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On $\phi-$Quasiconformally Symmetric $N(k)-$Contact Metric Manifolds

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Abstract : The object of the present paper is to study locally and globally $\phi-$quasiconformally symmetric $N(k)-$metric manifolds. We prove that a globally $\phi-$quasiconformally $N(k)-$contact metric manifold $M^{2n+1}(n \geq 1)$ is Sasakian. Some observations for a 3-dimensional locally $\phi-$symmetric $N(k)-$contact metric manifold are given. We also give an example of a 3-dimensional locally $\phi-$quasiconformally symmetric $N(k)-$contact metric manifold.

Keywords : $N(k)-$contact manifold; quasiconformal curvature tensor; $\eta-$Einstein manifold.
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1 Introduction

The notion of locally symmetry of a Riemannian manifold has been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, T. Takahasi [1] introduced the notion of locally $\phi-$symmetry, De et al. [2] introduced the notion of $\phi-$recurrent Sasakian manifold. In the context of contact geometry the notion of $\phi-$symmetry is introduced and studied by Boeckx, Bueken and Vanhecke [3] with several examples. In a recent paper De and Gazi

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[4] studied locally $\phi-$recurrent $N(k)-$contact metric manifolds. Also De, Özgür and Mondal [5] studied $\phi-$quasiconformally symmetric Sasakian manifolds. In the present paper we study $\phi-$quasiconformally symmetric $N(k)-$contact metric manifolds which generalizes the results of De, Özgür and Mondal [5] and also the result of Blair, Koufogiorgos and Sharma [6].

Let (M,g) be a $(2n+1),(n\geq 1)$-dimensional Riemannian manifold. The notion of the quasiconformal curvature tensor was introduced by Yano and Sawaki [7]. According to them a quasiconformal curvature tensor is defined by

$$\begin{aligned} C^*(X,Y)Z &= aR(X,Y)Z+b[S(Y,Z)X-S(X,Z)Y \\ &\quad +g(Y,Z)QX-g(X,Z)QY] \\ &\quad -\frac{r}{2n+1}[\frac{a}{2n}+2b][g(Y,Z)X-g(X,Z)Y], \end{aligned} \tag{1.1}$$

where a,b are constants, S is the Ricci tensor, Q is the Ricci operator defined by $S(X,Y)=g(QX,Y)$ and r is the scalar curvature of the manifold M. If $a=1$ and $b=-\frac{1}{2n-1}$, then (1.1) takes the form

$$\begin{aligned} C^*(X,Y)Z &= R(X,Y)Z-\frac{1}{2n-1}[S(Y,Z)X-S(X,Z)Y \\ &\quad +g(Y,Z)QX-g(X,Z)QY]+\frac{r}{2n(2n-1)}[g(Y,Z)X \\ &\quad -g(X,Z)Y]=C(X,Y)Z, \end{aligned} \tag{1.2}$$

where C is the conformal curvature tensor. In [8], De and Matsuyama studied quasiconformally flat Riemannian manifolds satisfying certain condition on the Ricci tensor. From Theorem 5 of [8], it can be proved that a 4-dimensional quasiconformally flat semi-Riemannian manifold is the Robertson-Walker spacetime, Robertson-Walker spacetime is the warped product $I\times_f M^*$, where M^* is a space of constant curvature and I is an open interval [9]. From (1.1) we obtain,

$$\begin{aligned} (\nabla_W C^*)(X,Y)Z &= a(\nabla_W R)(X,Y)Z+b[(\nabla_W S)(Y,Z)X \\ &\quad -(\nabla_W S)(X,Z)Y+g(Y,Z)(\nabla_W Q)X-g(X,Z)(\nabla_W Q)Y] \\ &\quad -\frac{dr(W)}{2n+1}[\frac{a}{2n}+2b][g(Y,Z)X-g(X,Z)Y]. \end{aligned} \tag{1.3}$$

If the condition $\nabla C^*=0$ holds on M, then M is called quasiconformally symmetric, where ∇ denotes the Levi-Civita connection on M. It is known [10] that a quasiconformally symmetric $N(k)-$contact metric manifold for $k\neq 0$ is a manifold of constant curvature k. This fact means that a quasiconformally symmetric condition is too strong for a $N(k)-$contact metric manifold. In [1], Takahashi introduced a weaker condition which is locally symmetry for a Sasakian manifold that satisfies the condition

$$\phi^2(\nabla_X R)(Y,Z)W=0, \tag{1.4}$$

where X, Y, Z, W are horizontal vector fields which means that it is horizontal with respect to the contact form η of the local fibering, namely, a horizontal vector is nothing but a vector which is orthogonal to ξ. In [6], Blair, Koufogirgos and Sharma studied locally $\phi-$symmetric 3-dimensional $N(k)-$contact metric manifolds.

In (1.4), if X, Y, Z, W are not horizontal vectors, then we call the manifold globally $\phi-$symmetric.

In this paper we introduce a weaker condition than quasiconformally symmetry that satisfies

$$\phi^2(\nabla_W C^*)(X,Y)Z = 0, \tag{1.5}$$

which is called globally $\phi-$quasiconformally symmetric for arbitrary vector fields X, Y, Z, W on M. If X, Y, Z, W are horizontal vectors, then the manifold is called locally $\phi-$quasiconformally symmetric.

The paper is organized as follows: After preliminaries in Section 3, we consider globally ϕ-quasiconformally symmetric $N(k)-$contact metric manifolds and prove that such a $N(k)$-contact metric manifold is Sasakian. Section 4 deals with 3-dimensional locally ϕ-quasiconformally symmetric $N(k)$-contact metric manifold. We prove that a 3-dimensional $N(k)$-contact metric manifold is locally ϕ-quasiconformally symmetric if and only if it is locally $\phi-$symmetric. Finally we construct an example of a 3-dimensional locally $\phi-$quasiconformally symmetric $N(k)-$contact metric manifold.

2 Preliminaries

A $(2n+1)$-dimensional manifold M is said to admit an almost contact metric structure if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$(a)\quad \phi^2 = -I + \eta\otimes\xi, \qquad (b)\quad \eta(\xi) = 1, \qquad (c)\quad \phi\xi = 0 \quad \text{and} \quad (d)\quad \eta\circ\phi = 0. \tag{2.1}$$

An almost contact metric structure is said to be normal if the induced almost complex structure J on the product manifold $M \times R$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$$

is integrable, where X is tangent to M, t is the coordinate of R and f is a smooth function on $M \times R$. Let g be a compatible Riemannian metric with almost contact structure (ϕ, ξ, η), that is

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.2}$$

Then M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). From (2.1) and (2.2) it can be easily seen that

$$(a)\quad g(X, \phi Y) = -g(\phi X, Y), \qquad (b)\quad g(X, \xi) = \eta(X), \tag{2.3}$$

for all vector fields X, Y. An almost contact metric structure becomes a contact metric structure if

$$g(X, \phi Y) = d\eta(X, Y), \tag{2.4}$$

for all vector fields X, Y. The 1-form η is then called a contact form and ξ is its characteristic vector field. We define a $(1,1)$ tensor field h by $h = \frac{1}{2}\pounds_\xi \phi$, where $\pounds$ denotes the Lie derivative. Then h is symmetric and satisfies $h\phi = -\phi h$. We have $Tr.h = Tr.\phi h = 0$ and $h\xi = 0$. Also

$$\nabla_X \xi = -\phi X - \phi h X \tag{2.5}$$

holds in a contact metric manifold. A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X, \qquad X, Y \in TM, \tag{2.6}$$

where ∇ is the Levi-Civita connection of the Riemannian metric g. A contact metric manifold $M(\phi, \xi, \eta, g)$ for which ξ is a Killing vector field is said to be a K-contact metric manifold. A Sasakian manifold is K-contact but not conversely. However a 3-dimensional K-contact manifold is Sasakian [11]. It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X, Y)\xi = 0$ [12]. On the other hand on a Sasakian manifold the following relation holds:

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y. \tag{2.7}$$

As a generalisation of both $R(X, Y)\xi = 0$ and the Sasakian case : D. E. Blair, T. Koufogiorgos and B. J. Papantoniou [13] introduced the (k, μ)- nullity distribution on a contact metric manifold and gave several reasons for studying it. The (k, μ)-nullity distribution $N(k, \mu)$ [13] of a contact metric manifold M is defined by

$$\begin{aligned} N(k, \mu) \quad : \quad & p \longrightarrow N_p(k, \mu) \\ & = \{W \in T_pM : R(X, Y)W = (kI + \mu h)(g(Y, W)X - g(X, W)Y)\}, \end{aligned}$$

for all $X, Y \in TM$, where $(k, \mu) \in \mathbb{R}^2$. A contact metric manifold M with $\xi \in N(k, \mu)$ is called a (k, μ)-contact manifold. In particular on a (k, μ)-contact manifold, we have

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{2.8}$$

On a (k, μ)-contact manifold $k \leq 1$. If $k = 1$, the structure is Sasakian ($h = 0$ and μ is indeterminant) and if $k < 1$, then the (k, μ)-nullity condition determines the curvature of M completely [13]. Infact, for a (k, μ)-contact manifold, the condition of being Sasakian, a K-contact manifold , $k = 1$ and $h = 0$ are all equivalent.

The k-nullity distribution $N(k)$ of a Riemannian manifold M is defined by [14]

$$N(k) : p \longrightarrow N_p(k) = \{Z \in T_pM : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\},$$

k being a constant. If the characteristic vector field $\xi \in N(k)$, then we call the manifold an $N(k)$-contact metric manifold [14]. If $k = 1$, then the manifold is Sasakian and if $k = 0$, then the manifold is locally isometric to the product $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$ [12]. In a (k, μ)-contact manifold if $\mu = 0$, then the manifold becomes an $N(k)$-contact manifold.

In [15], $N(k)$-contact metric manifold were studied in details. For more details we refer to ([6], [16]).

In a $(2n+1)$-dimensional $N(k)$-contact metric manifold M, the following relations hold:

$$h^2 = (k-1)\phi^2, \quad k \leq 1, \tag{2.9}$$

$$(\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.10}$$

$$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X], \tag{2.11}$$

$$S(X, \xi) = 2nk\eta(X), \tag{2.12}$$

$$\begin{aligned} S(X,Y) \quad = \quad & 2(n-1)g(X,Y) + 2(n-1)g(hX,Y) \\ & +[2(1-n) + 2nk]\eta(X)\eta(Y), \quad m \geq 1, \end{aligned} \tag{2.13}$$

$$r = 2n(2n - 2 + k), \tag{2.14}$$

$$S(\phi X, \phi Y) = S(X,Y) - 2nk\eta(X)\eta(Y) - 4(n-1)g(hX,Y), \tag{2.15}$$

$$(\nabla_X \eta)(Y) = g(X + hX, \phi Y), \tag{2.16}$$

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y], \tag{2.17}$$

$$\eta(R(X,Y)Z) = k[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.18}$$

for any vector fields X, Y, Z where R is the Riemannian curvature tensor and S is the Ricci tensor.

3 Globally ϕ-Quasiconformally Symmetric $N(k)-$Contact Metric Manifolds

Definition 3.1. A $N(k)-$contact metric manifold M is said to be globally $\phi-$quasiconformally symmetric if the quasiconformal curvature tensor C^* satisfies

$$\phi^2(\nabla_W C^*)(X,Y)Z = 0, \tag{3.1}$$

for all vector fields $X, Y, Z, W \in \chi(M)$.

A contact metric manifold is said to be an $\eta-$Einstein manifold if the Ricci tensor of the manifold is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{3.2}$$

where a, b are smooth functions on M and $X, Y \in \chi(M)$.

Here we state the following Lemma due to Baikoussis and Koufogiorgos [17]:

Lemma 3.2. *Let M be an η-Einstein manifold of dimension $(2n+1), (n \geq 1)$. If ξ belongs to the k-nullity distribution, then $k = 1$ and the structure is Sasakian.*

Let us suppose that the manifold M is globally $\phi-$quasiconformally symmetric $N(k)-$contact metric manifold. Then by definition

$$\phi^2(\nabla_W C^*)(X,Y)Z = 0. \tag{3.3}$$

Using (2.1)(a), we have

$$-(\nabla_W C^*)(X,Y)Z + \eta((\nabla_W C^*)(X,Y)Z)\xi = 0. \tag{3.4}$$

Using (1.3) in (3.4), it follows that

$$\begin{aligned}
&-ag((\nabla_W R)(X,Y)Z,U) - bg(X,U)(\nabla_W S)(Y,Z)\\
&+bg(Y,U)(\nabla_W S)(X,Z) - bg(Y,Z)g((\nabla_W Q)X,U)\\
&+bg(X,Z)g((\nabla_W Q)Y,U) + \frac{1}{2n+1}dr(W)[\frac{a}{2n} + 2b]\\
&[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)] + a\eta((\nabla_W R)(X,Y)Z)\eta(U)\\
&+b(\nabla_W S)(Y,Z)\eta(U)\eta(X) - b(\nabla_W S)(X,Z)\eta(Y)\eta(U)\\
&+bg(Y,Z)\eta((\nabla_W Q)X)\eta(U) - bg(X,Z)\eta((\nabla_W Q)Y)\eta(U)\\
&-\frac{1}{2n+1}dr(W)[\frac{a}{2n} + 2b][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\eta(U) = 0.
\end{aligned} \tag{3.5}$$

Put $X = U = e_i$, in (3.5), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, we get

$$\begin{aligned}
&-[a + (2n-1)b](\nabla_W S)(Y,Z) - \{bg((\nabla_W Q)e_i, e_i)\\
&-\frac{2n-1}{2n+1}dr(W)[\frac{a}{2n} + 2b] - b\eta((\nabla_W Q)\xi)\}g(Y,Z)\\
&+bg((\nabla_W Q)Y,Z) + a\eta((\nabla_W R)(\xi,Y)Z) - b(\nabla_W S)(\xi,Z)\eta(Y)\\
&-b\eta((\nabla_W Q)Y)\eta(Z) + \frac{1}{2n+1}dr(W)[\frac{a}{2n} + 2b]\eta(Y)\eta(Z) = 0.
\end{aligned} \tag{3.6}$$

Putting $Z = \xi$ in (3.6) and using (2.1)(a) and (2.3)(b), we obtain

$$\begin{aligned}
&-[a + (2n-1)b](\nabla_W S)(Y,\xi) - \{bdr(W) - \frac{2n-1}{2n+1}dr(W)[\frac{a}{2n} + 2b]\\
&-b\eta((\nabla_W Q)\xi)\}\eta(Y) + a\eta((\nabla_W R)(\xi,Y)\xi) - b(\nabla_W S)(\xi,\xi)\eta(Y)\\
&+\frac{1}{2n+1}dr(W)[\frac{a}{2n} + 2b]\eta(Y) = 0.
\end{aligned} \tag{3.7}$$

Now

$$\begin{aligned}
\eta((\nabla_W Q)\xi) &= g(\nabla_W Q\xi,\xi) - g(Q(\nabla_W \xi),\xi)\\
&= S(\phi X,\xi) + S(\phi hX,\xi)\\
&= 0,
\end{aligned} \tag{3.8}$$

Again

$$\begin{aligned} g((\nabla_W R)(\xi,Y)\xi,\xi) &= g(\nabla_W R(\xi,Y)\xi,\xi) - g(R(\nabla_W \xi,Y)\xi,\xi) \\ &- g(R(\xi,\nabla_W Y)\xi,\xi) - g(R(\xi,Y)\nabla_W \xi,\xi). \end{aligned} \quad (3.9)$$

From (2.11), we get by using (2.1)(b)

$$g(R(\xi,Y)\xi,\xi) = 0.$$

Since $\nabla g = 0$, we obtain from above

$$g(\nabla_W R(\xi,Y)\xi,\xi) + g(R(\xi,Y)\xi,\nabla_W \xi) = 0. \quad (3.10)$$

Again using (2.11), we have

$$\begin{aligned} g(R(\xi,\nabla_W Y)\xi,\xi) &= kg(\eta(\nabla_W Y)\xi - \nabla_W Y,\xi) \\ &= k[\eta(\nabla_W Y) - \eta(\nabla_W Y)] \\ &= 0. \end{aligned} \quad (3.11)$$

By using (2.5), 2.11) and (2.1)(d), we have

$$\begin{aligned} g(R(\nabla_W \xi,Y)\xi,\xi) &= g(R(-\phi W - \phi hW,Y)\xi,\xi) \\ &= -g(R(\phi W,Y)\xi,\xi) - g(R(\phi hW,Y)\xi,\xi) \\ &= -kg(\eta(Y)\phi W - \eta(\phi W)Y,\xi) - kg(\eta(Y)\phi hW - \eta(\phi hW)Y,\xi) \\ &= -k\eta(Y)g(\phi W,\xi) - k\eta(Y)g(\phi hW,\xi) \\ &= 0 \ \ \text{(since } \phi \text{ is skew symmetric and } \phi\xi = 0). \end{aligned} \quad (3.12)$$

Using (3.10), (3.11) and (3.12) in (3.9) yields

$$g((\nabla_W R)(\xi,Y)\xi,\xi) = 0. \quad (3.13)$$

From (2.11) by using (2.5) and $\phi\xi = 0$, we get

$$(\nabla_W S)(\xi,\xi) = \nabla_W S(\xi,\xi) - 2S(\nabla_W \xi,\xi) = -2S(-\phi W - \phi hW,\xi) = 0. \quad (3.14)$$

By the use of (3.8), (3.13) and (3.14), from (3.7), we obtain

$$(\nabla_W S)(Y,\xi) = \frac{1}{2n+1} dr(W)\eta(Y), \text{if} \quad a + (2n-1)b \neq 0. \quad (3.15)$$

Because $a + (2n-1)b = 0$ will imply $C^* = aC$, from (1.1). So, we can not take $a + (2n-1)b = 0$. Putting $Y = \xi$ in (3.15) we get $dr(W) = 0$. This implies r is constant. So from (3.15), we have

$$(\nabla_W S)(Y,\xi) = 0. \quad (3.16)$$

Now we have

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi).$$

Using (2.12) and (2.5) in the above relation, it follows that

$$(\nabla_W S)(Y,\xi) = 2nk(\nabla_W \eta)(Y) + S(Y, \phi W + \phi hW). \tag{3.17}$$

In virtue of (3.17), (2.16) and (2.3)(a), we get

$$(\nabla_W S)(Y,\xi) = -2nkg(\phi W + \phi hW, Y) + S(Y, \phi W + \phi hW). \tag{3.18}$$

By (3.16) and (3.18), we have

$$2nkg(\phi W + \phi hW, Y) - S(Y, \phi W + \phi hW) = 0. \tag{3.19}$$

Replacing Y by ϕY in (3.19) and using (2.1)(d), (2.2) and (2.15), we get

$$2nkg(\phi W + \phi hW, \phi Y) - S(\phi Y, \phi W + \phi hW) = 0$$

or,

$$\begin{aligned}&2nk[g(W + hW, Y) - \eta(W + hW)\eta(Y)] - S(Y, W + hW)\\ &+2nk\eta(W + hW)\eta(Y) + 4(n-1)g(hY, W + hW) = 0\end{aligned}$$

or,

$$\begin{aligned}&2nkg(Y,W) + 2nkg(Y,hW) - S(Y,W) - S(Y,hW)\\ &+4(n-1)g(Y,hW) + 4(n-1)g(Y,h^2W) = 0\end{aligned}$$

since $g(X,hY) = g(hX,Y)$. Now by (2.9), (2.13) and (2.1)(a) this implies

$$\begin{aligned}S(Y,W) + S(Y,hW) &= 2nkg(Y,W) + [2nk + 4(n-1)]g(Y,hW)\\ &\quad +4(n-1)(k-1)g(Y, -W + \eta(W)\xi)\end{aligned}$$

or,

$$\begin{aligned}&S(Y,W) + 2(n-1)g(Y,hW) - 2(n-1)(k-1)g(Y,W)\\ &+2(n-1)(k-1)\eta(Y)\eta(W) = [2nk - 4(n-1)(k-1)]g(Y,W)\\ &+[2nk + 4(n-1)]g(Y,hW) + 4(n-1)(k-1)\eta(Y)\eta(W),\end{aligned}$$

which implies,

$$\begin{aligned}S(Y,W) &= 2(n+k-1)g(Y,W) + 2(nk+n-1)g(Y,hW)\\ &\quad +2(n-1)(k-1)\eta(Y)\eta(W).\end{aligned} \tag{3.20}$$

Replacing W by hW and using (2.13), (2.9) and (2.1)(a), we get from (3.20)

$$-2kg(Y,hW) = -2nk(k-1)g(Y,W) + 2nk(k-1)\eta(Y)\eta(W).$$

Since we may assume that $k \neq 0$, this implies

$$g(Y,hW) = n(k-1)g(Y,W) - n(k-1)\eta(Y)\eta(W). \tag{3.21}$$

From (3.20) and (3.21), we get

$$S(Y,W) = Ag(Y,W) + B\eta(Y)\eta(W), \tag{3.22}$$

where $A = 2[(n+k-1) + n(k-1)(nk+n-1)]$

and $B = 2[(n-1)(k-1) - n(k-1)(nk+n-1)]$

are constants. So, the manifold is an η-Einstein manifold with constant coefficients.

Thus we state the following:

Proposition 3.3. *A $(2n+1)$-dimensional globally ϕ-quasiconformally symmetric $N(k)$-contact metric manifold is an η-Einstein manifold with constant coefficients.*

In view of Lemma 3.2 and Proposition 3.3 we have the following:

Theorem 3.4. *A $(2n+1)$-dimensional$(n \geq 1)$ globally ϕ-quasiconformally symmetric $N(k)$-contact metric manifold is a Sasakian manifold.*

If $k = 1$, then the manifold reduces to a Sasakian manifold. In this case from (3.22) it follows that the manifold is an Einstein manifold. Thus we obtain the following:

Proposition 3.5. *A $(2n+1)$-dimensional globally ϕ-quasiconformally symmetric Sasakian manifold is an Einstein manifold.*

The above proposition have been proved by De, Özgür and Mondal [5].

4 3-Dimensional Locally $\phi-$Quasiconformally Symmetric $N(k)-$Contact Metric Manifolds

In a 3-dimensional Riemannian manifold, we have

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ &\quad -S(X,Z)Y + \frac{r}{2}[g(X,Z)Y - g(Y,Z)X], \end{aligned} \tag{4.1}$$

where Q is the Ricci-operator, that is, $g(QX,Y) = S(X,Y)$ and r is the scalar curvature of the manifold. Now putting $Z = \xi$ in (4.1) and using (2.1)(a), (2.3)(b) and (2.12) we get

$$\begin{aligned} R(X,Y)\xi &= \eta(Y)QX - \eta(X)QY \\ &\quad +2k[\eta(Y)X - \eta(X)Y] + \frac{r}{2}[\eta(X)Y - \eta(Y)X]. \end{aligned} \tag{4.2}$$

Using (2.17) in (4.2), we get

$$(k-\frac{r}{2})[\eta(Y)X-\eta(X)Y]=\eta(X)QY-\eta(Y)QX. \tag{4.3}$$

Putting $Y=\xi$ in (4.3) and using (2.12), we get

$$QX=(\frac{r}{2}-k)X+(3k-\frac{r}{2})\eta(X)\xi. \tag{4.4}$$

Therefore it follows from (4.4) that

$$S(X,Y)=(\frac{r}{2}-k)g(X,Y)+(3k-\frac{r}{2})\eta(X)\eta(Y). \tag{4.5}$$

Using (4.1),(4.4) and (4.5) in (1.1) we get for $n=3$

$$\begin{aligned}
C^*(X,Y)Z &= [a(\frac{r}{2}-2k)+2b(\frac{r}{2}-k)-\frac{r}{3}(\frac{a}{2}+2b)][g(Y,Z)X-g(X,Z)Y]\\
&\quad+[a(3k-\frac{r}{2})+b(\frac{r}{2}-k)][g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi]\\
&\quad+\eta(Y)\eta(Z)X-\eta(X)\eta(Y)Z\\
&= (a+b)(r-2k)[g(Y,Z)X-g(X,Z)Y]\\
&\quad+[\frac{r}{2}(b-a)+k(3a-b)][g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi\\
&\quad+\eta(Y)\eta(Z)X-\eta(X)\eta(Y)Z].
\end{aligned} \tag{4.6}$$

Taking the covariant differentiation to the both sides of the equation (4.6), we have

$$\begin{aligned}
(\nabla_W C^*)(X,Y)Z &= dr(W)(a+b)[g(Y,Z)X-g(X,Z)Y]\\
&\quad+\frac{dr(W)}{2}(b-a)[g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi\\
&\quad+\eta(Y)\eta(Z)X-\eta(X)\eta(Y)Z]\\
&\quad+[\frac{r}{2}(b-a)+k(3a-b)][g(Y,Z)(\nabla_W\eta)(X)\xi\\
&\quad+g(Y,Z)\eta(X)\nabla_W\xi-g(X,Z)(\nabla_W\eta)(Y)\xi\\
&\quad-g(X,Z)\eta(Y)\nabla_W\xi+(\nabla_W\eta)(Y)\eta(Z)X\\
&\quad+\eta(Y)(\nabla_W\eta)(Z)X-(\nabla_W\eta)(X)\eta(Y)Z\\
&\quad-\eta(X)(\nabla_W\eta)(Y)Z].
\end{aligned} \tag{4.7}$$

Now, assume that X, Y and Z are horizontal vector fields. So, (4.7) becomes

$$\begin{aligned}
(\nabla_W C^*)(X,Y)Z &= dr(W)(a+b)[g(Y,Z)X-g(X,Z)Y]\\
&\quad+[\frac{r}{2}(b-a)+k(3a-b)][g(Y,Z)(\nabla_W\eta)(X)\xi\\
&\quad-g(X,Z)(\nabla_W\eta)(Y)\xi].
\end{aligned} \tag{4.8}$$

Using (2.1)(c) we obtain from (4.8)

$$\phi^2(\nabla_W C^*)(X,Y)Z = dr(W)(a+b)[g(X,Z)Y - g(Y,Z)X]. \quad (4.9)$$

Assume $\phi^2(\nabla_W C^*)(X,Y)Z = 0$. If $a+b=0$, then putting $a=-b$ into (1.1) we find for $n=3$

$$C^*(X,Y)Z = aC(X,Y)Z,$$

where C is the weyl conformal curvature tensor. But for a 3-dimensional Riemannian manifold since $C=0$, we obtain $C^*=0$. Therefore $a+b \neq 0$. Then the equation (4.9) implies $dr(W)=0$. Hence we conclude the following:

Theorem 4.1. *A 3-dimensional $N(k)$-contact metric manifold is locally ϕ-quasiconformally symmetric if and only if the scalar curvature r is constant.*

In [6] Blair et al proved the following:
A 3-dimensional $N(k)$-contact metric manifold is locally ϕ-symmetric if and only if the scalar curvature is constant.

Using the above result of Blair et al and Theorem 4.1, we state the following:

Theorem 4.2. *A 3-dimensional $N(k)$-contact metric manifold is locally ϕ-quasiconformally symmetric if and only if it is locally ϕ-symmetric.*

5 Example

In this section, we construct an example of a locally ϕ-quasiconformally symmetric 3-dimensional $N(k)$-contact manifold. We consider 3-dimensional manifold $M = \{(x,y,z) \in R^3\}$, where (x,y,z) are the standard coordinate in R^3. Let $\{e_1, e_2, e_3\}$ be linearly independent global frame on M given by

$$[e_2, e_3] = 2e_1, \qquad [e_3, e_1] = \frac{3}{2}e_2, \qquad [e_1, e_2] = \frac{1}{2}e_3.$$

Let g be the Riemannian metric defined by

$$g(e_1,e_3) = g(e_2,e_3) = g(e_1,e_2) = 0, \quad g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1.$$

Let η be the 1-form defined by

$$\eta(U) = g(U, e_1)$$

for any $U \in \chi(M)$. Let ϕ be the $(1,1)$-tensor field defined by

$$\phi e_1 = 0, \ \ \phi e_2 = e_3, \ \ \phi e_3 = -e_2.$$

Using the linearity of ϕ and g we have

$$\eta(e_1) = 1,$$

$$\phi^2(U) = -U + \eta(U)e_1$$

and

$$g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W)$$

for any $U, W \in \chi(M)$. Moreover

$$he_1 = 0, \quad he_2 = -\frac{1}{2}e_2 \text{ and } he_3 = \frac{1}{2}e_3.$$

The Riemannian connection ∇ of the metric tensor g is given by the Koszul's formulae as

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned}$$

We have

$$\begin{aligned} 2g(\nabla_{e_2} e_3, e_1) &= e_2 g(e_3, e_1) + e_3 g(e_1, e_2) - e_1 g(e_2, e_3) \\ &\quad -g(e_2, [e_3, e_1]) - g(e_3, [e_2, e_1]) + g(e_1, [e_2, e_3]) \\ &= 1 = 2g(\frac{1}{2}e_1, e_1). \end{aligned}$$

Similarly, we have

$$2g(\nabla_{e_2} e_3, e_2) = 0 = 2g(\frac{1}{2}e_1, e_2)$$

and

$$2g(\nabla_{e_2} e_3, e_3) = 0 = 2g(\frac{1}{2}e_1, e_3).$$

Therefore, we have $\nabla_{e_2} e_3 = \frac{1}{2}e_1$.
Similarly, we have

$$\begin{aligned} &\nabla_{e_1} e_1 = 0, \nabla_{e_1} e_2 = 0, \nabla_{e_1} e_3 = 0, \\ &\nabla_{e_2} e_1 = -\frac{1}{2}e_3, \nabla_{e_2} e_2 = 0, \nabla_{e_2} e_3 = \frac{1}{2}e_1, \\ &\nabla_{e_3} e_1 = \frac{3}{2}e_2, \nabla_{e_3} e_2 = -\frac{3}{2}e_1, \nabla_{e_3} e_3 = 0. \end{aligned}$$

Therefore, the manifold satisfies the relation

$$\nabla_{e_2} e_1 = -\phi e_2 - \phi(he_2)$$

and

$$\nabla_{e_3} e_1 = -\phi e_3 - \phi(he_3).$$

Hence we have

$$\nabla_X \xi = -\phi X - \phi hX,$$

for any vector field X. Hence the manifold is a contact metric manifold for $e_1 = \xi$. Now, we find the curvature tensors as

$$R(e_2, e_1)e_1 = \frac{3}{4}e_2, R(e_3, e_1)e_1 = \frac{3}{4}e_3, R(e_2, e_3)e_1 = 0,$$
$$R(e_2, e_3)e_2 = \frac{3}{4}e_3, R(e_2, e_1)e_3 = 0, R(e_2, e_3)e_3 = -\frac{3}{4}e_2,$$
$$R(e_1, e_2)e_2 = \frac{3}{4}e_1, R(e_1, e_3)e_3 = \frac{3}{4}e_1, R(e_1, e_3)e_2 = 0.$$

From the expressions of $R(e_2, e_1)e_1$ and $R(e_3, e_1)e_1$ we conclude the manifold is a $N(\frac{3}{4})$-contact metric manifold.
The Ricci tensors of this manifold are given as follows:

$$S(e_1, e_1) = \frac{3}{2}, S(e_2, e_2) = 0, S(e_3, e_3) = 0.$$

Hence the scalar curvature is

$$r = \frac{3}{2} = \text{constant}.$$

Therefore, in view of the Theorem 4.1, we can say that the manifold is locally ϕ-quasiconformally symmetric.

References

[1] T. Takahashi, Sasakian ϕ-symmetric spaces, Tohoku Math. J. 29 (1977) 91-113.

[2] U.C. De, A.A. Shaikh, S. Biswas, On ϕ-recurrent Sasakian manifolds, Novi sad J. Math. 33 (2003) 43-48.

[3] E. Boeckx, P. Bueken, L. Vanhecke, ϕ-symmetric contact metric spaces, Glasgow Math. J. 41 (1999) 409-416.

[4] U.C. De, A.K. Gazi, On ϕ-recurrent $N(k)$-contact metric manifolds, Math. J. Okayama Univ. 50 (2008) 101-112.

[5] U.C. De, C. Özgür, A.K. Mondal, On ϕ-quasiconformally symmetric Sasakian manifolds, Indagationes Mathematicae 20 (2009) 191-200.

[6] D.E. Blair, T. Koufogiorgos, R. Sharma, A classification of 3-dimensional contact metric manifolds with $Q\phi = \phi Q$, Kodai Math. J. 13 (1990) 391-401.

[7] K. Yano, S. Sawaki, Riemannian manifolds admitting a conformal transformation group, J. Diff. Geom. 2 (1968) 161-184.

[8] U.C. De, Y. Matsuyama, On quasiconformally flat manifolds, SUT. J. Math. 42 (2006) 295-303.

[9] B. O'Neill, Semi Riemannian Geometry with Applications to Relativity, Academy Press, Inc., 1983.

[10] U.C. De, A.A. Shaikh, Complex Manifolds and Contact Manifolds, Alpha Science Publishers, U.K., 2009.

[11] J.B. Jun, U.K. Kim, On 3-dimensional almost contact metric manifolds, Kyungpook Math. J. 34 (1994) 293-301.

[12] D.E. Blair, Two remarks on contact metric structures, Tohoku Math. J. 29 (1977) 319-324.

[13] D.E. Blair, T. Koufogiorgos, B.J. Papantoniou, Contact metric manifolds satisfying a nullity condition, Israel J. Math. 91 (1995) 189-214.

[14] S. Tanno, Ricci curvature of contact Riemannian manifolds, Tohoku Math. J. 40 (1988) 441-448.

[15] C. Baikoussis, D.E. Blair, T. Koufogiorgos, A Decomposition of the Curvature Tensor of a Contact Manifold satisfying $R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y)$, Mathematics Technical Report, University of Ioanniana, 1992.

[16] D.E. Blair, H. Chen, A classification of 3-dimensional contact metric manifolds with $Q\phi = \phi Q$, Bulletin of the Institute of Mathematics Academia Sinica 20 (1992) 379-383.

[17] C. Baikoussis, T. Koufogiorgos, On a type of contact manifolds, J. of Geometry 46 (1993) 1-9.

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