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Multivalued Coincidence Point Results in Partially Ordered Metric Spaces

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Abstract : Kannan type mappings hold an important position in metric fixed point theory. In this paper we define generalized multivalued Kannan type mappings and establish some coincidence point theorems for an arbitrary family of multivalued mappings with another singlevalued self mapping in partially ordered metric spaces. The corresponding singlevalued cases are discussed. One illustrative example is also given. The method of proofs here is a blending of order theoretic and analytic methodologies.

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1 Introduction and Preliminaries

The purpose of this paper is to establish some coincidence point results for an arbitrary family of multivalued mappings with another singlevalued self mapping in partially ordered metric spaces. Weakening of contractive inequalities began with the work of Alber et al. [1] where they established a weak version of the Banach contraction mapping principle in Hilbert spaces. Later it was proved by

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Rhoades [2] that the weak contraction introduced in [1] has necessarily a unique fixed point in any complete metric space. Many authors have created several types of weak contraction inequalities following this result. Fixed point results of functions satisfying these types of inequalities have been established in a number of works [3–6].

A contractive condition different from that of Banach's was given by Kannan [7, 8] which, like that of Banach, implies a unique fixed point in a complete metric space, but, unlike the Banach condition, there exist discontinuous functions satisfying the definition of Kannan. Following their appearance in [7,8], many persons created contractive conditions not requiring continuity of the mapping and established fixed point results for them. There is another reason for which the Kannan type mappings are considered to be important. Banach contraction principle does not characterize completeness. In fact there are examples of noncomplete spaces where every contraction has a fixed point [9]. It has been shown in [10, 11] that the necessary existence of fixed points for Kannan type mappings implies that the corresponding metric space is complete. The above are some, but not all, reasons for which the Kannan type mappings are considered important in mathematical analysis. There are several extensions and generalizations of Kannan type mappings in various spaces as, for instances, in the works noted in [12–15].

In the fixed point theory of setvalued maps, two types of distances are generally used. One is the Hausdorff distance. Nadler [16] had proved a multivalued version of the Banach contraction mapping principle by using the Hausdorff metric. There are many other results using this Hausdorff metric, some instances being [13,14,17]. The another distance is the δ - distance. This is not metric like the Hausdorff distance, but shares most of the properties of a metric.

In recent times, fixed point theory has developed rapidly in partially ordered metric spaces; that is, metric spaces endowed with a partial ordering. Some of these works are noted in [18–23]. A speciality of these problems is that they use both analytic and order theoretic methods. It is also one of the main reasons why they are considered interesting.

Khan et al. [24] initiated the use of a control function in metric fixed point theory which they called alternating distance function. Several works on fixed points have utilized this control function, some instances being [3, 4, 25].

We review below some essential concepts for our discussions in this paper. Let $(X,\ d)$ be a metric space. We denote the class of nonempty and bounded subsets of X by $B(X)$. For $A,\ B \in B(X)$, functions $D(A,\ B)$ and $\delta(A,\ B)$ are defined as $D(A,\ B) = \inf\{d(a,\ b) : a \in A,\ b \in B\}$ and $\delta(A,\ B) = \sup\{d(a,\ b) : a \in A,\ b \in B\}$. If $A = \{a\}$, then we write $D(A,\ B) = D(a,\ B)$ and $\delta(A,\ B) = \delta(a,\ B)$. Also in addition, if $B = \{b\}$, then $D(A,\ B) = d(a,\ b)$ and $\delta(A,\ B) = d(a,\ b)$. For all $A,\ B,\ C \in B(X)$, the definition of $\delta(A,\ B)$ yields that $\delta(A,\ B) = \delta(B,\ A)$, $\delta(A,\ B) \le \delta(A,\ C) + \delta(C,\ B)$, $\delta(A,\ B) = 0$ iff $A = B = \{a\}$, $\delta(A,\ A) =$ diam A [26]. There are several works which have utilized δ - distance [26–32].

Lemma 1.1 ([26]). *If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$, where (X,d) is a complete metric space and $\{A_n\} \to A$ and $\{B_n\} \to B$ where $A,B \in B(X)$ then*

$\delta(A_n, B_n) \to \delta(A, B)$ *as* $n \to \infty$.

Lemma 1.2 ([32]). *If* $\{A_n\}$ *is a sequence of bounded sets in a complete metric space* (X, d) *and if* $\lim_{n\to\infty} \delta(A_n, \{y\}) = 0$ *for some* $y \in X$, *then* $\{A_n\} \to \{y\}$.

Definition 1.3 ([29]). A setvalued mapping $T : X \to B(X)$, where $(X,\ d)$ is a metric space, is *continuous at a point* $x \in X$ if $\{x_n\}$ is a sequence in X converging to x, then the sequence $\{Tx_n\}$ in $B(X)$ converges to Tx. T is said to be *continuous* in X if it is continuous at each point $x \in X$.

Definition 1.4 ([33]). Two self maps g and T of a metric space (X, d) are said to be *compatible mappings* if $\lim_{n\to\infty} d(gTx_n, Tgx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$.

Definition 1.5 ([27]). The mappings $g : X \to X$ and $T : X \to B(X)$, where $(X,\ d)$ is a metric space, are δ- *compatible* if $\lim_{n\to\infty} \delta(Tgx_n, gTx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $gTx_n \in B(X)$ and $Tx_n \to \{t\}$, $gx_n \to t$, for some t in X.

Definition 1.6 ([27]). Let $(X,\ d)$ be a metric space and $g : X \to X$ and $T : X \to B(X)$. Then $u \in X$ is called a *coincidence point* of g and T if $\{gu\} = Tu$.

Definition 1.7 ([30]). Let A and B be two nonempty subsets of a partially ordered set $(X,\ \preceq)$. The relation between A and B is denoted and defined as follows:
$A \prec_1 B$, if for every $a \in A$ there exists $b \in B$ such that $a \preceq b$.

Definition 1.8 ([24]). A function $\psi : [0, \infty) \to [0, \infty)$ is called an *alternating distance function* if the following properties are satisfied:
(i) ψ is monotone increasing and continuous,
(ii) $\psi(t) = 0$ if and only if $t = 0$.

For $(x, y),\ (u, v) \in \mathbb{R} \times \mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers, we say $(x, y) \le (u, v)$ if and only if $x \le u$ and $y \le v$.

Definition 1.9. A function $\phi : [0, \infty)^2 \to [0, \infty)$ is said to be *monotone nondecreasing* if for $(x, y),\ (u, v) \in [0, \infty)^2$, $(x, y) \le (u, v) \implies \phi(x, y) \le \phi(u, v)$.

As already mentioned, we introduce here the definition of generalized multvalued Kannan type mapping in the following.

Definition 1.10 ([7,8]). A mapping $T : X \to X$, where (X, d) is a metric space, is called a *Kannan type mapping* if there exists $0 < k < \frac{1}{2}$ such that

$$d(Tx, Ty) \le k\ [d(x, Tx) + d(y, Ty)],\ \text{ for } x, y \in X. \tag{1.1}$$

Definition 1.11. A mapping $T : X \to X$, where $(X,\ d)$ is a metric space, is said to be a *generalized Kannan type mapping* if for all $x,\ y \in X$,

$$\psi(d(Tx,Ty)) \leq \psi\Big(\frac{1}{2}\ [d(x,Tx)+d(y,Ty)]\Big) - \phi(d(x,Tx),\ d(y,Ty)), \tag{1.2}$$

where ψ is an alternating distance function and $\phi : [0,\infty)^2 \to [0,\infty)$ is a continuous function with $\phi(s,\ t) = 0$ if and only if $(s,\ t) = (0,\ 0)$.

If one takes ψ to be the identity function and $\phi(s,\ t) = (\frac{1}{2} - k)(s+t)$, where $0 < k < \frac{1}{2}$, then (1.2) reduces to (1.1). Hence generalized Kannan type mappings are generalizations of Kannan type mappings.

Definition 1.12. A multivalued mapping $T : X \to B(X)$, where $(X,\ d)$ is a metric space, is said to be a *generalized multivalued Kannan type mapping* if for all $x,\ y \in X$,

$$\psi(\delta(Tx,Ty)) \leq \psi\Big(\frac{1}{2}\ [D(x,Tx)+D(y,Ty)]\Big) - \phi(\delta(x,Tx),\ \delta(y,Ty)), \tag{1.3}$$

where ψ is an alternating distance function and $\phi : [0,\infty)^2 \to [0,\infty)$ is a continuous function with $\phi(s,\ t) = 0$ if and only if $(s,\ t) = (0,\ 0)$.

If one treats T as a multivalued mapping in which case Tx is a singleton set for every $x \in X$, then (1.3) reduces to (1.2). Hence generalized Kannan type mappings are special cases of generalized multivalued Kannan type mappings.

In this paper we have proved some coincidence point results for an arbitrary family of multivalued mappings with another singlevalued self mapping using a control function in metric spaces having a partial order. The corresponding singlevalued cases have been discussed. One supporting example is given.

2 Main Results

Theorem 2.1. *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let $\phi : [0,\infty)^2 \to [0,\infty)$ be a monotone nondecreasing and continuous function with $\phi(s,t) = 0$ if and only if $(s,t) = (0,0)$ and ψ is an alternating distance function. Let $\{T_\alpha : X \to B(X) : \alpha \in \Lambda\}$ be a family of multivalued mappings. Let $g : X \to X$ be a mapping such that $g(X)$ is closed in X. Suppose that there exists $\alpha_0 \in \Lambda$ such that (i) T_{α_0} and g are continuous, (ii) $T_{\alpha_0}x \subseteq g(X)$ and $gT_{\alpha_0}x \in B(X)$, for every $x \in X$, (iii) there exists $x_0 \in X$ such that $\{gx_0\} \prec_1 T_{\alpha_0}x_0$, (iv) for $x,\ y \in X$, $gx \preceq gy$ implies $T_{\alpha_0}x \prec_1 T_{\alpha_0}y$, (v) the pair $(g,\ T_{\alpha_0})$ is δ - compatible, (vi) $\psi(\delta(T_{\alpha_0}x,\ T_\alpha y)) \leq \psi(\frac{1}{2}\ [D(gx,T_{\alpha_0}x)+D(gy,T_\alpha y)]) - \phi(\delta(gx,T_{\alpha_0}x),\ \delta(gy,T_\alpha y))$, where $x,\ y \in X$ such that gx and gy are comparable and $\alpha \in \Lambda$. Then g and $\{T_\alpha : \alpha \in \Lambda\}$ have a coincidence point.*

Proof. First we establish that any coincidence point of g and T_{α_0} is a coincidence point of g and $\{T_\alpha : \alpha \in \Lambda\}$ and conversely. Suppose that $p \in X$ be a coincidence point of g and T_{α_0}. Then $\{gp\} = T_{\alpha_0}p$. From (vi) and using the monotone property of ψ, we have

$$\begin{aligned}\psi(\delta(gp,\ T_\alpha p)) &\leq \psi(\delta(T_{\alpha_0}p,\ T_\alpha p))\\ &\leq \psi\Big(\tfrac{1}{2}\ [D(gp, T_{\alpha_0}p) + D(gp, T_\alpha p)]\Big) - \phi(\delta(gp, T_{\alpha_0}p),\ \delta(gp, T_\alpha p))\\ &\leq \psi\Big(\tfrac{1}{2}\ D(gp, T_\alpha p)\Big)\ (\text{ by a property of } \phi\ ).\end{aligned}$$

Again using the monotone property of ψ, we have

$$\delta(gp,\ T_\alpha p) \leq \frac{1}{2}\ D(gp, T_\alpha p) \leq \frac{1}{2}\ \delta(gp, T_\alpha p),$$

which implies that $\delta(gp,\ T_\alpha p) = 0$, that is, $\{gp\} = T_\alpha p$, for all $\alpha \in \Lambda$. Hence p is a coincidence point of g and $\{T_\alpha : \alpha \in \Lambda\}$. Converse part is trivial.

Now it is sufficient to prove that g and T_{α_0} have coincidence point. Let $x_0 \in X$ be such that $\{gx_0\} \prec_1 T_{\alpha_0}x_0$. Then there exists $u \in T_{\alpha_0}x_0$ such that $gx_0 \preceq u$. Since $T_{\alpha_0}x_0 \subseteq g(X)$ and $u \in T_{\alpha_0}x_0$, there exists $x_1 \in X$ such that $gx_1 = u$. So $gx_0 \preceq gx_1$. Then by the assumption (iii), $T_{\alpha_0}x_0 \prec_1 T_{\alpha_0}x_1$. Since $u = gx_1 \in T_{\alpha_0}x_0$, there exists $v \in T_{\alpha_0}x_1$ such that $gx_1 \preceq v$. As $T_{\alpha_0}x_1 \subseteq g(X)$ and $v \in T_{\alpha_0}x_1$, there exists $x_2 \in X$ such that $gx_2 = v$. So $gx_1 \preceq gx_2$. Continuing this process we construct a sequence $\{x_n\}$ in X such that

$$gx_{n+1} \in T_{\alpha_0}x_n,\ \text{for all } n \geq 0 \tag{2.1}$$

and

$$gx_0 \preceq gx_1 \preceq gx_2 \preceq \ldots \preceq gx_n \preceq gx_{n+1} \ldots. \tag{2.2}$$

Since $gx_n \preceq gx_{n+1}$, putting $\alpha = \alpha_0$, $x = x_{n+1}$ and $y = x_n$ in (vi) and using the monotone properties of ψ and ϕ, we have

$$\begin{aligned}\psi(d(gx_{n+2},\ gx_{n+1})) &\leq \psi(\delta(T_{\alpha_0}x_{n+1},\ T_{\alpha_0}x_n))\\ &\leq \psi\Big(\frac{1}{2}\ [D(gx_{n+1}, T_{\alpha_0}x_{n+1}) + D(gx_n, T_{\alpha_0}x_n)]\Big)\\ &\qquad - \phi(\delta(gx_{n+1}, T_{\alpha_0}x_{n+1}),\ \delta(gx_n, T_{\alpha_0}x_n))\\ &\leq \psi\Big(\frac{1}{2}[d(gx_{n+1}, gx_{n+2}) + d(gx_n, gx_{n+1})]\Big)\\ &\qquad - \phi(d(gx_{n+1}, gx_{n+2}), d(gx_n, gx_{n+1})), \end{aligned}\tag{2.3}$$

which, by a property of ϕ, implies that

$$\psi(d(gx_{n+2},\ gx_{n+1})) \leq \psi\Big(\frac{1}{2}[d(gx_{n+1},\ gx_{n+2}) + d(gx_n,\ gx_{n+1})]\Big).$$

Using the monotone property of ψ, we have

$$d(gx_{n+2},\ gx_{n+1}) \leq \frac{1}{2}[d(gx_{n+1},\ gx_{n+2}) + d(gx_n,\ gx_{n+1})],$$

that is,

$$d(gx_{n+2},\ gx_{n+1}) \leq d(gx_{n+1},\ gx_n).$$

Therefore, $\{d(gx_{n+1},\ gx_n)\}$ is a monotone decreasing sequence of non-negative real numbers. Hence there exists an $r \geq 0$ such that

$$\lim_{n\to\infty} d(gx_{n+1}, gx_n) = r. \tag{2.4}$$

Taking limit as $n \to \infty$ in (2.3), using (2.4) and the continuities of ψ and ϕ, we have

$$\psi(r) \leq \psi(r) - \phi(r,\ r),$$

which is a contradiction unless $r = 0$. Hence

$$\lim_{n\to\infty} d(gx_{n+1},\ gx_n) = 0. \tag{2.5}$$

Next we show that $\{gx_n\}$ is a Cauchy sequence. If $\{gx_n\}$ is not a Cauchy sequence, then there exists an $\epsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, $n(k) > m(k) > k$ and $d(gx_{n(k)},\ gx_{m(k)}) \geq \epsilon$. Assuming that $n(k)$ is the smallest such positive integer, we get

$$n(k) > m(k) > k,\ d(gx_{n(k)},\ gx_{m(k)}) \geq \epsilon \text{ and } d(gx_{n(k)-1},\ gx_{m(k)}) < \epsilon.$$

Now, $\epsilon \leq d(gx_{n(k)},\ gx_{m(k)}) \leq d(gx_{n(k)},\ gx_{n(k)-1}) + d(gx_{n(k)-1},\ gx_{m(k)})$, that is,

$$\epsilon \leq d(gx_{n(k)},\ gx_{m(k)}) < d(gx_{n(k)},\ gx_{n(k)-1}) + \epsilon.$$

Taking limit as $k \to \infty$ in the above inequality and using (2.5), we have

$$\lim_{k\to\infty} d(gx_{n(k)},\ gx_{m(k)}) = \epsilon. \tag{2.6}$$

Again,

$$\begin{aligned} d(gx_{n(k)},\ gx_{m(k)}) \quad &\leq \quad d(gx_{n(k)},\ gx_{n(k)+1}) + d(gx_{n(k)+1},\ gx_{m(k)+1}) \\ &\qquad + d(gx_{m(k)+1},\ gx_{m(k)}) \end{aligned}$$

and

$$\begin{aligned} d(gx_{n(k)+1},\ gx_{m(k)+1}) \quad &\leq \quad d(gx_{n(k)+1},\ gx_{n(k)}) + d(gx_{n(k)},\ gx_{m(k)}) \\ &\qquad + d(gx_{m(k)},\ gx_{m(k)+1}). \end{aligned}$$

Taking limit as $k \to \infty$ in above inequalities, using (2.5) and (2.6), we have

$$\lim_{k\to\infty} d(gx_{n(k)+1},\ gx_{m(k)+1}) = \epsilon. \tag{2.7}$$

For each positive integer k, $gx_{m(k)}$ and $gx_{n(k)}$ are comparable. Then putting $\alpha = \alpha_0$, $x = x_{n(k)}$ and $y = x_{m(k)}$ in (vi) and using the monotone properties of ψ

and ϕ, we have

$$\begin{aligned}\psi(d(x_{n(k)+1},x_{m(k)+1})) &\leq \psi(\delta(T_{\alpha_0}x_{n(k)},\ T_{\alpha_0}x_{m(k)}))\\ &\leq \psi\Big(\frac{1}{2}[D(gx_{n(k)},\ T_{\alpha_0}x_{n(k)})+D(gx_{m(k)},\ T_{\alpha_0}x_{m(k)})]\Big)\\ &\qquad -\phi(\delta(gx_{n(k)},\ T_{\alpha_0}x_{n(k)}),\ \delta(gx_{m(k)},\ T_{\alpha_0}x_{m(k)}))\\ &\leq \psi\Big(\frac{1}{2}[d(gx_{n(k)},\ gx_{n(k)+1})+d(gx_{m(k)},\ gx_{m(k)+1})]\Big)\\ &\qquad -\phi(d(gx_{n(k)},\ gx_{n(k)+1}),\ d(gx_{m(k)},\ gx_{m(k)+1})).\end{aligned}$$

Letting $k\to\infty$ in the above inequality, using (2.5), (2.7) and the properties of ϕ and ψ, we have $\psi(\epsilon)\leq 0$, which is a contradiction by virtue of a property of ψ. Hence $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Since X is complete and $g(X)$ is closed in X, there exists $u\in g(X)$ such that $gx_n\to u$ as $n\to\infty$. Since $u\in g(X)$, there exists $z\in X$ such that $u=gz$. Then

$$gx_n\to u=gz \text{ as } n\to\infty. \tag{2.8}$$

Using (2.3) and the properties of ψ and ϕ, we have

$$d(gx_{n+2},gx_{n+1})\leq\delta(T_{\alpha_0}x_{n+1},T_{\alpha_0}x_n)\leq\frac{1}{2}[d(gx_{n+1},gx_{n+2})+d(gx_n,gx_{n+1})].$$

Taking $n\to\infty$ in the above inequality and using (2.8), we have

$$\lim_{n\to\infty}\delta(T_{\alpha_0}x_{n+1},\ T_{\alpha_0}x_n)=0. \tag{2.9}$$

Now,

$$\begin{aligned}\delta(T_{\alpha_0}x_n,\ \{u\}) &\leq \delta(T_{\alpha_0}x_n,\ gx_n)+\delta(gx_n,\ \{u\})\\ &\leq \delta(T_{\alpha_0}x_n,\ T_{\alpha_0}x_{n-1})+d(gx_n,\ u).\end{aligned}$$

Letting $n\to\infty$ in the above inequality using (2.8) and (2.9), we have

$$\lim_{n\to\infty}\delta(T_{\alpha_0}x_n,\ \{u\})=0,$$

which, by Lemma 1.2, implies that

$$T_{\alpha_0}x_n\to\{u\},\ \text{as } n\to\infty. \tag{2.10}$$

Since the pair (g,T_{α_0}) is δ - compatible, from (2.8) and (2.10), we have

$$\lim_{n\to\infty}\delta(T_{\alpha_0}gx_n,\ gT_{\alpha_0}x_n)=0.$$

As g and T_{α_0} are continuous, it follows that $\delta(T_{\alpha_0}u,\ gu)=0$, that is, $T_{\alpha_0}u=\{gu\}$. Hence $u\in g(X)\subseteq X$ is a coincidence point of g and T_{α_0}. By what we have already proved, u is a coincidence point of g and $\{T_\alpha:\alpha\in\Lambda\}$. $\square$

In our next theorem, we relax the continuity assumption on T_{α_0} and g by imposing an order condition. We also relax the condition that $gT_{\alpha_0}x \in B(X)$, for every $x \in X$.

Theorem 2.2. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Assume that if $x_n \to x$ is a nondecreasing sequence in X, then $x_n \preceq x$, for all n. Let $\phi : [0, \infty)^2 \to [0, \infty)$ be a monotone nondecreasing and continuous function with $\phi(s, t) = 0$ if and only if $(s, t) = (0, 0)$ and ψ is an alternating distance function. Let $\{T_\alpha : X \to B(X) : \alpha \in \Lambda\}$ be a family of multivalued mappings. Let $g : X \to X$ be a mapping such that $g(X)$ is closed in X. Suppose that there exists $\alpha_0 \in \Lambda$ such that (i) $T_{\alpha_0}x \subseteq g(X)$ for every $x \in X$, (ii) there exists $x_0 \in X$ such that $\{gx_0\} \prec_1 T_{\alpha_0}x_0$, (iii) for $x,\ y \in X$, $gx \preceq gy$ implies $T_{\alpha_0}x \prec_1 T_{\alpha_0}y$, (iv) $\psi(\delta(T_{\alpha_0}x,\ T_\alpha y)) \leq \psi(\frac{1}{2}\ [D(gx, T_{\alpha_0}x) + D(gy, T_\alpha y)]) - \phi(\delta(gx, T_{\alpha_0}x),\ \delta(gy, T_\alpha y))$, where $x,\ y \in X$ such that gx and gy are comparable and $\alpha \in \Lambda$. Then g and $\{T_\alpha : \alpha \in \Lambda\}$ have a coincidence point.*

Proof. We take the same sequence $\{gx_n\}$ as in the proof of Theorem 2.1. Then we have $gx_{n+1} \in T_{\alpha_0}x_n$, for all $n \geq 0$, $\{gx_n\}$ is monotonic nondecreasing and $gx_n \to gz$ as $n \to \infty$. By the order condition of the metric space, we have $gx_n \preceq gz$, for all n. Using by the monotone properties of ψ and ϕ and the condition (iv), we have

$$\begin{aligned}
\psi(\delta(gx_{n+1},\ T_\alpha z)) &\leq \psi(\delta(T_{\alpha_0}x_n,\ T_\alpha z)) \\
&\leq \psi\Big(\frac{1}{2}\ [D(gx_n,\ T_{\alpha_0}x_n) + D(gz,\ T_\alpha z)]\Big) \\
&\qquad - \phi(\delta(gx_n,\ T_{\alpha_0}x_n),\ \delta(gz,\ T_\alpha z)) \\
&\leq \psi\Big(\frac{1}{2}\ [d(gx_n,\ gx_{n+1}) + D(gz,\ T_\alpha z)]\Big) \\
&\qquad - \phi(d(gx_n,\ gx_{n+1}),\ \delta(gz,\ T_\alpha z)).
\end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality and using the continuities of ϕ and ψ, we have

$$\psi(\delta(gz,\ T_\alpha z)) \leq \psi\Big(\frac{1}{2}\ D(gz,\ T_\alpha z)\Big) - \phi(0, \delta(gz,\ T_\alpha z)),$$

which implies that

$$\psi(\delta(gz,\ T_\alpha z)) \leq \psi\Big(\frac{1}{2}\ D(gz,\ T_\alpha z)\Big) \text{ (by a property of } \phi).$$

Using the monotone property of ψ, we have

$$\delta(gz,\ T_\alpha z) \leq \frac{1}{2}\ D(gz,\ T_\alpha z) \leq \frac{1}{2}\ \delta(gz,\ T_\alpha z),$$

which implies that $\delta(gz,\ T_\alpha z) = 0$, that is, $\{gz\} = T_\alpha z$, for all $\alpha \in \Lambda$. Hence z is a coincidence point of g and $\{T_\alpha : \alpha \in \Lambda\}$. □

Considering $\{T_\alpha : X \to B(X) : \alpha \in \Lambda\} = \{T\}$ in Theorem 2.1, we have the following corollary.

Corollary 2.3. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $\phi : [0,\infty)^2 \to [0,\infty)$ be a monotone nondecreasing and continuous function with $\phi(s,t) = 0$ if and only if $(s,t) = (0,0)$ and ψ is an alternating distance function. Let $T : X \to B(X)$ be a multivalued mapping and $g : X \to X$ be a mapping such that (i) T and g are continuous, (ii) $Tx \subseteq g(X)$ and $gTx \in B(X)$, for every $x \in X$, and $g(X)$ is closed in X, (iii) there exists $x_0 \in X$ such that $\{gx_0\} \prec_1 Tx_0$, (iv) for $x,\ y \in X$, $gx \preceq gy$ implies $Tx \prec_1 Ty$, (v) the pair $(g,\ T)$ is δ - compatible, (vi) $\psi(\delta(Tx,\ Ty)) \leq \psi(\frac{1}{2}\ [D(gx,Tx) + D(gy,Ty)]) - \phi(\delta(gx,Tx),\ \delta(gy,Ty))$, where $x,\ y \in X$ such that gx and gy are comparable. Then g and T have a coincidence point.*

Considering $\{T_\alpha : X \to B(X) : \alpha \in \Lambda\} = \{T\}$ in Theorem 2.2, we have the following corollary.

Corollary 2.4. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Assume that if $x_n \to x$ is a nondecreasing sequence in X, then $x_n \preceq x$, for all n. Let $\phi : [0,\infty)^2 \to [0,\infty)$ be a monotone nondecreasing and continuous function with $\phi(s,t) = 0$ if and only if $(s,t) = (0,0)$ and ψ is an alternating distance function. Let $T : X \longrightarrow B(X)$ be a multivalued mapping and $g : X \to X$ be a mapping such that (i) $Tx \subseteq g(X)$ for every $x \in X$, and $g(X)$ is closed in X, (ii) there exists $x_0 \in X$ such that $\{gx_0\} \prec_1 Tx_0$, (iii) for $x,\ y \in X$, $gx \preceq gy$ implies $Tx \prec_1 Ty$, (iv) $\psi(\delta(Tx,\ Ty)) \leq \psi(\frac{1}{2}\ [D(gx,Tx) + D(gy,Ty)]) - \phi(\delta(gx,Tx),\ \delta(gy,Ty))$, where $x,\ y \in X$ such that gx and gy are comparable. Then g and T have a coincidence point.*

The following theorems are singlevalued cases of the Theorems 2.1 and 2.2 respectively. Here we treat T as a multivalued mapping in which case Tx is a singleton set for every $x \in X$. For the following theorems function ϕ need not to be monotone nondecreasing.

Theorem 2.5. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $\phi : [0,\infty)^2 \to [0,\infty)$ be a continuous function with $\phi(s,t) = 0$ if and only if $(s,t) = (0,0)$ and ψ is an alternating distance function. Let $\{T_\alpha : X \to X : \alpha \in \Lambda\}$ be a family of mappings. Let $g : X \to X$ be a mapping such that $g(X)$ is closed in X. Suppose that there exists $\alpha_0 \in \Lambda$ such that (i) T_{α_0} and g are continuous, (ii) $T_{\alpha_0}(X) \subseteq g(X)$, (iii) there exists $x_0 \in X$ such that $gx_0 \preceq T_{\alpha_0}x_0$, (iv) for $x,\ y \in X$, $gx \preceq gy$ implies $T_{\alpha_0}x \preceq T_{\alpha_0}y$, (v) the pair $(g,\ T_{\alpha_0})$ is compatible, (vi) $\psi(d(T_{\alpha_0}x,\ T_\alpha y)) \leq \psi(\frac{1}{2}\ [d(gx, T_{\alpha_0}x) + d(gy, T_\alpha y)]) - \phi(d(gx, T_{\alpha_0}x),\ d(gy, T_\alpha y))$, where $x,\ y \in X$ such that gx and gy are comparable and $\alpha \in \Lambda$. Then g and $\{T_\alpha : \alpha \in \Lambda\}$ have a coincidence point.*

Theorem 2.6. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Assume that if $x_n \to x$ is a nondecreasing sequence in X, then $x_n \preceq x$, for all n. Let $\phi : [0, \infty)^2 \to [0, \infty)$ be a continuous function with $\phi(s, t) = 0$ if and only if $(s, t) = (0, 0)$ and ψ is an alternating distance function. Let $\{T_\alpha : X \to X : \alpha \in \Lambda\}$ be a family of mappings. Let $g : X \to X$ be a mapping such that $g(X)$ is closed in X. Suppose that there exists $\alpha_0 \in \Lambda$ such that (i) $T_{\alpha_0}(X) \subseteq g(X)$, (ii) there exists $x_0 \in X$ such that $gx_0 \preceq T_{\alpha_0}x_0$, (iii) for $x, y \in X$, $gx \preceq gy$ implies $T_{\alpha_0}x \preceq T_{\alpha_0}y$, (iv) $\psi(d(T_{\alpha_0}x, T_\alpha y)) \leq \psi(\frac{1}{2}[d(gx, T_{\alpha_0}x) + d(gy, T_\alpha y)]) - \phi(d(gx, T_{\alpha_0}x), d(gy, T_\alpha y))$, where $x, y \in X$ such that gx and gy are comparable and $\alpha \in \Lambda$. Then g and $\{T_\alpha : \alpha \in \Lambda\}$ have a coincidence point.*

Example 2.7. Let $X = [0, \infty)$ be equipped with usual order '$\leq$' and usual metric 'd'. Let $g : X \to X$ be defined as $gx = 8x$, for $x \in X$. Let $\Lambda = \{1, 2, 3, ...\}$. Let the family of mappings $\{T_\alpha : X \to B(X) : \alpha \in \Lambda\}$ be defined as $T_1x = \{0\}$, for $x \in X$, and for $\alpha \geq 2$, $T_\alpha x = \begin{cases} \{0\}, & \text{if } 0 \leq x \leq 1, \\ \{0, \frac{\alpha}{\alpha+1}\}, & \text{if } x > 1. \end{cases}$ Let $\psi : [0, \infty) \to [0, \infty)$ and $\phi : [0, \infty)^2 \to [0, \infty)$ be respectively defined as $\psi(t) = t^2$, for $t \in [0, \infty)$ and $\phi(x, y) = \frac{z}{100}$, for $(x, y) \in [0, \infty)^2$ with $z = \max\{x, y\}$. Here all the conditions of Theorems 2.1 and 2.2 are satisfied and 0 is a coincidence point of g and $\{T_\alpha : \alpha \in \Lambda\}$.

Note. In the above example if one takes $g : X \to X$ as $gx = \begin{cases} \frac{x}{2}, & \text{if } 0 \leq x \leq 1, \\ 200, & \text{if } x > 1. \end{cases}$ Then the above example is still applicable to Theorem 2.2 but not applicable to Theorem 2.1 because g is not continuous and hence does not satisfy required conditions mentioned in Theorem 2.1.

Remark 2.8. In the above example $\{T_\alpha : \alpha \in \Lambda\}$ contains infinitely many functions and so Corollaries 2.3 and 2.4 can not be applied to it. This shows that Theorems 2.1 and 2.2 properly contain their Corollaries 2.3 and 2.4 respectively. Also, in the above example $\{T_\alpha : \alpha \in \Lambda\}$ is a family of multivalued mappings and hence Theorems 2.5 and 2.6 are not applicable to it.

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