k - \Delta^{(m)}\eta_k|}{\epsilon\alpha_k}\right) \\ &\leq \frac{1}{2} \sum_{k=p_0+1}^\infty f_k\left(\frac{2|\Delta^{(m)}\xi_k|}{\epsilon\alpha_k}\right) + \frac{1}{2} \sum_{k=p_0+1}^\infty f_k\left(\frac{2|\Delta^{(m)}\eta_k|}{\epsilon\alpha_k}\right) \\ &\leq 1, \end{aligned}$$

which gives $\|x - u\|_{\alpha, \Delta^{(m)}} \leq \epsilon$. Thus $N(i)$ is everywhere dense in $\Delta^{(m)}l_\alpha\{f_n\}$ for each $i = 1, 2, \dots$ and hence its complement $M(i)$ is nowhere dense in $\Delta^{(m)}l_\alpha\{f_n\}$. Therefore $\Delta^{(m)}l_\alpha\{f_n\} \cap \Delta^{(m)}l^\alpha\{g_n\}$ is a dense F_σ -set of the first Baire category in $\Delta^{(m)}l_\alpha\{f_n\}$. \square

Let $\{f_n\}_{n=0}^\infty$ and $\{g_n\}_{n=0}^\infty$ be two sequences of non degenerate Orlicz functions satisfying the condition (T). Then from Theorem 5.4, the following results hold:

Corollary 5.5. (i) If $\Delta^{(m)}l_\alpha\{f_n\} \cap \Delta^{(m)}l^\alpha\{g_n\} \neq \Delta^{(m)}l^\alpha\{g_n\}$, then the set $\Delta^{(m)}l_\alpha\{f_n\} \cap \Delta^{(m)}l^\alpha\{g_n\}$ is a dense F_σ -set of the first Baire category in $\Delta^{(m)}l^\alpha\{g_n\}$.

(ii) If $m = 0, \alpha = \{\alpha_n\}_{n=0}^\infty = \{1, 1, \dots, 1, \dots\}$, then $l\{f_n\} \cap l\{g_n\}$ is a dense F_σ -set of the first Baire category in $l\{f_n\}$ [8].

(iii) If $m = 0, f_n = f$ and $g_n = g$ for each n , then $l_f^\alpha \cap l_g^\alpha$ is a dense F_σ -set of the first Baire category in l_f^α . These spaces l_f^α and l_g^α were defined by Gupta & Pradhan [17].

(iv) If $m = 0, \alpha \in l_\infty, \{\alpha^{-1}\}$ is unbounded and $f_n = f = g_n$ for each n , then $l_f^\alpha \subset l_f^\alpha$ [17, Theorem 3.1(ii)]. Hence l_f^α is a dense F_σ -set of the first Baire category in l_f^α .

(v) It is immediate from Proposition 4.3(ii) that the given assumption on α , both the spaces $\Delta^{(m)}l_\alpha\{f_n\}$ and $\Delta^{(m)}l^\alpha\{g_n\}$ are dense F_σ -set of the first Baire category in $\Delta^{(m)}l_p$.

(vi) Further if $m = 0, f_n = f$ for each n and $\alpha_n^{\frac{1}{n}} \rightarrow \infty$, then it is easy to show that $l_f^\alpha \subset \delta$, the space of all entire sequences defined as $\delta = \{x = \{\xi_n\}_{n=0}^\infty \in \omega : \lim_{n \rightarrow \infty} |\xi_n|^{\frac{1}{n}} = 0\}$. The inclusion is strict for $\alpha = \{\alpha_n\}_{n=0}^\infty = \{(n+1)^n\}_{n=0}^\infty$ and $x = \{\xi_n\}_{n=0}^\infty = \left\{\frac{1}{(n+1)^n}\right\}_{n=0}^\infty$. Then l_f^α is a dense F_σ -set of the first Baire category in δ .

Remark 5.6. By the Cauchy criterion of a series of real numbers the set $\Delta^{(m)}l_\alpha\{f_n\} \cap \Delta^{(m)}l^\alpha\{g_n\}$ can be written as

$$\Delta^{(m)}l_\alpha\{f_n\} \cap \Delta^{(m)}l^\alpha\{g_n\} = \bigcap_{p=1}^\infty \bigcup_{N \geq 1} \bigcap_{N \leq i < j} \left\{x \in \Delta^{(m)}l_\alpha\{f_n\} : \right.$$

$$\left| \sum_{k=i}^j g_k \left(\alpha_k |\Delta^{(m)} \xi_k| \right) \right| \leq \frac{1}{p} \}.$$

Hence the set $\Delta^{(m)} l_\alpha \{f_n\} \cap \Delta^{(m)} l_\alpha \{g_n\}$ is an $F_{\sigma\delta}$ -set in $\Delta^{(m)} l_\alpha \{f_n\}$.

6 Conclusion

The application of the Baire category results in sequence spaces were studied by Šalát [6], Ewert and Šalát [8] and others. In the present paper, the Baire category results in modular sequence spaces $\Delta^{(m)} l_\alpha \{f_n\}$ & $\Delta^{(m)} l_\alpha \{g_n\}$ defined by using difference sequences and a sequence $\{\alpha_n\}_{n=0}^\infty (> 0)$ of real numbers are studied. For distinct choices of the sequence $\{\alpha_n\}_{n=0}^\infty$ and the difference operator $\Delta^{(m)}$, the results related to dense F_σ -set of the first Baire category in s of the spaces introduced by Gupta and Pradhan [17], Altay [16] are obtained. The work of Ewert and Šalát [8] has also been generalized.

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