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On the Kernel of the Black-Scholes Equation for the Option Price on Future Related to the Black-Scholes Formula

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Abstract : In this paper, we studied the Kernel of of the Black-Scholes Equation for the option price on the future and obtained the new results of the interesting properties. Moreover such Kernel can be related to cover the Black-Scholes Formula. However, we hope that such results of this paper may be useful in the research area of Financial Mathematics.

Keywords : kernel; Black-Scholes equation; option price.
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1 Introduction

It is concepted that the Black-Scholes Formula is useful for computing the option price of the stock price, particularly the option price on future. Actually the Black-Scholes Formula is the solution of the Black-Scholes Equation, see [1,

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pp. 637-659]. But unfortunately such Black-Scholes Formula is too complicated to be derived directly from the Black-Scholes Equation. So many books only use the variation method to show that such Black-Scholes Formula is the solution of the Black-Scholes Equation. So in our work we try to find the other solutions of the Black-scholes Equation and luckily we found such solution in the form of kernel and can be related to cover the Black-Scholes formula for the option price on future which is given by

$$\frac{\partial}{\partial t}C(F,t)+\frac{1}{2}\sigma^2F^2\frac{\partial^2}{\partial F^2}C(F,t)-rC(F,t)=0, \tag{1.1}$$

see [2, pp. 118-119] with the payoff

$$C(F_T,T)=\max(F_T-p,0)\equiv(F_T-p)^+ \tag{1.2}$$

where $F=se^{r(T-t)}$ is the stock price on future, $C(F,t)$ is the option price on future, σ is the volatility of the stock, T is the expiration time, F_T is the stock price at time T, r is the interest rate and p is the strike price.

In fact $F_T=s_Te^{r(T-T)}=S_T$ then (1.2) becomes $C(s_T,T)=(s_T-p)^+$ where s_T is the price of stock at the expiration time T.

Now the solution of (1.1) is the Black-Scholes Formula which is given by

$$C(F,t)=e^{-r(T-t)}(FN(d_1)-pN(d_2) \tag{1.3}$$

where

$$d_1=\frac{\ln\left(\frac{F}{p}\right)+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \tag{1.4}$$

and

$$d_2=\frac{\ln\left(\frac{F}{p}\right)-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \tag{1.5}$$

and denote $N(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{y^2}{2}}dy$.

Now let $R=\ln F$ and $\tau=T-t$ and write $C(F,t)=v(R,\tau)$ and substitute into (1.1). Then (1.1) is transformed to the equation

$$\frac{\partial}{\partial\tau}v(R,\tau)+\frac{1}{2}\sigma^2\frac{\partial}{\partial R}v(R,\tau)-\frac{1}{2}\sigma^2\frac{\partial^2}{\partial R^2}v(R,\tau)+rv(R,\tau)=0 \tag{1.6}$$

with the Call payoff or the initial condition

$$v(R,0)=C(F_T,T)=(F_T-p)^+=(e^R-p)^+$$

where $\tau = 0$ corresponds to $t = T$ and let

$$v(R,0) = (e^R - p)^+ = f(R) \tag{1.7}$$

where f is the continuous function of R. In this paper, we study the solution of (1.6) and then relate to (1.1). By taking the Fourier transform to (1.6) and then we obtain

$$v(R,\tau) = \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}} \int_{-\infty}^{\infty} \exp\left[-\frac{(R - \frac{\sigma^2}{2}\tau - y)^2}{2\sigma^2\tau} \right] f(y)dy \tag{1.8}$$

as the solution of (1.6).
Or in the convolution form

$$V(R,\tau) = K(R,\tau) * f(R) \tag{1.9}$$

where

$$K(R,\tau) = \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}} \exp\left[-\frac{(R - \frac{\sigma^2}{2})^2}{\sigma^2\tau} \right] \tag{1.10}$$

is the kernel of (1.6).

Moreover we can relate (1.8) to cover the Black-Scholes Formula in (1.3), (1.4) and (1.5).

2 Preliminaries

The following definitions and some lemmas are needed.

Definition 2.1. Let f be locally integrable function, then the Fourier transform of f is defined by

$$\mathscr{F} f(x) = \widehat{f(\omega)} = \int_{-\infty}^{\infty} e^{-i\omega x} f(x)dx \tag{2.1}$$

and the inverse Fourier transform is also defined by

$$f(x) = \mathscr{F}^{-1}\widehat{f(\omega)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} \widehat{f(\omega)} d\omega. \tag{2.2}$$

Definition 2.2. (The Dirac-delta distribution)
The Dirac-delta distribution or the impluse function is denoted by δ and also defined by

$$< \delta(x), \varphi(x) > \equiv \int_{-\infty}^{\infty} \delta(x)\varphi(x)dx = \varphi(0) \tag{2.3}$$

where $\varphi(x)$ is the testing function of infinitely differentiable with compact support.

Lemma 2.3. *Recall the equation in* (1.6)

$$\frac{\partial}{\partial\tau}v(R,\tau)+\frac{1}{2}\sigma^2\frac{\partial}{\partial R}v(R,\tau)-\frac{1}{2}\sigma^2\frac{\partial^2}{\partial R^2}v(R,\tau)+rv(R,\tau)=0 \tag{2.4}$$

with the call payoff in (1.7)

$$v(R,0)=f(R). \tag{2.5}$$

Then (2.4) *has*

$$v(R,\tau)=K(R,\tau)*f(R) \tag{2.6}$$

as the solution in the convolution form where

$$K(R,\tau)=\frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}}\exp\left[-\frac{(R-\frac{\sigma^2}{2}\tau)^2}{2\sigma^2}\right] \tag{2.7}$$

is the kernel.

Proof. Take the Fourier transform defined by (2.1) with respect to R to (2.4). Then

$$\frac{\partial}{\partial\tau}\widehat{v(\omega,\tau)}+\frac{1}{2}\sigma^2 i\omega\widehat{v(\omega,\tau)}+\frac{1}{2}\sigma^2\omega^2\widehat{v(\omega,\tau)}+r\widehat{v(\omega,\tau)}=0$$

whose solution is

$$\widehat{v(\omega,\tau)}=C(\omega)\exp\left[(-\frac{1}{2}\sigma^2\omega^2-\frac{1}{2}i\omega-r)\tau\right]$$

since from (2.5) $\widehat{v(\omega,0)}=\widehat{f(\omega)}$, hence $\widehat{v(\omega,0)}=\widehat{f(\omega)}=C(\omega)$. Thus we have

$$\widehat{v(\omega,\tau)}=\widehat{f(\omega)}\exp\left[(-\frac{1}{2}\sigma^2\omega^2-\frac{1}{2}\sigma^2 i\omega-r)\tau\right].$$

Now from (2.2),

$$V(R,\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega R}\widehat{v(\omega,\tau)}d\omega.$$

Thus

$$\begin{aligned}v(R,\tau)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega R}\widehat{f(\omega)}\exp\left[(-\frac{1}{2}\sigma^2\omega^2-\frac{1}{2}i\omega-r)\tau\right]d\omega\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i\omega R}e^{-i\omega y}\exp\left[(-\frac{1}{2}\sigma^2\omega^2-\frac{1}{2}\sigma^2 i\omega-r)\tau\right]f(y)dyd\omega\end{aligned}$$

where $\widehat{f(\omega)} = \int_{-\infty}^{\infty} e^{-i\omega y} f(y)dy$. Thus

$$v(R,\tau) = \frac{e^{-r\tau}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\sigma^2\omega^2\tau - (\frac{1}{2}\sigma^2\tau - R + y)i\omega\right] f(y)dyd\omega$$

$$= \frac{e^{-r\tau}}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\sigma^2\tau(\omega + i\frac{\frac{\sigma^2\tau}{2} - R + y}{\sigma^2\tau})^2\right]d\omega\right)\times$$

$$\exp\left[-\frac{(\frac{\sigma^2\tau}{2} - R + y)^2}{2\sigma^2\tau}\right] f(y)dy.$$

Put $u = \sqrt{\frac{1}{2}\sigma^2\tau}\left(\omega + i\frac{\frac{\sigma^2\tau}{2} - R + y}{\sigma^2\tau}\right)$ then $d\omega = \frac{\sqrt{2}}{\sqrt{\sigma^2\tau}}du$. Thus

$$v(R,\tau) = \frac{e^{-r\tau}}{2\tau}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-u^2}\frac{1}{\sigma}\sqrt{\frac{2}{\tau}}du\right)\times$$

$$\exp\left[-\frac{(\frac{\sigma^2\tau}{2} - R + y)^2}{2\sigma^2\tau}\right] f(y)dy$$

$$= \frac{e^{-r\tau}}{2\pi}\frac{1}{\sigma}\sqrt{\frac{2}{\tau}}\sqrt{\pi}\int_{-\infty}^{\infty}\exp\left[-\frac{(\frac{\sigma^2\tau}{2} - R + y)^2}{2\sigma^2\tau}\right] f(y)dy$$

(Note that $\int_{-\infty}^{\infty} e^{-u^2}du = \sqrt{\pi}$). So we have

$$v(R,\tau) = \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}}\int_{-\infty}^{\infty}\exp\left[-\frac{(R - \frac{\sigma^2\tau}{2} - y)^2}{2\sigma^2\tau}\right] f(y)dy \tag{2.8}$$

or $v(R,\tau) = K(R,\tau) * f(R)$ where

$$K(R,\tau) = \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}}\exp\left[-\frac{(R - \frac{\sigma^2\tau}{2} - y)^2}{2\sigma\tau}\right]. \tag{2.9}$$

Thus we obtain (2.6) and (2.7) as required. Moreover we can show that $\lim_{\tau\to 0} K(R,\tau)$

$= \delta(R)$, see [3, pp. 36-37]. It follows that

$$v(R,0) = \delta(R) * f(R) = \int_{-\infty}^{\infty} \delta(y) f(R-y)dy = f(R-0) = f(R).$$

By the definition of convolution and (2.3) of Definition 2.2. Thus (2.5) holds. □

Lemma 2.4. *The solution* $v(R,\tau)$ *in* (2.8) *can be computed as*

$$v(R,\tau) = e^{-r\tau}(e^R - p) \tag{2.10}$$

and can be also related to cover the Black-Scholes formula in (1.3), (1.4) *and* (1.5).

Proof. Since we have in (2.8) that

$$\begin{aligned} v(R,\tau) &= \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}} \int_{-\infty}^{\infty} \exp\left[-\frac{(R - \frac{\sigma^2\tau}{2} - y)^2}{2\sigma^2\tau}\right] f(y)dy \\ &= \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}} \left(\int_{-\infty}^{\infty} \exp\left[-\frac{(R - \frac{\sigma^2\tau}{2} - y)^2}{2\sigma^2\tau}\right] e^y dy \right. \\ &\quad \left. -p \int_{-\infty}^{\infty} \exp\left[-\frac{(R - \frac{\sigma^2\tau}{2} - y)^2}{2\sigma^2\tau}\right] dy \right) \end{aligned} \tag{2.11}$$

where $f(y) = e^y - p$ from (1.7). Now consider the second integral. Put $u = \frac{1}{\sqrt{2\sigma^2\tau}}(R - \frac{\sigma^2\tau}{2} - y)$ then $dy = d(-y) = \sqrt{2\sigma^2\tau}du$ and thus

$$\int_{-\infty}^{\infty} \exp\left[-\frac{(R - \frac{\sigma^2\tau - y}{2})^2}{2\sigma^2\tau}\right] dy = \sqrt{2\sigma^2\tau} \int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{2\sigma^2\tau}\sqrt{\pi} = \sqrt{2\pi\sigma^2\tau}.$$

For the first integral

$$\int_{-\infty}^{\infty} \exp\left[-\frac{(R - \frac{\sigma^2\tau}{2} - y)^2}{2\sigma^2\tau}\right] e^y dy = \int_{-\infty}^{\infty} \exp\left[-\frac{(y - R + \frac{\sigma^2\tau}{2})^2}{2\sigma^2\tau}\right] e^y dy.$$

Put $w = \frac{1}{\sqrt{2\sigma^2\tau}}(y - R + \frac{\sigma^2\tau}{2})$ then $dy = \sqrt{2\sigma^2\tau}dw$ and $y = \sqrt{2\sigma^2\tau}w + R - \frac{\sigma^2\tau}{2}$.

Thus

$$\int_{-\infty}^{\infty}\exp\left[-\frac{(R-\frac{\sigma^2\tau}{2}-y)^2}{2\sigma^2\tau}\right]e^y dy = \int_{-\infty}^{\infty}e^{-w^2}\sqrt{2\sigma^2\tau}\exp\left[\sqrt{2\sigma^2\tau}w + R - \frac{\sigma^2\tau}{2}\right]dw$$

$$= \sqrt{2\sigma^2\tau}e^{R-\frac{\sigma^2\tau}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-w^2+\sqrt{2\sigma^2\tau}w}dw$$

$$= \sqrt{2\sigma^2\tau}e^{(R-\frac{\sigma^2\tau}{2})}e^{\frac{\sigma^2\tau}{2}}\int_{-\infty}^{\infty}e^{-(w-\sqrt{\frac{\sigma^2\tau}{2}})^2}d(w-\sqrt{\frac{\sigma^2\tau}{2}})$$

$$= \sqrt{2\sigma^2\tau}\sqrt{\pi}e^R.$$

So we have

$$v(R,\tau) = \frac{e^{-R\tau}e^R\sqrt{2\pi\sigma^2\tau}}{\sqrt{2\pi\sigma^2\tau}} - \frac{e^{-r\tau}\sqrt{2\pi\sigma^2\tau}p}{\sqrt{2\pi\sigma^2\tau}}.$$

It follows that $v(R,\tau) = e^{-R\tau}(e^R - p)$ thus we obtain (2.10). To show that $v(R,\tau)$ in (2.8) can be related to cover the Black-Scholes Formula in (1.3), (1.4) and (1.5). Now from (2.11)

$$v(R,\tau) = \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}}\left(\int_{-\infty}^{\infty}Ady - p\int_{-\infty}^{\infty}Bdy\right)$$

where $A = \exp\left[-\frac{(R-\frac{\sigma^2\tau}{2}-y)^2}{2\sigma^2\tau}\right]e^y$ and $B = \exp\left[-\frac{(R-\frac{\sigma^2\tau}{2}-y)^2}{2\sigma^2\tau}\right]$. Now we consider the integral $\int_{-\infty}^{\infty}Bdy$. Put $\frac{u}{\sqrt{2}} = \frac{1}{\sqrt{2\sigma^2\tau}}(R-\frac{\sigma^2\tau}{2}-y)$ and choose $y \geq \ln p$. Then $-\infty < u \leq \frac{1}{\sqrt{\sigma^2\tau}}(R-\frac{\sigma^2\tau}{2}-\ln p)$ and $d(y) = d(-y) = \sqrt{\sigma^2\tau}du$. Thus $\int_{-\infty}^{\infty}Bdy = \sqrt{\sigma^2\tau}\int_{-\infty}^{a}e^{-\frac{u^2}{2}}du$ where $a = \frac{R-\frac{\sigma^2\tau}{2}-\ln p}{\sigma\sqrt{\tau}}$.
Next consider the integral $\int_{-\infty}^{\infty}Ady$. Put $w = \frac{1}{\sqrt{2\sigma^2\tau}}(R-\frac{\sigma^2\tau}{2}-y)$ and choose $y \geq \ln p$ then $d(y) = d(-y) = \sqrt{2\sigma^2\tau}dw$ and $-\infty < w \leq \frac{1}{\sqrt{2\sigma^2\tau}}(R-\frac{\sigma^2\tau}{2}-\ln p)$.

Thus

$$\int_{-\infty}^{\infty} Ady = \int_{-\infty}^{b} e^{-w^2}\sqrt{2\sigma^2\tau}\exp\left[R - \frac{\sigma^2\tau}{2} - \sqrt{2\sigma^2\tau}w\right]dw$$
$$= \sqrt{2\sigma^2\tau}e^{R-\frac{\sigma^2\tau}{2}}\int_{-\infty}^{\infty} e^{-w^2-\sqrt{2\sigma^2\tau}w}dw$$
$$= \sqrt{2\sigma^2\tau}e^{R-\dfrac{\sigma^2\tau}{2}}e^{\frac{\sigma^2\tau}{2}}\int_{-\infty}^{b} e^{-(w+\sqrt{\frac{\sigma^2\tau}{2}})^2}dw$$

where $b = \dfrac{R - \dfrac{\sigma^2\tau}{2} - \ln p}{\sqrt{2\sigma^2\tau}}$. Now put $\dfrac{\alpha}{\sqrt{2}} = w + \sqrt{\dfrac{\sigma^2\tau}{2}}$ then $dw = \dfrac{1}{\sqrt{2}}d\alpha$ and $\alpha = \sqrt{2}w + \sqrt{\sigma^2\tau}$. Since $-\infty < w \leq \dfrac{1}{\sqrt{2\sigma^2\tau}}(R - \dfrac{\sigma^2\tau}{2} - \ln p)$ hence

$$-\infty < \alpha \leq \frac{1}{\sqrt{\sigma^2\tau}}(R - \frac{\sigma^2\tau}{2} - \ln p) + \sqrt{\sigma^2\tau}$$
$$-\infty < \alpha \leq \frac{(R + \dfrac{\sigma^2\tau}{2} - \ln p)}{\sqrt{\sigma^2\tau}}.$$

Thus

$$\int_{-\infty}^{\infty} Ady = \sqrt{2\sigma^2\tau}e^{R-\frac{\sigma^2\tau}{2}}e^{\frac{\sigma^2\tau}{2}}\frac{1}{\sqrt{2}}\int_{-\infty}^{c} e^{-\frac{\alpha^2}{2}}d\alpha = \sqrt{\sigma^2\tau}e^{R}\int_{-\infty}^{c} e^{-\frac{\alpha^2}{2}}d\alpha$$

where $c = \dfrac{R + \dfrac{\sigma^2\tau}{2} - \ln p}{\sigma\sqrt{\tau}}$. So we have

$$v(R,\tau) = \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}}\sqrt{\sigma^2\tau}e^R\int_{-\infty}^{C} e^{-\frac{\alpha^2}{2}}d\alpha - \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}}\sqrt{\sigma^2\tau}p\int_{-\infty}^{a} e^{-\frac{u^2}{2}}du$$
$$= e^{-r\tau}e^R\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{R+\frac{\sigma^2\tau}{2}-\ln p}{\sigma\sqrt{\tau}}} e^{-\frac{\alpha^2}{2}}\alpha\right) - e^{r\tau}p\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{R+\frac{\sigma^2\tau}{2}-\ln p}{\sigma\sqrt{\tau}}} e^{-\frac{u^2}{2}}du\right).$$

Since we write $C(F,t) = v(R,\tau)$ where $C(F,t)$ is the solution of (1.1) with

$R = \ln F$ and $\tau = T - t$. Thus

$$\begin{aligned} C(F,t) &= v(\ln F, T-t) \\ &= e^{-r(T-t)} F \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln(\frac{F}{p}) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{\alpha^2}{2}} d\alpha \right) \\ &\quad - e^{-r(T-t)} p \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln(\frac{F}{p}) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{\alpha^2}{2}} d\alpha \right). \end{aligned}$$

It follows that $C(F,t) = e^{-r(T-t)}(FN(d_1) - pN(d_2))$ which is the Black-Scholes Formula in (1.3) and d_1 is defined by (1.4) and d_2 is defined by (1.5). □

3 Main Results

Theorem 3.1. *Recall the equation* (1.1) *that*

$$\frac{\partial}{\partial t} C(F,t) + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2}{\partial F^2} C(F,t) - rC(F,t) = 0 \tag{3.1}$$

with the call payoff

$$C(F_T, T) = (F_T - p)^+ \tag{3.2}$$

then (3.1) *has* $C(F,t)$ *as the solution in the forms*

(i) $C(F,t) = \dfrac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}} \displaystyle\int_{-\infty}^{\infty} \exp\left[-\frac{(\ln F - \frac{\sigma^2}{2}(T-t) - y)^2}{2\sigma^2(T-t)} \right] f(y) dy$

or the convolution form $C(F,t) = K(\ln F, T-t) * f(\ln F)$ *where*

$$K(logF, T-t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2\tau}} \exp\left[-\frac{(\ln F - \frac{\sigma^2}{2}(T-t) - y)^2}{2\sigma^2(T-t)} \right]$$

is the kernel.

(ii) $C(F,t) = e^{-r(T-t)}(F - p)$.

(iii) $C(F,t)$ *can be related to cover the Black-Scholes Formula given by* (1.3), (1.4) *and* (1.5).

Proof. (i) From (2.8) of Lemma 2.3 and

$$C(F,t) = v(R,\tau) = v(\ln F, T-t).$$

Thus

$$C(F,t) = v(R,\tau) = \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}} \int_{-\infty}^{\infty} \exp\left[-\frac{(\ln F - \frac{\sigma^2}{2}(T-t)) - y)^2}{2\sigma^2(T-t)}\right] f(y)dy.$$

Moreover $C(F,t) = K(\ln F, T-t) * f(\ln F)$ and from (2.9)

$$K(\ln F, T-t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}} \exp\left[-\frac{(\ln F - \frac{\sigma^2}{2}(T-t))^2}{2\sigma^2(T-t)}\right].$$

(ii) From (2.10) of Lemma 2.4

$$C(F,t) = v(R,\tau) = v(\ln F, T-t) = e^{-r(T-t)}(F-p).$$

(iii) From Lemma 2.4 $C(F,t)$ can be related to cover the Black-Scholes formula given by (1.3), (1.4) and (1.5).

□

Theorem 3.2. (The properties of $K(\ln F, T-t)$)
The kernel $K(\ln F, T-t)$ have the following properties

(i) $K(\ln F, T-t)$ *satisfies the equation* (3.1).

(ii) $K(\ln F, T-t) > 0$ *for* $0 < t \leq T$.

(iii) $K(\ln F, T-t)$ *is the tempered distribution, that is* $K(\ln F, T-t) \in S'(\mathbb{R})$ *is the space of tempered distribution on the set of real number* $\mathbb{R}$.

(iv) $e^{r(T-t)} \int_{-\infty}^{\infty} K(\ln F, T-t) d(\ln F) = 1$.

(v) $\lim_{t \to T} K(\ln F, T-t) = \delta(\ln F)$.

(vi) $K(\ln F, T-t)$ *is Gaussian function or Normal distribuition with mean*

$e^{-r(T-t)}\frac{\sigma^2}{2}(T-t)$ *and variance* $e^{-2r(T-t)}\sigma^2(T-t)$.

Proof. (i) Since $K(\ln F, T-t)$ is the kernel of (3.1) which is the solution of (3.1). So we can compute directly that $K(\ln F, T-t)$ satisfies (3.1).

(ii) $K(\ln F, T-t) > 0$ for $0 < t \leq T$ is obvious.

(iii) $K(\ln F, T-t) \in S'(\mathbb{R})$, see [4, pp. 135-136].

(iv) Since

$$e^{r(T-t)}\int_{-\infty}^{\infty} K(\ln F, T-t)d(\ln F) =$$

$$\frac{1}{\sqrt{2\pi\sigma^2(T-t)}}\int_{-\infty}^{\infty}\exp\left[-\frac{(\ln F-\frac{\sigma^2}{2}(T-t))^2}{2\sigma^2(T-t)}\right]d(\ln F).$$

Put $u = \frac{1}{\sqrt{2\sigma^2(T-t)}}\left(\ln F - \frac{\sigma^2}{2}(T-t)\right)$ then $d(\ln F) = \sqrt{2\sigma^2(T-t)}du$. Thus

$$\begin{aligned} e^{r(T-t)}\int_{-\infty}^{\infty} K(\ln F, T-t)d(\ln F) &= \frac{\sqrt{2\sigma^2(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}}\int_{-\infty}^{\infty} e^{-u^2}du \\ &= \frac{\sqrt{2\sigma^2(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}}\sqrt{\pi} = 1. \end{aligned}$$

(v) $\lim_{t\to T} K(\ln F, T-t) = \delta(\ln F)$, see [3, pp. 36-37].

(vi)

$$\begin{aligned} \text{mean} &= E(K(\ln F, T-t)) \\ &= e^{-r(T-t)}E\left(\frac{1}{\sqrt{2\pi\sigma^2(T-t)}}\exp\left[-\frac{(\ln F-\frac{\sigma^2}{2}(T-t))^2}{2\sigma^2(T-t)}\right]\right) \\ &= e^{-r(T-t)}\frac{\sigma^2}{2}(T-t) \end{aligned}$$

where E is expectation and

$$\begin{aligned} \text{variance} &= V\left(K(\ln F, T-t)\right) \\ &= e^{-2r(T-t)}V\left(\frac{1}{\sqrt{2\pi\sigma^2(T-t)}}\exp\left[-\frac{(\ln F-\frac{\sigma^2}{2}(T-t))^2}{2\sigma^2(T-t)}\right]\right) \\ &= e^{2r(T-t)}\sigma^2(T-t). \end{aligned}$$

Where V is variance.
Or shortly denote $K(\ln F, T-t)$ is $N(e^{-r(T-r)}\frac{\sigma^2}{2}(T-t), \sigma^2(T-t))$.

□

4 Conclusion

At present many people are interested in investing the option price on future of the stock prices and the Black-Schole Formula is needed for computing such the option price on future. Unfortunately such Black-Scholes Formula is too complicated and cannot be derived directly from the Black-Scholes Equation. So the main purpose of this work is to find the another solution of the Black-Scholes Equation and then we obtained such solution in the form of kernel which is simple formula for the option price on future. Moreover we can related such solution to cover the Black-Scholes Formula that shown in Theorem 3.1.

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