Thai Journal of Mathematics
Volume 16 (2018) Number 3 : 683–692
http://thaijmath.in.cmu.ac.th
Online ISSN 1686-0209

Some Inequalities for the q-Gamma and the q-Polygamma Functions

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Abstract : In this paper, the complete monotonicity property for functions related to the q-gamma and the q-polygamma functions, where q is a positive real number, is proved and exploited to establish some inequalities for the q-gamma and the q-polygamma functions.

Keywords : inequalities; q-gamma function; q-polygamma functions; completely monotonic function.
2010 Mathematics Subject Classification : 33D05; 26D07; 26A48.

1 Introduction

Batir [1] presented a sharp double inequality

$$x^x e^{-x}\sqrt{2\pi(x+a)} < \Gamma(x+1) < x^x e^{-x}\sqrt{2\pi(x+b)} \tag{1.1}$$

for the gamma function. He proved that the inequality (1.1) is valid for $x > 1$ with the best possible constant $a = \frac{1}{6}$ and $b = \frac{e^2}{2\pi} - 1$. His proof depended on strictly decreasing monotone of the function

$$g(x) = \frac{(\Gamma(x+1))^2}{2\pi x^{2x} e^{-2x}} - x, \qquad x > 1 \tag{1.2}$$

and the inequality

$$\sqrt{\pi}x^x e^{-x}(8x^3+4x^2+x+\frac{1}{100})^{\frac{1}{6}} < \Gamma(x+1) < \sqrt{\pi}x^x e^{-x}(8x^3+4x^2+x+\frac{1}{30})^{\frac{1}{6}}. \qquad (1.3)$$

One of the important aims of this paper is to extend the double inequality (1.1) to the q-gamma function for all positive real numbers x and q under slightly different conditions by means of studying the complete monotonicity property for the function

$$F_a(x;q) = \log\Gamma_q(x+1) - x\log[x]_q - \frac{\mathrm{Li}_2(1-q^x)}{\log q} - C_{\hat{q}} - \frac{1}{2}(1-a)H(q-1)\log q - \frac{1}{2}\log[x+a]_q. \qquad (1.4)$$

where $H(\cdot)$ denotes the Heaviside step function, $[x]_q = (1-q^x)/(1-q)$, $\mathrm{Li}_2(z)$ is the dilogarithm function defined for complex argument z as [2]

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t}dt, \qquad z \notin (1,\infty) \qquad (1.5)$$

$\Gamma_q(x)$ is the q-gamma function defined for all positive real variable x as

$$\Gamma_q(x) = (1-q)^{1-x}\prod_{n=0}^{\infty}\frac{1-q^{n+1}}{1-q^{n+x}}, \qquad 0<q<1, \qquad (1.6)$$

$$= (q-1)^{1-x}q^{\frac{x(x-1)}{2}}\prod_{n=0}^{\infty}\frac{1-q^{-(n+1)}}{1-q^{-(n+x)}}, \qquad q>1, \qquad (1.7)$$

$$\hat{q} = \begin{cases} q & \text{if } \ 0<q\le 1 \\ q^{-1} & \text{if } \ q \ge 1 \end{cases}$$

and

$$C_q = \frac{1}{2}\log(2\pi) + \frac{1}{2}\log\left(\frac{q-1}{\log q}\right) - \frac{1}{24}\log q + \log\left(\sum_{m=-\infty}^{\infty}\left(r^{m(6m+1)} - r^{(2m+1)(3m+1)}\right)\right) \qquad (1.8)$$

where $r = \exp(4\pi^2/\log q)$.

From the previous definitions, for a positive x and $q \ge 1$, we get

$$\Gamma_q(x) = q^{\frac{(x-1)(x-2)}{2}}\Gamma_{q^{-1}}(x). \qquad (1.9)$$

Many of the classical facts about the ordinary gamma function have been extended to the q-gamma function (see [3–6] and the references given therein). An important fact for gamma function in applied mathematics as well as in probability is the Stirling formula that gives a pretty accurate idea about the size of gamma function. With the Euler-Maclaurin formula, Moak [5] obtained the following q-analogue of Stirling formula (see also [7])

$$\log\Gamma_q(x) \sim \left(x-\frac{1}{2}\right)\log[x]_q + \frac{\mathrm{Li}_2(1-q^x)}{\log q} + \frac{1}{2}H(q-1)\log q + C_{\hat{q}} + \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left(\frac{\log\hat{q}}{\hat{q}^x-1}\right)^{2k-1}\hat{q}^x P_{2k-3}(\hat{q}^x), \qquad x\to\infty \tag{1.10}$$

where $B_k,\ k\in\mathbb{N}$ are the Bernoulli numbers and P_k is a polynomial of degree k satisfying

$$P_k(z) = (z-z^2)P'_{k-1}(z) + (kz+1)P_{k-1}(z), \qquad P_0 = P_{-1} = 1, \quad k\in\mathbb{N}. \tag{1.11}$$

It is easy to see that

$$\lim_{q\to 1} C_q = C_1 = \frac{1}{2}\log(2\pi), \quad \lim_{q\to 1}\frac{\mathrm{Li}_2(1-q^x)}{\log q} = -x \quad \text{and} \quad P_k(1) = (k+1)! \tag{1.12}$$

and so (1.4) when letting $q\to 1$, tends to the ordinary Stirling formula [2]

$$\log\Gamma(x) \sim \left(x-\frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) + \sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)}\frac{1}{x^{2k-1}}, \qquad x\to\infty. \tag{1.13}$$

An important function related to q-gamma function is the so-called q-digamma function $\psi_q(x)$ which defined as the logarithmic derivative of the q-gamma function

$$\psi_q(x) = \frac{d}{dx}(\log\Gamma_q(x)) = \frac{\Gamma'_q(x)}{\Gamma_q(x)} \tag{1.14}$$

The q-digamma function $\psi_q(x)$ appeared in the work of Krattenthaler and Srivastava [8], when they studied the summations for basic hypergeometric series. Some of its properties have been presented and proved in their work. Also, in their work, they proved that $\psi_q(x)$ tends to the digamma function $\psi(x)$ when letting $q\to 1$. Some inequalities involve the q-gamma function and some of its related functions (q-beta, q-digamma and q-polygamma functions) have been introduced in [9–21]. For more details on the q-digamma function (see [22] and the references given therein).

For all positive real variable x, (1.6) gives

$$\psi_q(x) = -\log(1-q) + \log q\sum_{k=1}^{\infty}\frac{q^{xk}}{1-q^k}, \qquad 0<q<1 \tag{1.15}$$

and (1.7) gives

$$\psi_q(x) = -\log(q-1) + \log q \left[x - \frac{1}{2} - \sum_{k=1}^{\infty} \frac{q^{-xk}}{1-q^{-k}} \right], \qquad q > 1. \tag{1.16}$$

An important recursive formula that we need, obtained in [11] as

$$\psi_q(x+1) = \psi_q(x) - \frac{q^x \log q}{1-q^x}, \qquad q > 0;\ x > 0. \tag{1.17}$$

2 The Complete Monotone Functions

In this section, the complete monotonicity property for the function $F_a(x;q)$ mentioned in (1.4) is proved by means of studying the complete monotonicity of its derivative with respect to x. As a consequence of these results, some inequalities for the q-gamma and the q-polygamma functions are established.

Theorem 2.1. *Let x and q be positive real. Then, the function*

$$G_a(x;q) = \psi_q(x+1) - \log[x]_q + \frac{1}{2} \frac{q^{x+a} \log q}{1-q^{x+a}} \tag{2.1}$$

is strictly completely monotonic on $(0,\infty)$ if $a \geq \frac{\sqrt{259}-7}{42} \simeq 0.216511...$; and the function $-G_b(x;q)$ is strictly completely monotonic on $(-b,\infty)$ if $b \leq 0$.

Proof. When $0 < q < 1$, (1.15), Taylor series for logarithm function and binomial theorem give

$$G_a(x;q) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{q^{xk}}{k(1-q^k)} f(a,y), \qquad y = q^k$$

where

$$f(a,y) = 2y \log y + 2(1-y) + y^a (1-y) \log y.$$

It is obvious that the function $a \mapsto f(a,y)$ is increasing on $\mathbb{R}$ for all $0 < y < 1$. When $a = 0$, the function $f(a,y)$ can be rewritten after simple calculations in the form

$$f(0,y) = -y \sum_{n=3}^{\infty} \frac{\log^n(1/y)}{n!}(n-2) < 0$$

which means that $f(a,y) < 0$ if $a \leq 0$ for all $0 < y < 1$.

When $a > 0$, the function $f(a,y)$ can be rewritten in the form

$$\begin{aligned} f(a,y) = {} & 2y^{a+1} \log y e^{a\log(1/y)} + 2y^{a+1} e^{a\log(1/y)} (e^{a\log(1/y)} - 1) \\ & + y^{a+1} \log y (e^{a\log(1/y)} - 1). \end{aligned}$$

Using the series expansion of the exponential function and Cauchy product rule would yield

$$\begin{aligned} f(a,y) &= y^{a+1}\sum_{n=3}^{\infty}\frac{\log^n(1/y)}{n!}\left(2\sum_{r=0}^{n-1}\binom{n}{r}a^r - 2na^{n-1} - n\right) \\ &= y^{a+1}\sum_{n=3}^{\infty}\frac{\log^n(1/y)}{n!}\left(2\sum_{r=0}^{n-2}\binom{n}{r}a^r - n\right) \\ &= y^{a+1}\sum_{n=3}^{\infty}\frac{\log^n(1/y)}{n!}g_n(a). \end{aligned}$$

It is obvious that $g_n(a)$ is a polynomial of a with degree $n-2$ and all its roots depend on n. According to Descartes' rules of sign, the polynomial $g_n(a)$ has only one positive root, say $a(n)$, depends on n for all $n \geq 3$. It is very difficult to determine this root due to that n has infinite values start with $n = 3$ and thus we suffice to identify a suitable upper bound for $a(n)$ to be close as much as possible to the highest root of $g_n(a)$. Let us now rewrite $g_n(a)$ to be

$$g_n(a) = 2\sum_{r=3}^{n-2}\binom{n}{r}a^r + n(n-1)a^2 + 2na + 2 - n.$$

It is clear that the quadratic polynomial $n(n-1)a^2 + 2na + 2 - n$ has only one positive root depends on n at

$$a(n) = \frac{-n + \sqrt{n^3 - 2n^2 + 2n}}{n(n-1)}, \qquad \text{for all} \quad n \geq 3.$$

Therefore, the polynomial $g_n(a)$ is greater than zero if $a \geq a(n)$ for all $n \geq 3$ and so is the function $f(a,y)$. By differentiating $a(n)$, we get $a'(n) > 0$ if $3 \leq n \leq 6$ and $a'(n) < 0$ if $a \geq 7$ and thus $a(n)$ is decreasing for all $n \geq 7$ which reveals that

$$a(n) \leq a(7) = \frac{\sqrt{259} - 7}{42} \simeq 0.216511... = \alpha, \qquad \text{for all} \quad n \geq 7.$$

Since $a(n) \leq a(6) = (2\sqrt{39} - 6)/30 \simeq 0.216333... < \alpha$ for all $3 \leq n \leq 6$, then $a(n) \leq \alpha$ for all $n \geq 3$ which leads to $g_n(a) > 0$ for all $a > \alpha$ with $n \geq 3$ and so is the function $f(a,y)$. In view of the previous results, we conclude that $G_a(x;q) < 0$ if $a \leq 0$ and $G_a(x;q) > 0$ if $a \geq \alpha$.

When $q \geq 1$, (1.9) and (2.1) give

$$G_a(x;q) = \psi_{q^{-1}}(x+1) - \log[x]_{q^{-1}} + \frac{1}{2}\frac{q^{-(x+a)}\log q^{-1}}{1 - q^{-(x+a)}} = G_a(x;q^{-1}).$$

This completes the proof. □

Corollary 2.2. *Let x and q be positive real with $x > \{0, -b\}$. Then, the double inequalities*

$$\log[x]_q + \frac{q^x \log q}{1-q^x} - \frac{1}{2}\frac{q^{x+a}\log q}{1-q^{x+a}} < \psi_q(x) < \log[x]_q + \frac{q^x \log q}{1-q^x} - \frac{1}{2}\frac{q^{x+b}\log q}{1-q^{x+b}} \tag{2.2}$$

hold true for all $a \geq \frac{\sqrt{259}-7}{42} \simeq 0.216511...$ and $b \leq 0$.
Moreover, for all positive integer n, the class of inequalities

$$\begin{aligned}
&(-1)^n \left(\frac{\log q}{1-q^x}\right)^{n+1} q^x P_{n-1}(q^x) - (-1)^n \left(\frac{\log q}{1-q^x}\right)^{n} q^x P_{n-2}(q^x) \\
&\quad - (-1)^n \frac{1}{2}\left(\frac{\log q}{1-q^{x+a}}\right)^{n+1} q^{x+a} P_{n-1}(q^{x+a}) \\
&< (-1)^n \psi_q^{(n)}(x) \\
&< (-1)^n \left(\frac{\log q}{1-q^x}\right)^{n+1} q^x P_{n-1}(q^x) - (-1)^n \left(\frac{\log q}{1-q^x}\right)^{n} q^x P_{n-2}(q^x)) \\
&\quad - (-1)^n \frac{1}{2}\left(\frac{\log q}{1-q^{x+b}}\right)^{n+1} q^{x+b} P_{n-1}(q^{x+b})
\end{aligned} \tag{2.3}$$

is valid for all $a \geq \frac{\sqrt{259}-7}{42} \simeq 0.216511...$ and $b \leq 0$, where P_n is the polynomial mentioned in Section 1.

Proof. Theorem 2.1 tells that $G_b(x;q) < 0 < G_a(x;q)$ which is equivalent (2.2) with using the identity (1.17), and

$$(-1)^n G_b^{(n)}(x;q) < 0 < (-1)^n G_a^{(n)}(x;q), \qquad n \in \mathbb{N}$$

which is equivalent (2.3) with using the identities (1.17) and

$$\frac{d^n}{dx^n}\left[\frac{q^x \log q}{1-q^x}\right] = \left(\frac{\log q}{1-q^x}\right)^{n+1} q^x P_{n-1}(q^x), \qquad n = 0, 1, 2, \cdots$$

which was proved by Moak [5]. □

Remark 2.3. *When letting $q \to 1$, we have the two sided-inequality*

$$\log x - \frac{1}{x} + \frac{1}{2(x+a)} < \psi(x) < \log x - \frac{1}{2x}, \qquad x > 0 \tag{2.4}$$

holds for $a = \frac{\sqrt{259}-7}{42} \simeq 0.216511...$. The left hand side refines the inequality

$$\log x - \frac{1}{x} < \psi(x) < \log x - \frac{1}{2x} \tag{2.5}$$

which obtained by Anderson and Qiu [23] for all $x > 0$.

Theorem 2.4. *Let x and q be positive real. Then, the function $F_a(x;q)$ defined as in* (1.4) *is strictly completely monotonic on $(-a,\infty)$ if $a \le 0$ and is strictly increasing on $(0,\infty)$ if $a \ge \frac{\sqrt{259}-7}{42} \simeq 0.216511....$*

Proof. The function $F_a(x;q)$ defined as in (1.4) can be read as

$$F_a(x;q) = \mu_q(x) + \nu_a(x;q) \tag{2.6}$$

where

$$\mu_q(x) = \log\Gamma_q(x) - \left(x - \frac{1}{2}\right)\log[x]_q - \frac{\mathrm{Li}_2(1-q^x)}{\log q} - C_{\hat{q}} - \frac{1}{2}H(q-1)\log q \tag{2.7}$$

and

$$\nu_a(x;q) = \frac{1}{2}\left(\log(1-q^x) - \log(1-q^{x+a}) + aH(q-1)\log q\right). \tag{2.8}$$

Using Moak formula (1.10) yields $\lim_{x\to\infty}\mu_q(x) = 0$ for all $q > 0$. Obviously, $\lim_{x\to\infty}\nu_a(x;q) = 0$ if $0 < q < 1$ and when $q > 1$, we get

$$\begin{aligned}\lim_{x\to\infty}\nu_a(x;q) &= \frac{1}{2}\Big(x\log q + \log(1-q^{-x}) - (x+a)\log q - \log(1-q^{-(x+a)}) + a\log q\Big)\\ &= 0.\end{aligned}$$

These conclude that $\lim_{x\to\infty}F_a(x;q) = 0$ for all $q > 0$ and consequently from Theorem 2.1, we obtain the proof of this theorem. □

Corollary 2.5. *Let x and q be positive real. Then, the q-gamma function holds the two-sided inequalities*

$$\begin{aligned}&[x]_q^x q^{\frac{1}{2}(1-b)H(q-1)} S_{\hat{q}}\sqrt{2\pi[x+b]_q}\exp\left(\frac{Li_2(1-q^x)}{\log q}\right) < \Gamma_q(x+1)\\ &< [x]_q^x q^{\frac{1}{2}(1-a)H(q-1)} S_{\hat{q}}\sqrt{2\pi[x+a]_q}\exp\left(\frac{Li_2(1-q^x)}{\log q}\right), \quad x > \max\{0,-b\}\end{aligned} \tag{2.9}$$

and the one-sided inequalities

$$\Gamma_q(x+1) \ge [x]^x\sqrt{\frac{1-q^{x+a}}{1-q^a}}\exp\left(\frac{Li_2(1-q^x)}{\log q}\right), \qquad x > 0 \tag{2.10}$$

for all $a \ge \frac{\sqrt{259}-7}{42} \simeq 0.216511...$ and $b \le 0$ with the best possible constants $a = \frac{\sqrt{259}-7}{42}$ and $b = 0$, where

$$S_q = q^{\frac{-1}{24}}\sqrt{\frac{q-1}{\log q}}\sum_{m=-\infty}^{\infty}\left(r^{m(6m+1)} - r^{(2m+1)(3m+1)}\right).$$

Proof. The monotonicity properties of the function $F_a(x;q)$ in Theorem 2.4 give $F_a(x;q) < 0 < F_b(x;q)$ which is equivalent (2.9) and

$$F_a(x;q) > F_a(0;q) = -C_{\hat{q}} - \frac{1}{2}H(q-1)\log q - \frac{1}{2}\log[a]_q$$

which is equivalent (2.10). □

Remark 2.6. *When letting $q \to 1$, (2.9) tends to*

$$x^x e^{-x}\sqrt{2\pi(x+b)} < \Gamma(x+1) < x^x e^{-x}\sqrt{2\pi(x+a)} \tag{2.11}$$

which is valid for all $x > 0$ with the best possible constants $a = \frac{\sqrt{259}-7}{42}$ and $b = 0$. Although the values of the constants a, b in (1.1) are better than here but we extend the values of x to start with zero. Also, when letting $q \to 1$, (2.10) tends to

$$\Gamma(x+1) \geq x^x e^{-x}\sqrt{\frac{x+a}{a}}, \qquad x > 0 \tag{2.12}$$

with the best possible constants $a = \frac{\sqrt{259}-7}{42}$. The inequality (2.12) for the gamma function appears to be a new.

3 Conclusion

In this paper, the Moak formula (1.10) is used to prove the complete monotonicity property for the function $F_a(x;q)$ for all positive real q. The inequalities (2.2), (2.3), (2.9) and (2.10) come as an application of the complete monotonicity property for the function $F_a(x;q)$ and $G_a(x;q)$. When letting $q \to 1$, these inequalities give the inequalities (2.4), (2.5), (2.11) and (2.12). Some of them are considered refinement of existing inequalities and (2.12) is shown to be new.

Acknowledgements : This project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, under grant no. (G:150-130-1439). The author, therefore, acknowledges with thanks DSR for technical and financial support.

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(Received 9 January 2013)
(Accepted 11 February 2016)