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New Inequalities of Ostrowski Type for Mappings whose Derivatives are (α, m)-Convex via Fractional Integrals

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Abstract : New identity similar to an identity of [1] for fractional integrals have been defined. Then making use of this identity, some new Ostrowski type inequalities for Riemann-Liouville fractional integral have been developed. Our results have some relationships with the results of Özdemir et. al., proved in [1] and the analysis used in the proofs is simple.

Keywords : (α, m)-Convex mappings; Hermite-Hadamard inequality; Ostrowski inequality.

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1 Introduction

In 1938, A.M. Ostrowski proved the following interesting and useful integral inequality (see also [2, page 468]):

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Theorem 1.1. *Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a mapping differentiable in the interior I° of I, and let $a, b \in I^{\circ}$ with $a < b$. If $|f'(x)| \leq M$ for all $x \in [a, b]$, then the following inequality holds:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right]$$

for all $x \in [a, b]$.

The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller one.

This inequality gives an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(t)\,dt$ by the value $f(x)$ at point $x \in [a, b]$. In recent years, such inequalities were studied extensively by many researchers and numerious generalizations, extensions and variants of them appeared in a number of papers, see [1, 3–5].

In [6], V.G. Miheşan defined (α, m)-convexity as the following:

Definition 1.2. The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be (α, m)-*convex*, where $(\alpha, m) \in [0, 1]^2$, if we have

$$f(tx + m(1-t)y) \leq t^{\alpha} f(x) + m(1 - t^{\alpha}) f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Denote by $K_m^{\alpha}(b)$ the class of all (α, m)-convex functions on $[0, b]$ for which $f(0) \leq 0$.

It can be easily seen that for $(\alpha, m) = (1, m)$, (α, m) convexity reduces to m-convexity; $(\alpha, m) = (\alpha, 1)$, (α, m)-convexity reduces to α-convexity and for $(\alpha, m) = (1, 1)$, (α, m)-convexity reduces to the concept of usual convexity defined on $[0, b]$, $b > 0$. For recent results and generalizations concerning (α, m)-convex functions, see [1, 7, 8].

In order to prove our results we need the following equality which was given in [1, page 372] by Özdemir et al.:

$$\begin{aligned} mf(x) - \frac{1}{b-a}\int_{ma}^{mb} f(u)\,du \;=\; & \frac{(x - ma)^2}{b-a}\int_0^1 t f'(tx + m(1-t)a)\,dt \\ & - \frac{(mb - x)^2}{b-a}\int_0^1 t f'(tx + m(1-t)b)\,dt \qquad (1.1) \end{aligned}$$

which is a special case of Lemma 1 in [9] with $ma \to a$ and $mb \to b$.

Using the inequality in (1.1), Özdemir et al. in [1] established the following results which holds for (α, m)-convex functions.

Theorem 1.3. *Let I be an open interval such that $[0, \infty) \subset I$ and $f : I \to \mathbb{R}$ be a differentiable function on I such that $f' \in L[ma, mb]$ where $ma, mb \in I$ with*

$a<b$. If $|f'|^q$ is (α,m)-convex on $[ma,mb]$ for $(\alpha,m)\in[0,1]^2$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $|f'(x)|\leq M$, $x\in[ma,mb]$, then the following inequality holds:

$$\begin{aligned}&\left|mf(x)-\frac{1}{b-a}\int_{ma}^{mb}f(u)\,du\right|\\ \leq\;& M\left(\frac{\alpha m+1}{\alpha+1}\right)^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{(x-ma)^2+(mb-x)^2}{b-a}\end{aligned}$$

for each $x\in[ma,mb]$.

Theorem 1.4. *Let I be an open interval such that $[0,\infty)\subset I$ and $f:I\to\mathbb{R}$ be a differentiable function on I such that $f'\in L[ma,mb]$ where $ma,mb\in I$ with $a<b$. If $|f'|^q$ is (α,m)-convex on $[ma,mb]$ for $(\alpha,m)\in[0,1]^2$, $q\in[1,\infty)$ and $|f'(x)|\leq M$, then the following inequality holds:*

$$\begin{aligned}&\left|mf(x)-\frac{1}{b-a}\int_{ma}^{mb}f(u)\,du\right|\\ \leq\;& M\left(\frac{2+\alpha m}{\alpha+2}\right)^{\frac{1}{q}}\frac{(x-ma)^{\alpha+1}+(mb-x)^{\alpha+1}}{(b-a)\,2}\end{aligned}$$

for each $x\in[ma,mb]$.

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 1.5. Let $f\in L_1[a,b]$. The *Riemann-Liouville integrals* $J_{a^+}^{\alpha}(f)$ and $J_{b^-}^{\alpha}(f)$ of order $\alpha>0$ with $a\geq 0$ are defined by

$$J_{a^+}^{\alpha}f(x)=\tfrac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)\,dt,\quad x>a$$

and

$$J_{b^-}^{\alpha}f(x)=\tfrac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)\,dt,\quad x<b$$

where $\Gamma(\alpha)=\int_0^{\infty}e^{-t}u^{\alpha-1}du$. Here $J_{a^+}^{0}f(x)=J_{b^-}^{0}f(x)=f(x)$.

In the case of $\alpha=1$, the fractional integral reduces to the classical integral. Some recent results and properties concerning this operator can be found in [10–16].

We establish new Ostrowski type inequalities for (α,m)-convex functions via Riemann-Liouville fractional integral. An interesting feature of our results is that they provide new estimates on these types of inequalities for fractional integrals.

2 Ostrowski Type Inequalities for Fractional Integrals

In order to prove our main results we need the following identity:

Lemma 2.1. *Let I be an open real interval such that $[0,\infty)\subset I$ and $f: I\to\mathbb{R}$ be a differentiable function on I such that $f'\in L([ma,mb])$, where $[ma,mb]\in I$ with $a<b$ then for all $x\in(ma,mb)$ and $\alpha>0$ we have:*

$$\begin{aligned}
&\frac{(x-ma)^{\alpha}+(mb-x)^{\alpha}}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^{\alpha}f(ma)+J_{x^+}^{\alpha}f(mb)\right]\\
=&\ \frac{(x-ma)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}f'(tx+m(1-t)a)\,dt\\
&-\frac{(mb-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}f'(tx+m(1-t)b)\,dt
\end{aligned}$$

where $\Gamma(\alpha)=\int_0^{\infty}e^{-t}u^{\alpha-1}du$.

Proof. Integration by parts we have

$$\begin{aligned}
&\int_0^1 t^{\alpha}f'(tx+m(1-t)a)\,dt\\
=&\ t^{\alpha}\frac{f(tx+m(1-t)a)}{x-ma}\bigg|_0^1-\int_0^1\alpha t^{\alpha-1}\frac{f(tx+m(1-t)a)}{x-ma}dt\\
=&\ \frac{f(x)}{x-ma}-\frac{\alpha}{x-ma}\int_{ma}^{x}\left(\frac{u-ma}{x-ma}\right)^{\alpha-1}f(u)\frac{du}{x-ma}\\
=&\ \frac{f(x)}{x-ma}-\frac{\Gamma(\alpha+1)}{(x-ma)^{\alpha+1}}\frac{1}{\Gamma(\alpha)}\int_{ma}^{x}(u-ma)^{\alpha-1}f(u)\,du\\
=&\ \frac{f(x)}{x-ma}-\frac{\Gamma(\alpha+1)}{(x-ma)^{\alpha+1}}J_{x^-}^{\alpha}f(ma)
\end{aligned}\tag{2.1}$$

and similarly

$$\begin{aligned}
&\int_0^1 t^{\alpha}f'(tx+m(1-t)b)\,dt\\
=&\ t^{\alpha}\frac{f(tx+m(1-t)b)}{x-mb}\bigg|_0^1-\int_0^1\alpha t^{\alpha-1}\frac{f(tx+m(1-t)b)}{x-mb}dt\\
=&\ \frac{f(x)}{x-mb}+\frac{\alpha}{(mb-x)^2}\int_{x}^{mb}\left(\frac{mb-u}{mb-x}\right)^{\alpha-1}f(u)\,du\\
=&\ \frac{f(x)}{x-mb}+\frac{\Gamma(\alpha+1)}{(mb-x)^{\alpha+1}}\frac{1}{\Gamma(\alpha)}\int_{x}^{mb}(mb-u)^{\alpha-1}f(u)\,du\\
=&\ \frac{f(x)}{x-mb}+\frac{\Gamma(\alpha+1)}{(mb-x)^{\alpha+1}}J_{x^+}^{\alpha}f(mb).
\end{aligned}\tag{2.2}$$

Multiplying the both sides of 2.1 and 2.2 by $\frac{(x-ma)^{\alpha+1}}{b-a}$ and $\frac{(mb-x)^{\alpha+1}}{b-a}$, respectively, we have

$$\frac{(x-ma)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha} f'(tx+m(1-t)a)\,dt=\frac{(x-ma)^{\alpha}}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{b-a}J_{x^-}^{\alpha}f(ma) \tag{2.3}$$

and

$$\frac{(mb-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha} f'(tx+m(1-t)b)\,dt=-\frac{(mb-x)^{\alpha}}{b-a}f(x)+\frac{\Gamma(\alpha+1)}{b-a}J_{x^+}^{\alpha}f(mb). \tag{2.4}$$

If we add the inequalities in 2.3 and 2.4, we get the desired result. □

Using Lemma 2.1, we can obtain the following fractional integral inequalities:

Theorem 2.2. *Let I be an open interval such that $[0,\infty)\subset I$ and $f:I\to\mathbb{R}$ be a differentiable function on I such that $f'\in L[ma,mb]$ where $ma,mb\in I$ with $a<b$. If $|f'|$ is (α,m)-convex on $[ma,mb]$ for $(\alpha,m)\in[0,1]^2$ and $|f'(x)|\le M$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{(x-ma)^{\alpha}+(mb-x)^{\alpha}}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^{\alpha}f(ma)+J_{x^+}^{\alpha}f(mb)\right]\right|\\ \le\ & M\left(\frac{1+m\alpha}{2\alpha+1}\right)\frac{(x-ma)^{\alpha+1}+(mb-x)^{\alpha+1}}{b-a}\end{aligned}$$

for all $x\in[ma,mb]$.

Proof. From Lemma 2.1 and using the (α,m)-convexity of $|f'|$, we have

$$\begin{aligned}&\left|\frac{(x-ma)^{\alpha}+(mb-x)^{\alpha}}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^{\alpha}f(ma)+J_{x^+}^{\alpha}f(mb)\right]\right|\\
\le\ & \frac{(x-ma)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left|f'(tx+m(1-t)a)\right|dt\\
&+\frac{(mb-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left|f'(tx+m(1-t)b)\right|dt\\
\le\ & \frac{(x-ma)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left[t^{\alpha}|f'(x)|+m(1-t^{\alpha})|f'(a)|\right]dt\\
&+\frac{(mb-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left[t^{\alpha}|f'(x)|+m(1-t^{\alpha})|f'(b)|\right]dt\\
\le\ & M\frac{(x-ma)^{\alpha+1}}{b-a}\int_0^1\left[t^{2\alpha}+m\left(t^{\alpha}-t^{2\alpha}\right)\right]dt\\
&+M\frac{(mb-x)^{\alpha+1}}{b-a}\int_0^1\left[t^{2\alpha}+m\left(t^{\alpha}-t^{2\alpha}\right)\right]dt\\
\le\ & M\left(\frac{1+m\alpha}{2\alpha+1}\right)\frac{(x-ma)^{\alpha+1}+(mb-x)^{\alpha+1}}{b-a}\end{aligned}$$

where we have used the fact that

$$\int_0^1 \left[t^{2\alpha} + m\left(t^\alpha - t^{2\alpha}\right)\right] dt = \frac{1+m\alpha}{2\alpha+1}.$$

The proof is completed. □

The corresponding version for powers of the absolute value of the first derivative is incorporated in the following results:

Theorem 2.3. *Let I be an open interval such that $[0,\infty) \subset I$ and $f : I \to \mathbb{R}$ be a differentiable function on I such that $f' \in L[ma, mb]$ where $ma, mb \in I$ with $a < b$. If $|f'|^q$ is (α, m)-convex on $[ma, mb]$ for $(\alpha, m) \in [0,1]^2$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $|f'(x)| \le M$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{(x-ma)^\alpha + (mb-x)^\alpha}{b-a} f(x) - \frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^\alpha f(ma) + J_{x^+}^\alpha f(mb)\right]\right| \\
\le\ & M\left(\frac{1+m\alpha}{\alpha+1}\right)^{\frac{1}{q}}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}} \frac{(x-ma)^{\alpha+1} + (mb-x)^{\alpha+1}}{b-a}
\end{aligned}$$

for all $x \in [ma, mb]$.

Proof. Suppose that $p > 1$. From Lemma 2.1 and using the Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{(x-ma)^\alpha + (mb-x)^\alpha}{b-a} f(x) - \frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^\alpha f(ma) + J_{x^+}^\alpha f(mb)\right]\right| \\
\le\ & \frac{(x-ma)^{\alpha+1}}{b-a}\int_0^1 t^\alpha \left|f'(tx + m(1-t)a)\right| dt \\
& + \frac{(mb-x)^{\alpha+1}}{b-a}\int_0^1 t^\alpha \left|f'(tx + m(1-t)b)\right| dt \\
\le\ & \frac{(x-ma)^{\alpha+1}}{b-a}\left(\int_0^1 t^{\alpha p} dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(tx + m(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
& + \frac{(mb-x)^{\alpha+1}}{b-a}\left(\int_0^1 t^{\alpha p} dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(tx + m(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|^q$ is (α, m)-convex function and $|f'(x)| \le M$, then we have

$$\begin{aligned}
\left(\int_0^1 \left|f'(tx + m(1-t)a)\right|^q dt\right)^{\frac{1}{q}} &\le \left(\int_0^1 \left[t^\alpha \left|f'(x)\right|^q + m(1-t^\alpha)\left|f'(a)\right|^q\right] dt\right)^{\frac{1}{q}} \\
&= M\left(\frac{1+\alpha m}{\alpha+1}\right)^{\frac{1}{q}}
\end{aligned}$$

and similarly

$$\left(\int_0^1 |f'(tx+m(1-t)b)|^q\,dt\right)^{\frac{1}{q}} \leq \left(\int_0^1 \left[t^\alpha |f'(x)|^q + m(1-t^\alpha)|f'(b)|^q\right]dt\right)^{\frac{1}{q}} = M\left(\frac{1+\alpha m}{\alpha+1}\right)^{\frac{1}{q}}.$$

Therefore, we have

$$\begin{aligned}
&\left|\frac{(x-ma)^\alpha+(mb-x)^\alpha}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^\alpha f(ma)+J_{x^+}^\alpha f(mb)\right]\right| \\
\leq\ & \frac{(x-ma)^{\alpha+1}}{b-a}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(M^q\frac{1+\alpha m}{\alpha+1}\right)^{\frac{1}{q}} \\
&+\frac{(mb-x)^{\alpha+1}}{b-a}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(M^q\frac{1+\alpha m}{\alpha+1}\right)^{\frac{1}{q}} \\
=\ & M\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{1+\alpha m}{\alpha+1}\right)^{\frac{1}{q}}\frac{(x-ma)^{\alpha+1}+(mb-x)^{\alpha+1}}{b-a}.
\end{aligned}$$

□

Theorem 2.4. *Let I be an open interval such that $[0,\infty)\subset I$ and $f: I\to\mathbb{R}$ be a differentiable function on I such that $f'\in L[ma,mb]$ where $ma, mb\in I$ with $a<b$. If $|f'|^q$ is (α,m)-convex on $[ma,mb]$ for $(\alpha,m)\in[0,1]^2$, $q\geq 1$ and $|f'(x)|\leq M$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{(x-ma)^\alpha+(mb-x)^\alpha}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^\alpha f(ma)+J_{x^+}^\alpha f(mb)\right]\right| \\
\leq\ & M\left(\frac{\alpha(m+1)+1}{2\alpha+1}\right)^{\frac{1}{q}}\frac{(x-ma)^{\alpha+1}+(mb-x)^{\alpha+1}}{(b-a)(\alpha+1)}
\end{aligned}$$

for all $x\in[ma,mb]$.

Proof. From Lemma 2.1 and using the well known power mean inequality, we have

$$\begin{aligned}
&\left|\frac{(x-ma)^\alpha+(mb-x)^\alpha}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^\alpha f(ma)+J_{x^+}^\alpha f(mb)\right]\right| \\
\leq\ & \frac{(x-ma)^{\alpha+1}}{b-a}\int_0^1 t^\alpha |f'(tx+m(1-t)a)|\,dt \\
&+\frac{(mb-x)^{\alpha+1}}{b-a}\int_0^1 t^\alpha |f'(tx+m(1-t)b)|\,dt \qquad (2.5)\\
\leq\ & \frac{(x-ma)^{\alpha+1}}{b-a}\left(\int_0^1 t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^\alpha |f'(tx+m(1-t)a)|^q\,dt\right)^{\frac{1}{q}} \\
&+\frac{(mb-x)^{\alpha+1}}{b-a}\left(\int_0^1 t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^\alpha |f'(tx+m(1-t)b)|^q\,dt\right)^{\frac{1}{q}}
\end{aligned}$$

Since $|f'|^q$ is (α, m)-convex function and $|f'(x)| \leq M$, then we have

$$\begin{aligned} \int_0^1 t^{\alpha} \left|f'\left(tx+m\left(1-t\right)a\right)\right|^q dt &= \int_0^1 t^{\alpha} \left|f'\left(tx+m\left(1-t\right)b\right)\right| dt \\ &\leq M^q \frac{\alpha\left(m+1\right)+1}{\left(2\alpha+1\right)\left(\alpha+1\right)}. \end{aligned} \tag{2.6}$$

Then using the inequality (2.6) in (2.5) and computing the integrals in (2.5), we get the desired result. □

Remark 2.5.

(i) *In Theorem* 2.3*, if we choose* $\alpha = 1$ *we get the result in Theorem* 1.3 *with* $\alpha = 1$.

(ii) *In Theorem* 2.4*, if we choose* $\alpha = 1$ *we get the result in Theorem* 1.4 *with* $\alpha = 1$.

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