Thai Journal of Mathematics
Volume 16 (2018) Number 3 : 693–707
http://thaijmath.in.cmu.ac.th
Online ISSN 1686-0209

On Special Weakly Riccisymmetric and Generalized Ricci-Recurrent Trans-Sasakian Structures

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Abstract : The present paper deals with the trans-Sasakian structure admitting an $m-$projective curvature tensor. In the last, the properties of special weakly Riccisymmetric and generalized Ricci-recurrent trans-Sasakian structures are studied.

Keywords : trans-Sasakian structures; $m-$projective curvature tensor; special weakly Riccisymmetric and generalized Ricci-recurrent trans-Sasakian structures; Ricci solitons.
2010 Mathematics Subject Classification : 53C25; 53C35; 53D10.

1 Introduction

The class of co Kähler, Sasakian and Kenmotsu manifolds are precisely the three classes which occur in a classification theorem of connected almost Hermitian manifolds M^{2m+1} for the automorphism groups having maximum dimension $(m+1)^2$ [1]. In 1969, Tanno [2] classified connected almost contact metric manifold whose automorphism group possess the maximum dimension. In such manifold, the sectional curvature of plane sections containing ξ is constant, say c, and thus it can be classified into three classes:

(i) homogeneous normal contact Riemannian manifolds with $c > 0$,

(ii) global Riemannian products of a line or a circle with a Kähler manifold of constant holomorphic sectional curvature, if $c = 0$ and

(iii) a wrapped product space $R \times_f C$, if $c < 0$.

It is well known that the manifold of class (i) is characterized by admitting a Sasakian structure. The manifold of class (ii) is characterized by a tensorial relation admitting a cosymplectic structure. In [3], Kenmotsu characterized the differential geometric properties of the manifolds of class (iii); the structure so obtained is now called a Kenmotsu structure. In general, the structures defined by Kenmotsu in [3] are not Sasakian. In the Grey Hervella classification of almost Hermitian manifolds [4], there appears a class W_4 of Hermitian manifolds, which are closely related to locally conformal Kähler manifolds [5]. An almost contact metric structure is called a trans-Sasakian structure [6] if the product manifold $M \times R$ belongs to the class W_4. In [7], the class $C_6 \otimes C_5$ coincides with the class of trans-Sasakian structure of type (α, β). A trans-Sasakian structure of type $(0, 0)$, $(0, \beta)$ and $(\alpha, 0)$ are known as cosymplectic [8], $\beta-$Kenmotsu and $\alpha-$Sasakian manifolds [9], respectively. The different geometrical properties of trans-Sasakian manifolds have studied by De and Tripathi [10], Bagewadi and Venkatesha [11], Bagewadi and Girish [12], De and Sarkar [13], China and Gonzales [14], Kim, Prasad and Tripathi [15], Pankaj et. al [16] and others. In the present paper, we study the properties of trans-Sasakian structures. The present work is structured as follows:

The second section is a brief review of trans-Sasakian structure, $m-$projective curvature tensor and Ricci solitons. In Section 3, we give an example of a trans-Sasakian structure of dimension 3 which validates the existence of such structure. Next section is devoted to the study of $m-$projectively flat trans-Sasakian structures and find some geometrical results. The properties of trans-Sasakian structures with $R(X, Y).W^* = 0$ are studied in Section 5. Section 6 is the study of special weakly Riccisymmetric trans-Sasakian structures. In next section, we deal with generalized Ricci-recurrent trans-Sasakian structures.

2 Preliminaries

If on an $n-$dimensional differentiable manifold M_n, $(n = 2m + 1)$, of differentiability class C^{r+1}, there exists a vector valued real linear function ϕ, an associated $1-$form η, the characteristic vector field ξ and the Riemannian metric g satisfying

$$(a) \quad \phi^2 X = -X + \eta(X)\xi, \quad (b) \quad \eta(\phi X) = 0 \tag{2.1}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

for arbitrary vector fields X and Y on M_n, then (M_n, g) is said to be an almost contact metric manifold and the structure $\{\phi, \eta, \xi, g\}$ is called an almost contact metric structure to M_n [8].

In view of (2.1) (a), (2.1) (b) and (2.2), we conclude

$$\eta(\xi) = 1, \quad g(X, \xi) = \eta(X), \quad \phi(\xi) = 0 \tag{2.3}$$

and

$$g(X, \phi Y) + g(\phi X, Y) = 0. \tag{2.4}$$

An almost contact metric manifold $(M_n, \phi, \xi, \eta, g)$ is called a trans-Sasakian structure [6] if $(M_n \otimes R, J, G)$ belongs to the class W_4 of the Gray Hervella classification of almost Hermitian manifolds [4], where J is the complex structure and G be the Hermitian metric on $(M_n \otimes R)$. This can be expressed by the condition

$$(D_X \phi)(Y) = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \tag{2.5}$$

for smooth functions α and β on M_n, and we say that the trans-Sasakian structure of type (α, β) [17]. Here D_X denotes the covariant derivative along the vector field X. In particular, if $\alpha = 0$ and $\beta = 0$, then (2.5) gives

$$(D_X \phi)(Y) = 0,$$

which is characterized by cosymplectic structure [8]. Also if $\alpha = 0$ and $\beta = 1$; $\alpha = 1$ and $\beta = 0$, then (2.5) becomes

$$(D_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X;$$

$$(D_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X,$$

which are characterized by Kenmotsu and Sasakian structures [17], respectively. From (2.5), it follows that

$$D_X \xi = -\alpha \phi X + \beta(X - \eta(X)\xi). \tag{2.6}$$

Also the following relations hold on trans-Sasakian structures

$$(D_X \eta)(Y) = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y), \tag{2.7}$$

$$\begin{aligned} R(X, Y)\xi = {} & (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\phi X - \eta(X)\phi Y) \\ & + (Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y, \end{aligned} \tag{2.8}$$

$$\eta(R(X, Y)Z) = (\alpha^2 - \beta^2)\{\eta(X)g(Y, Z) - \eta(Y)g(X, Z)\}\xi, \tag{2.9}$$

$$2\alpha\beta + \xi\alpha = 0, \tag{2.10}$$

$$S(\phi X, \phi Y) = S(X, Y) - 2m(\alpha^2 - \beta^2 - \xi\beta)\eta(X)\eta(Y), \tag{2.11}$$

$$S(X, \xi) = (2m(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - (2m - 1)X\beta - (\phi X)\alpha, \tag{2.12}$$

$$Q\xi = (2m(\alpha^2 - \beta^2) - \xi\beta)\xi - (2m - 1)grad\beta + \phi(grad\alpha), \tag{2.13}$$

for arbitrary vector fields X, Y and Z on M_n. Here Q is a symmetric endomorphism of the tangent space corresponding to Ricci tensor S, i.e., $S(X,Y) = g(QX,Y)$ and R represents the curvature tensor of the manifold M_n. If

$$(2m-1)grad\beta = \phi(grad\alpha), \tag{2.14}$$

then (2.11), (2.12) and (2.13) become

$$S(\phi X, \phi Y) = S(X,Y) - 2m(\alpha^2 - \beta^2)\eta(X)\eta(Y), \tag{2.15}$$

$$S(X,\xi) = 2m(\alpha^2 - \beta^2)\eta(X), \tag{2.16}$$

$$Q\xi = 2m(\alpha^2 - \beta^2)\xi. \tag{2.17}$$

A Riemannian manifold M_n is said to be $\eta-$Einstein if its Ricci tensor S assumes the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{2.18}$$

for arbitrary vector fields X and Y of M_n, where a and b are smooth functions on (M_n, g) [8].
In 1971, Pokhariyal and Mishra [18] defined a tensor W^* of type $(1,3)$ on a Riemannian manifold as

$$W^*(X,Y)Z = R(X,Y)Z - \frac{1}{4m}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \tag{2.19}$$

so that $'W^*(X,Y,Z,U) \overset{\text{def}}{=} g(W^*(X,Y)Z,U) =' W^*(Z,U,X,Y)$ and $'W^*_{ijkl}w^{ij}w^{kl} = {'W_{ijkl}}w^{ij}w^{kl}$, where $'W^*_{ijkl}$ and $'W_{ijkl}$ are components of $'W^*$ and $'W$, w^{kl} is a skew-symmetric tensor [19, 20], Q is the Ricci operator of type $(1,1)$, defined by $S(X,Y) \overset{\text{def}}{=} g(QX,Y)$ and S is the Ricci tensor for arbitrary vector fields X, Y, Z. Such a tensor field W^* is known as $m-$projective curvature tensor. Ojha [19, 21] studied the properties of $m-$projective curvature tensor in Sasakian and Kähler manifolds. He has also shown that it bridges the gap between conformal curvature tensor, conharmonic curvature tensor and concircular curvature tensor on one side and $H-$projective curvature tensor on the other. The properties of $m-$projective curvature tensor studied by Chaubey and Ojha [22], Chaubey, Prakash and Nivas [23], Taleshian and Asghari [24], De and Mallick [25], Chaubey [26–29] and other geometers.

Hamilton [30] introduced the notion of Ricci flow to obtain a canonical metric on a differentiable manifold in the beginning of $80's$. After that it became a powerful tool to study Riemannian manifolds of positive curvature. To prove Poincaré conjecture, Perelman [31,32] used Ricci flow and its surgery. Also Brendle and Schoen [33] proved the differentiable sphere theorem by using Ricci flow. The evolution equation for metrics on a Riemannian manifold, called Ricci flow and defined as

$$\frac{\partial}{\partial t} g_{ij}(t) = -2S_{ij},$$

where S_{ij} denotes the components of Ricci tensor. The solutions of Ricci flow are called Ricci solitons if they are governed by a one parameter family of diffeomorphisms and scalings. A triplet (g, V, λ) on a Riemannian manifold (M, g) is called Ricci soliton [34], natural generalized of Einstein metric, and satisfies

$$\frac{1}{2}L_V g + S = \lambda g, \tag{2.20}$$

where S is the Ricci tensor, $L_V g$ denotes the Lie derivative of Riemannian metric g along the vector field V on M and λ is a real constant [34]. A Ricci soliton is said to be steady, expanding and shrinking if $\lambda = 0, < 0$ and > 0, respectively. From equations (2.3), (2.4) and (2.6), we have

$$(L_\xi g)(X, Y) = 2\beta\{g(X, Y) - \eta(X)\eta(Y)\}, \tag{2.21}$$

for arbitrary vector fields X and Y on M_n.

3 Example of Trans-Sasakian Structure

Let

$$M_3 = \{(x, y, z) \in \mathfrak{R}^3 : x, y, z(\neq 0) \in \mathfrak{R}\}$$

be a differentiable manifold of dimension 3, where (x, y, z) denotes the standard coordinate of a point in $\mathfrak{R}^3$. Let us assume that

$$e_1 = e^z \frac{\partial}{\partial x}, \quad e_2 = e^z(\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}), \quad e_3 = \frac{\partial}{\partial z}$$

represent a set of linearly independent vector fields at each point of the manifold M_3 and therefore it forms a basis for the tangent space of M_3. We define the associated Riemannian metric g of the manifold M_3 as $g(e_i, e_j) = \delta_{ij}$, where $i, j = 1, 2, 3$ and δ_{ij} denotes the components of Kronecker delta. Also the associated $1-$form η corresponding to Riemannian metric g defined as $\eta(Z) = g(Z, e_3)$ for arbitrary vector field Z of M_3. Again we define the structure tensor field ϕ of type $(1, 1)$ as

$$\phi e_1 = e_2, \quad \phi e_2 = -e_1, \quad \phi e_3 = 0.$$

From the above results and linearity properties of g and ϕ, we can prove that

$$\phi^2 Z = -Z + \eta(Z)e_3, \quad \eta(e_3) = 1,$$

hold for all vector field Z in M_3. This reflects that $e_3 = \xi$ and the structure (ϕ, ξ, η, g) defines an almost contact structure on M_3. We can also verify that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for arbitrary vector fields X and Y of M_3. Thus M_3 together with g gives an almost contact metric manifold of dimension 3. From above relations, we find that

$$[e_1, e_2] = e^{2z}e_3 - xe^z e_1, \quad [e_1, e_3] = -e_1, \quad [e_2, e_3] = -e_2,$$

where $[.,.]$ denotes the Lie bracket. By using above results and Koszul's formula, defined by

$$2g(D_XY,Z) = Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]), \tag{3.1}$$

for arbitrary vector fields X, Y, Z, we find that

$$\begin{aligned}
&D_{e_1}e_1 = xe^z e_2 + e_3, && D_{e_1}e_2 = -xe^z e_1 + \frac{e^{2z}}{2}e_3, && D_{e_1}e_3 = -e_1 - \frac{e^{2z}}{2}e_2,\\
&D_{e_2}e_1 = -\frac{e^{2z}}{2}e_3, && D_{e_2}e_2 = e_3, && D_{e_2}e_3 = \frac{e^{2z}}{2}e_1 - e_2,\\
&D_{e_3}e_1 = -\frac{e^{2z}}{2}e_2, && D_{e_3}e_2 = \frac{e^{2z}}{2}e_1, && D_{e_3}e_3 = 0,
\end{aligned}$$

where D represents the covariant derivative. From the above calculations, we can observe that $D_X\xi = -\alpha\phi X + \beta(X - \eta(X)\xi)$ satisfies for $\xi = e_3$, $\alpha = \frac{e^{2z}}{2}$ and $\beta = -1$. Thus the structure (ϕ, ξ, η, g) is a $3-$dimensional trans-Sasakian structure of type $(\frac{e^{2z}}{2}, -1)$. We also observe that the equations (2.10) and (2.14) satisfies on M_3.

4 $m-$Projectively Flat Trans-Sasakian Structures

In view of $W^* = 0$, (2.19) becomes

$$R(X,Y)Z = \frac{1}{4m}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \tag{4.1}$$

Replacing Z by ξ in (4.1) and then using (2.3), (2.8) and (2.16), we obtain

$$\begin{aligned}
&(\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\phi X - \eta(X)\phi Y)\\
&+(Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y\\
&= \frac{1}{4m}[2m(\alpha^2 - \beta^2)\{\eta(Y)X - \eta(X)Y\} + \eta(Y)QX - \eta(X)QY].
\end{aligned}$$

Again substituting ξ in place of X in the above relation and using (2.1), (2.3), (2.10) and (2.17), we have

$$QY = 2m[(\alpha^2 - \beta^2) - 2\xi\beta]Y + 4m(\xi\beta)\eta(Y)\xi, \tag{4.2}$$

which gives

$$S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z), \tag{4.3}$$

where

$$a = 2m(\alpha^2 - \beta^2 - 2\xi\beta), \qquad b = 4m(\xi\beta). \tag{4.4}$$

From (2.20), it is clear that for the vector field V, we have two situations: the first is that $V \in Span(\xi)$ and second $V \perp \xi$. From the analysis point of view, second situation becomes complex and therefore we are going to consider the first case, *i.e.*, $V = \xi$. Define,

$$\mu(X,Y) = \frac{1}{2}(L_\xi g)(X,Y) + S(X,Y), \tag{4.5}$$

which becomes

$$\mu(X,Y) = (a+\beta)g(X,Y) + (b-\beta)\eta(X)\eta(Y), \tag{4.6}$$

by virtue of (2.21) and (4.3). From (2.3), (2.20) and (4.5), we obtain

$$\mu(X,Y) = \lambda g(X,Y). \tag{4.7}$$

Replacing X and Y with ξ in (4.6) and then use of (2.3) gives

$$\lambda = \mu(\xi,\xi) = 2m(\alpha^2 - \beta^2).$$

Hence if $\frac{1}{2}L_\xi g + S$ is parallel on an n-dimensional m-projectively flat trans-Sasakian structure of type (α,β), then the Ricci soliton (g,ξ,λ) is shrinking, expanding and steady accordingly $\alpha^2 >, <$ and $= \beta^2$, respectively. Thus we have the following theorem.

Theorem 4.1. *A Ricci soliton on an n-dimensional m-projectively flat trans-Sasakian structure of type (α,β) with parallel tensor $\frac{1}{2}L_\xi g + S$ is shrinking, expanding and steady accordingly $\alpha^2 >, <$ and $= \beta^2$, respectively.*

In particular, If we consider that α is a non-zero constant, then (2.10) shows that $\beta = 0$ and therefore $\lambda = 2m\alpha^2 > 0$. Thus we state the following corollaries.

Corollary 4.2. *If $\frac{1}{2}L_\xi g + S$ is parallel on an n-dimensional m-projectively flat α-Sasakian manifold, then the Ricci soliton (g,ξ,λ) is shrinking.*

Corollary 4.3. *If $\frac{1}{2}L_\xi g + S$ is parallel on an n-dimensional m-projectively flat β-Kenmotsu manifold, then the Ricci soliton (g,ξ,λ) is expanding.*

5 Trans-Sasakian Structures satisfying $R.W^* = 0$

In consequence of (2.3) and (2.9), (2.19) becomes

$$\begin{aligned}\eta(W^*(X,Y)Z) = (\alpha^2 - \beta^2)\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\} \\ - \frac{1}{4m}\{\eta(X)S(Y,Z) - \eta(Y)S(X,Z) + g(Y,Z)\eta(QX) \\ - g(X,Z)\eta(QY)\}.\end{aligned} \tag{5.1}$$

Replacing Z by ξ in (5.1) and using (2.1), (2.3) and (2.16), we obtain

$$\eta(W^*(X,Y)\xi) = 0. \tag{5.2}$$

Again substituting $X=\xi$ in (5.1) and using (2.3), (2.16) and (2.17), we find

$$\eta(W^*(\xi,Y)Z)=\frac{1}{2}(\alpha^2-\beta^2)g(Y,Z)-\frac{1}{4m}S(Y,Z). \tag{5.3}$$

We have,

$$\begin{aligned}(R(X,Y).W^*)(Z,U)V &= R(X,Y)W^*(Z,U)V-W^*(R(X,Y)Z,U)V\\ &\quad -W^*(Z,R(X,Y)U)V-W^*(Z,U)R(X,Y)V.\end{aligned} \tag{5.4}$$

Let us suppose that trans-Sasakian structure is $m-$projectively semisymmetric, $i.e.$, $R.W^*=0$ and thus (5.4) becomes

$$\begin{aligned}&R(X,Y)W^*(Z,U)V-W^*(R(X,Y)Z,U)V\\ &-W^*(Z,R(X,Y)U)V-W^*(Z,U)R(X,Y)V=0.\end{aligned}$$

Taking inner product of above equation with ξ, we obtain

$$\begin{aligned}&g(R(X,Y)W^*(Z,U)V,\xi)-g(W^*(R(X,Y)Z,U)V,\xi)\\ &-g(W^*(Z,R(X,Y)U)V,\xi)-g(W^*(Z,U)R(X,Y)V,\xi)=0.\end{aligned}$$

Putting X by ξ in the above equation and using (2.3), (2.8), (2.9) and (5.1), we obtain

$$\begin{aligned}&-'W^*(Z,U,V,Y)+\eta(Y)\eta(W^*(Z,U)V)-\eta(Z)\eta(W^*(Y,U)V)\\ &-\eta(U)\eta(W^*(Z,Y)V)+g(Y,Z)\eta(W^*(\xi,U)V)+g(Y,U)\eta(W^*(Z,\xi)V)\\ &-\eta(V)\eta(W^*(Z,U)Y)=0.\end{aligned} \tag{5.5}$$

Substituting $Y=Z=e_i$ in (5.5) and using (5.2) and (5.3), where $\{e_i, i=1,2,3,...,(2m+1)\}$ be an orthonormal basis of the tangent space at each point of the manifold and then taking the sum for i, $1\le i\le(2m+1)$, we obtain

$$\begin{aligned}&S(U,V)\\ &=\frac{1}{4m+1}[\{r+4m^2(\alpha^2-\beta^2)\}g(U,V)-\{r-2m(2m+1)(\alpha^2-\beta^2)\}\eta(U)\eta(V)],\end{aligned} \tag{5.6}$$

which gives

$$QU=\frac{1}{4m+1}[\{r+4m^2(\alpha^2-\beta^2)\}U-\{r-2m(2m+1)(\alpha^2-\beta^2)\}\eta(U)\xi], \tag{5.7}$$

and

$$r=2m(2m+1)(\alpha^2-\beta^2). \tag{5.8}$$

With the help of (5.8), (5.6) reflects that

$$S(U,V)=2m(\alpha^2-\beta^2)g(U,V). \tag{5.9}$$

Thus the $m-$projectively semisymmetric trans-Sasakian structure is an Einstein manifold. Again equations (5.1), (5.7), (5.8) and (5.9) give $\eta(W^*(X,Y)Z)=0$ and therefore (5.5) shows that the manifold is $m-$projectively flat, $i.e.$, $W^*=0$. Converse part is obvious with (5.4) and (5.5). Thus we have the following result.

Theorem 5.1. *An $n-$dimensional trans-Sasakian structure of type (α, β) is $m-$projectively semisymmetric if and only if it is $m-$projectively flat.*

In view of Theorem 5.1, we have the following corollary.

Corollary 5.2. *An $n-$dimensional $m-$projectively semisymmetric trans-Sasakian structure of type (α, β) is conformally, projectively and concircularly flat.*

Proof. By considering (2.3), (4.1), (5.8) and (5.9), we can easily find that the $m-$projectively semisymmetric trans-Sasakian structure is conformally, projectively and concircularly flat. □

6 On Special Weakly Riccisymmetric Trans-Sasakian Structures

The notion of special weakly Riccisymmetric Riemannian manifold was introduced and studied by Singh and Khan [35]. An $n-$dimensional trans-Sasakian manifold (M_n, g) is called a special weakly Ricci-symmetric manifold $(SWRS)_n$ if

$$(D_X S)(Y, Z) = 2\pi(X)S(Y, Z) + \pi(Y)S(X, Z) + \pi(Z)S(X, Y), \tag{6.1}$$

where π is a $1-$form and is defined by $\pi(X) = g(X, \rho)$ for associated vector field ρ [35, 36]. Taking $Z = \xi$ in (6.1) and using (2.3) and (2.16), we get

$$(D_X S)(Y, \xi) = 2m(\alpha^2 - \beta^2)\{2\pi(X)\eta(Y) + \pi(Y)\eta(X)\} + \pi(\xi)S(X, Y). \tag{6.2}$$

We also know that,

$$(D_X S)(Y, \xi) = D_X S(Y, \xi) - S(D_X Y, \xi) - S(Y, D_X \xi).$$

In consequence of (2.6) and (2.16), above equation becomes

$$\begin{aligned}(D_X S)(Y, \xi) = D_X\{2m(\alpha^2 - \beta^2)\eta(Y)\} - 2m(\alpha^2 - \beta^2)\eta(D_X Y)\\ - S(Y, -\alpha\phi X + \beta(X - \eta(X)\xi)).\end{aligned} \tag{6.3}$$

which is equivalent to

$$\begin{aligned}(D_X S)(Y, \xi) = 4m(\alpha d\alpha(X) - \beta d\beta(X))\eta(Y) - 2m\alpha(\alpha^2 - \beta^2)g(\phi X, Y)\\ + 2m\beta(\alpha^2 - \beta^2)g(X, Y) + \alpha S(Y, \phi X) - \beta S(X, Y).\end{aligned} \tag{6.4}$$

We have from equations (6.2) and (6.4)

$$\begin{aligned}(\beta + \pi(\xi))S(X, Y) = 4m(\alpha d\alpha(X) - \beta d\beta(X))\eta(Y) - 2m\alpha(\alpha^2 - \beta^2)g(\phi X, Y)\\ + 2m\beta(\alpha^2 - \beta^2)g(X, Y) + \alpha S(Y, \phi X)\\ - 2m(\alpha^2 - \beta^2)\{2\pi(X)\eta(Y) + \pi(Y)\eta(X)\}.\end{aligned} \tag{6.5}$$

Setting $Y=\xi$ in (6.5) and then using (2.1), (2.3) and (2.16), we get

$$\alpha d\alpha(X)-\beta d\beta(X)=(\alpha^2-\beta^2)\{\pi(\xi)\eta(X)+\pi(X)\}. \tag{6.6}$$

In view of (6.6), (6.5) takes the form

$$\begin{aligned} S(X,Y)=\frac{2m(\alpha^2-\beta^2)}{\beta+\pi(\xi)}&\{2\pi(\xi)\eta(X)\eta(Y)-\alpha g(\phi X,Y)\\ &+\beta g(X,Y)-\pi(Y)\eta(X)\}+\frac{\alpha}{\beta+\pi(\xi)}S(Y,\phi X), \end{aligned} \tag{6.7}$$

provided $\beta+\pi(\xi)\neq 0$. Substituting $X=\xi$ in (6.7) and using (2.1), (2.3) and (2.16), we obtain

$$\pi(Y)=\pi(\xi)\eta(Y). \tag{6.8}$$

Equations (6.7) and (6.8) gives

$$\begin{aligned} S(X,Y)=\frac{2m(\alpha^2-\beta^2)}{\beta+\pi(\xi)}&\{\pi(\xi)\eta(X)\eta(Y)+\beta g(X,Y)\\ &-\alpha g(\phi X,Y)\}+\frac{\alpha}{\beta+\pi(\xi)}S(Y,\phi X). \end{aligned} \tag{6.9}$$

Interchanging X and Y in (6.9), we get

$$\begin{aligned} S(Y,X)=\frac{2m(\alpha^2-\beta^2)}{\beta+\pi(\xi)}&\{\pi(\xi)\eta(X)\eta(Y)+\beta g(X,Y)\\ &-\alpha g(\phi Y,X)\}+\frac{\alpha}{\beta+\pi(\xi)}S(X,\phi Y). \end{aligned} \tag{6.10}$$

Changing Y with ϕY in (2.15) and using (2.1), (2.3) and (2.16), we have

$$S(\phi X,Y)+S(X,\phi Y)=0.$$

Adding (6.9) and (6.10) and using (2.4), above equation and symmetric properties of Ricci tensor, we find

$$S=a_1 g+b_1\eta\otimes\eta, \tag{6.11}$$

where $a_1=\frac{2m\beta(\alpha^2-\beta^2)}{\beta+\pi(\xi)}$ and $b_1=\frac{2m(\alpha^2-\beta^2)\pi(\xi)}{\beta+\pi(\xi)}$. From (6.11) it is clear that the special weakly Riccisymmetric trans-Sasakian structure is an $\eta-$Einstein. Thus we state:

Theorem 6.1. *An $n-$dimensional special weakly Riccisymmetric trans-Sasakian structure is an $\eta-$Einstein manifold.*

From (2.21), (4.5) and (6.11), we have

$$\mu(X,Y)=(a_1+\beta)g(X,Y)+(b_1-\beta)\eta(X)\eta(Y). \tag{6.12}$$

Setting X and Y with ξ in (6.12) and using (2.3) and (4.7), we obtain

$$\lambda = 2m(\alpha^2 - \beta^2). \tag{6.13}$$

If $\frac{1}{2}L_\xi g+S$ is parallel on a special weakly Riccisymmetric trans-Sasakian structure, then (6.13) implies that the Ricci soliton (g, ξ, λ) on special weakly Riccisymmetric trans-Sasakian structure is shrinking, expanding and steady as $\alpha^2 - \beta^2 >, <$ and $= 0$, respectively. Thus we have the following result.

Theorem 6.2. *If $\frac{1}{2}L_\xi g + S$ is parallel on an $n-$dimensional special weakly Riccisymmetric trans-Sasakian structure, then the Ricci soliton (g, ξ, λ) on it is shrinking, expanding and steady accordingly $\alpha^2 - \beta^2 >, <$ and $= 0$, respectively.*

In view of Theorem 6.2, we state the following corollaries.

Corollary 6.3. *The Ricci soliton (g, ξ, λ) on an $n-$dimensional special weakly Riccisymmetric Sasakian manifold with parallel $\frac{1}{2}L_\xi g + S$ is shrinking.*

Corollary 6.4. *The Ricci soliton (g, ξ, λ) on an $n-$dimensional special weakly Riccisymmetric Kenmotsu manifold with parallel $\frac{1}{2}L_\xi g + S$ is expanding.*

Corollary 6.5. *The Ricci soliton (g, ξ, λ) on an $n-$dimensional special weakly Riccisymmetric cosymplectic manifold with parallel $\frac{1}{2}L_\xi g + S$ is steady.*

7 Generalized Ricci-Recurrent Trans-Sasakian Structures

A non-flat Riemannian manifold M_n of dimension greater than two is called a generalized Ricci-recurrent manifold [37] if its Ricci tensor S satisfies the condition

$$(D_X S)(Y, Z) = A(X)S(Y, Z) + B(X)S(Y, Z), \tag{7.1}$$

where D is the Riemannian connection of the Riemannian metric g and A, B are $1-$forms associated with the vector fields P_1, P_2, respectively on M, *i.e.*

$$A(X) = g(X, P_1); \quad B(X) = g(X, P_2), \tag{7.2}$$

for arbitrary vector fields X, Y and Z. If the $1-$form B vanishes identically, the manifold M_n reduces to the well know Ricci-recurrent manifold [38].

Let M_n be a generalized Ricci-recurrent trans-Sasakian manifold. Putting $Z = \xi$ in (7.1) and using (2.3) and (2.16), we have

$$(D_X S)(Y, \xi) = 2m(\alpha^2 - \beta^2)\{A(X) + B(X)\}\eta(Y), \tag{7.3}$$

In view of (6.4), (7.3) takes the form

$$\begin{aligned} &2m(\alpha^2 - \beta^2)\{A(X) + B(X)\}\eta(Y) \\ &\quad = 4m(\alpha d\alpha(X) - \beta d\beta(X))\eta(Y) - \beta S(X, Y) \\ &\qquad - 2m\alpha(\alpha^2 - \beta^2)g(\phi X, Y) + 2m\beta(\alpha^2 - \beta^2)g(X, Y) + \alpha S(Y, \phi X). \end{aligned} \tag{7.4}$$

Setting $Y=\xi$ in (7.4) and using (2.1), (2.3) and (2.16), we find

$$(\alpha^2-\beta^2)[A(X)+B(X)-\beta\eta(X)]=2(\alpha d\alpha(X)-\beta d\beta(X)) \tag{7.5}$$

and therefore (7.4) assumes the form

$$\begin{aligned}\beta S(X,Y)=&\,2m\beta(\alpha^2-\beta^2)\{g(X,Y)-\eta(X)\eta(Y)\}\\ &-2m\alpha(\alpha^2-\beta^2)g(\phi X,Y)+\alpha S(Y,\phi X).\end{aligned} \tag{7.6}$$

If α is a non-zero constant, then from (2.10) and (7.5), it is clear that the associated vector fields of the $1-$forms A and B are in the opposite directions. Interchanging X and Y in (7.6), we get

$$\begin{aligned}\beta S(Y,X)=&\,2m\beta(\alpha^2-\beta^2)\{g(Y,X)-\eta(X)\eta(Y)\}\\ &-2m\alpha(\alpha^2-\beta^2)g(\phi Y,X)+\alpha S(X,\phi Y).\end{aligned} \tag{7.7}$$

Adding (7.6) and (7.7) and using (2.4), $S(\phi X,Y)+S(X,\phi Y)=0$, we get

$$S(X,Y)=2m(\alpha^2-\beta^2)\{g(X,Y)-\eta(X)\eta(Y)\}. \tag{7.8}$$

Thus we can state the following theorem.

Theorem 7.1. *An $n-$dimensional generalized Ricci-recurrent trans-Sasakian structure of type (α,β) is an $\eta-$Einstein manifold.*

In consequence of (2.3), (2.21), (4.5) and (7.8), we observe that

$$\lambda=0.$$

Thus we have the following result.

Theorem 7.2. *A Ricci soliton (g,ξ,λ) on an $n-$dimensional generalized Ricci-recurrent trans-Sasakian structure of type (α,β) with parallel vector $\frac{1}{2}L_\xi+S$ is steady.*

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(Received 31 July 2012)
(Accepted 17 October 2017)

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