Thai Journal of Mathematics
Volume 16 (2018) Number 3 : 675–681

http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

ϵ-Closed Sets

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Abstract : Our goal in this paper is to introduce the relatively new notions of ϵ-closed and ϵ-generalized closed sets. Several properties and connections to other well-known weak and strong closed sets are discussed. ϵ-generalized continuous and ϵ-generalized irresolute functions and their basic properties and relations to other continuities are explored.

Keywords : ϵ-open set; ϵ-closed set; ϵ-generalized closed set; ϵ-generalized continuous function.
2010 Mathematics Subject Classification : 54C08; 54H05.

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1 Introduction

Let $(X, \mathfrak{T})$ be a topological space (or simply, a space). If $A \subseteq X$, then the closure of A and the interior of A will be denoted by $Cl(A)$ and $Int(A)$, respectively. A subset $A \subseteq X$ is called *semi-open* [1] if there exists an open set $O \in \mathfrak{T}$ such that $O \subseteq A \subseteq Cl(O)$. Clearly A is a semi-open set if and only if $A \subseteq Cl(Int(A))$. A complement of a semi-open set is called *semi-closed*. A is a *generalized closed* (= *g-closed*) set [2] if $A \subseteq U$and $U \in \mathfrak{T}$ implies that $Cl(A) \subseteq U$. For more on the preceding notions, the reader is referred to [3–24].

A function $f : (X, \mathfrak{T}) \to (Y, \mathfrak{T}')$ is called *g-continuous* [25] if $f^{-1}(V)$ is g-closed in $(X, \mathfrak{T})$ for every closed set V of $(Y, \mathfrak{T}')$ and *contra-semi-continuous* [26] if $f^{-1}(V)$ is semi-open in $(X, \mathfrak{T})$ for every closed set V of $(Y, \mathfrak{T}')$.

We introduce the relatively new notions of ϵ-closed sets, which is closely related to the class of closed subsets. We investigate several characterizations of ϵ-open and ϵ-closed notions via the operations of interior and closure. In Section 3, we introduce the notion of ϵ-generalized closed sets and study connections to other weak and strong forms of generalized closed sets. In addition several interesting

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properties and constructions of ϵ-generalized closed sets are discussed. Section 4 is devoted to introducing and studying ϵ-generalized continuous and ϵ-generalized irresolute functions and connections to other similar forms of continuity.

2 ϵ-Closed Sets

We begin this section by introducing the notions of ϵ-open and ϵ-closed subsets.

Definition 2.1. Let A be a subset of a space $(X, \mathfrak{T})$. The *ϵ-interior* of A is the union of all open subsets of X whose closures are contained in $Cl(A)$, and is denoted by $Int_\epsilon(A)$. A is called *ϵ-open* if $A = Int_\epsilon(A)$.The complement of a ϵ-open subset is called *ϵ-closed.* Equivalently, a subset A of X is ϵ-closed if $A = Cl_\epsilon(A)$, where $Cl_\epsilon(A) = \{x \in X : Cl(U) \cap Cl(A) \neq \emptyset, U \in \mathfrak{T}, x \in U\}$.

Clearly $Int(A) \subseteq Int_\epsilon(A) \subseteq Cl(A)$ and $A \subseteq Cl(A) \subseteq Cl_\epsilon(A)$ and hence every ϵ-closed set is closed, but the converses need not be true.

Example 2.2. Let $X = \{a, b, c\}$ and $\mathfrak{T} = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Set $A = \{a, c\}$. Then A is closed, but not ϵ-closed as $Cl_\epsilon(A) = X$.

Even the intersection of two ϵ-open subsets needs not be ϵ-open.

Example 2.3. Let $X = \{a, b, c, d\}$ and $\mathfrak{T} = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}, \{a, b\}, \{a, b, c\}\}$. Then $A = \{a, b\}$ and $B = \{a, c\}$ are ϵ-open subsets, but $A \cap B = \{c\}$ is not ϵ-open.

Next, we show that arbitrary union of ϵ-open subsets are ϵ-open.

Theorem 2.4. *If $(X, \mathfrak{T})$ is a space, then arbitrary union of ϵ-open subsets are ϵ-open.*

Proof. If $\{A_\alpha : \alpha \in \Delta\}$ is a collection of ϵ-open subsets of X, then for every $\alpha \in \Delta$, $Int_\epsilon(A\alpha) = A_\alpha$. Hence

$$\begin{aligned} Int_\epsilon(\cup_{\alpha\in\Delta} A\alpha) &= \cup\{U \in \mathfrak{T} : Cl(U) \subseteq Cl(\cup_{\alpha\in\Delta} A\alpha)\} \\ &= \cup\{U \in \mathfrak{T} : Cl(U) \subseteq \cup_{\alpha\in\Delta} Cl(A\alpha)\} \\ &= \cup_{\alpha\in\Delta}(Int_\epsilon(A\alpha)) \\ &= \cup_{\alpha\in\Delta} A_\alpha. \end{aligned}$$

Hence $\cup_{\alpha\in\Delta} A\alpha$ is ϵ-open. □

Corollary 2.5. *Arbitrary intersection of ϵ-closed subsets are ϵ-closed, while finite unions of ϵ-closed subsets need not be ϵ-closed.*

Next we show that $A \subseteq Int_\epsilon(A)$ and $Int_\epsilon(A) \subseteq A$ need not be true.

Example 2.6. Consider the space in Example 2.2. Then $\{c\} \subsetneq Int_\epsilon(\{c\}) = \emptyset$ and $Int_\epsilon(\{a, b\}) = X \subsetneq \{a, b\}$.

Remark 2.7. *If A is a dense subset of X, then $Int_\epsilon(A) = X$.*

Lemma 2.8. *The intersection of a closed set with a ϵ-closed set is closed.*

Proof. Let A be a closed set and B be a ϵ-closed set. For all $x \in Cl(A \cap B)$, the for every open set U containing x, $U \cap (A \cap B) \neq \emptyset$. Hence $U \cap A \neq \emptyset$ and $Cl(U) \cap Cl(B) \neq \emptyset$. Thus $x \in Cl(A) \cap Cl_\epsilon(B) = A \cap B$.Therefore, $A \cap B$ is closed. □

Corollary 2.9. *The union of an open set with a ϵ-open set is open.*

Lemma 2.10. *If A is a semi-closed subset of an E.D. space X, then $Cl(A) = Cl_\epsilon(A)$.*

Proof. We only need to show $Cl_\epsilon(A) \subseteq Cl(A)$ when A is semi-closed. For all $x \in Cl_\epsilon(A)$ and all U open set containing x, we have $Cl(U) \cap Cl(A) \neq \emptyset$. As X is E.D., $Cl(A) = Int(Cl(A))$ and hence $Cl(U) \cap Int(Cl(A)) \neq \emptyset$. Thus there exists $y \in$Cl(U) and $y \in Int(Cl(A))$ which is open. Hence $U \cap Int(Cl(A)) \neq \emptyset$ and as A is semi-closed, $U \cap A \neq \emptyset$. Therefore $Cl_\epsilon(A) \subseteq Cl(A)$. □

We remark that X being an E.D. space is necessary in Lemma 2.10.

Example 2.11. Consider the space in Example 2.3. Then $Cl_\epsilon(\{b\}) = X \neq \{a, b, d\} = Cl(\{b\})$.

Corollary 2.12. *In an E.D. space, a subset is closed if and only if it is ϵ-closed.*

3 ϵ-Generalized Closed Sets

In this section, we introduce the notion of ϵ-generalized closed set. Moreover, several interesting properties and constructions of these subsets are discussed.

Definition 3.1. A subset A of a space X is called *ϵ-generalized closed* (ϵ-g-closed) if whenever U is an open subset containing A, we have $Cl_\epsilon(A) \subseteq U$. A is ϵ-g-open if $X \backslash A$ is ϵ-g-closed.

Theorem 3.2. *A subset A of $(X, \mathfrak{T})$ is ϵ-g-open if and only if $F \subseteq Int\epsilon(A)$, whenever $F \subseteq A$ and F is closed in $(X, \mathfrak{T})$.*

Proof. Let A be an ϵ-g-open set and F be a closed subset such that $F \subseteq A$. Then $X \backslash A \subseteq X \backslash F$. As $X \backslash A$ is ϵ-g-closed and as $X \backslash F$ is open, $Cl_\epsilon(X \backslash A) \subseteq X \backslash F$. So $F \subseteq X \backslash Cl_\epsilon(X \backslash A) = Int_\epsilon(A)$.

Conversely, if $X \backslash A \subseteq U$ where U is open, then the closed set $X \backslash U \subseteq A$. Thus $X \backslash U \subseteq Int_\epsilon(A) = X \backslash Cl_\epsilon(X \backslash A)$ and so $Cl_\epsilon(X \backslash A) \subseteq U$. □

Next we show the class of ϵ-g-closed sets is properly placed between the classes of closed and ϵ-closed sets. Moreover, the class g-closed sets is properly placed between the classes of closed sets and ϵ-g-closed sets. A closed set is trivially g-closed and clearly every ϵ-closed set is closed and every ϵ-g-closed set is g-closed as $Cl(A) \subseteq Cl_\epsilon(A)$ for every subset A of a space X. In Example 2.2, $A = \{a, c\}$ is a closed set that is not ϵ-closed. In Example 2.3, $A = \{a, b, d\}$ is not ϵ-closed, but as the only super set of A is X, A is ϵ-g-closed.

Example 3.3. Consider the space $X = \{a, b, c, d\}$ and $\mathfrak{T} = \{\emptyset, X, \{a, b, d\}, \{c, d\}, \{d\}\}$. Then $A = \{a, b\}$ is closed and hence is g-closed, but not $\epsilon-$g-closed as $Cl_\epsilon(A) = X$.

The following is an immediate result from Lemma 2.10:

Theorem 3.4. *If A is a semi-closed subset of an E.D. space X, the following are equivalent:*

(1) A is ϵ-g-closed;

(2) A is g-closed

Its clear that if $A \subseteq B$, then $Int_\epsilon(A) \subseteq Int_\epsilon(B)$ and $Cl_\epsilon(A) \subseteq Cl_\epsilon(B)$.

Lemma 3.5. *If A and B are subsets of a space X, then $Cl_\epsilon(A \cup B) = Cl_\epsilon(A) \cup Cl_\epsilon(B)$ and $Cl_\epsilon(A \cap B) \subseteq Cl_\epsilon(A) \cap Cl_\epsilon(B)$.*

Proof. Since A and B are subsets of $A \cup B$, $Cl_\epsilon(A) \cup Cl_\epsilon(B) \subseteq Cl_\epsilon(A \cup B)$. On the other hand, if $x \in Cl_\epsilon(A \cup B)$ and U is an open set containing x, then $Cl(U) \cap Int(A \cup B) \neq \emptyset$. Hence either $Cl(U) \cap Cl(A) \neq \emptyset$ or $Cl(U) \cap Int(B) \neq \emptyset$. Thus $x \in Cl_\epsilon(A) \cup Cl_\epsilon(B)$.

Finally since $A \cap B$ is a subset of A and B, $Cl_\epsilon(A \cap B) \subseteq Cl_\epsilon(A) \cap Cl_\epsilon(B)$. □

Corollary 3.6. *Finite union of ϵ-g-closed sets is ϵ-g-closed.*

While the finite intersection of ϵ-g-closed sets needs not be ϵ-g-closed.

Example 3.7. Let $X = \{a, b, c, d, e\}$ and $\mathfrak{T} = \{\emptyset, X, \{a, b\}, \{c\}, \{a, b, c\}\}$. Then $A = \{a, c, d\}$ and $B = \{b, c, e\}$ are ϵ-g-closed sets as the only super set of them is X, but $A \cap B = \{c\}$ is not ϵ-g-closed.

Theorem 3.8. *The intersection of an ϵ-g-closed set with a ϵ-closed set is ϵ-g-closed.*

Proof. Let A be a ϵ-g-closed set and B be a ϵ-closed set. Let U be an open set containing $A \cap B$. Then $A \subseteq U \cup X \backslash B$. Since $X \backslash B$ is ϵ-open, by Corollary 2.9, $U \cup X \backslash B$ is open and since A is ϵ-g-closed, $Cl_\epsilon(A \cap B) \subseteq Cl_\epsilon(A) \cap Cl_\epsilon(B) = Cl_\epsilon(A) \cap B \subseteq (U \cup X \backslash B) \cap B = U \cap B \subseteq U$. □

4 ϵ-g-Continuous and ϵ-g-Irresolute Functions

Definition 4.1. A function $f : (X, \mathfrak{T}) \to (Y, \mathfrak{T}')$ is called

(1) *ϵ-g-continuous* if $f^{-1}(V)$ is ϵ-g-closed in $(X, \mathfrak{T})$ for every closed set V of $(Y, \mathfrak{T}')$,

(2) *ϵ-g-irresolute* if $f^{-1}(V)$ is ϵ-g-closed in $(X, \mathfrak{T})$ for every ϵ-g-closed set V of $(Y, \mathfrak{T}')$.

Lemma 4.2. *Let $f : (X, \mathfrak{T}) \to (Y, \mathfrak{T}')$ be ϵ-g-continuous. Then f is g-continuous but not conversely.*

Proof. Follows from the fact that every ϵ-g-closed set is g-closed. □

Example 4.3. Consider the space $(X, \mathfrak{T})$ in Example 3.3 and the identity function $f : (X, \mathfrak{T}) \to (X, \mathfrak{T}')$ where $\mathfrak{T}' = \{\emptyset, X, \{c, d\}\}$. Since $f^{-1}(\{a, b\}) = \{a, b\} \neq Cl_\epsilon(\{a, b\})$, f is not ϵ-g-continuous, but f is continuous and hence g-continuous.

Even the composition of ϵ-g-continuous functions needs not be ϵ-g-continuous.

Example 4.4. Let f be the function in Example 3.7. Let $\mathfrak{T}'' = \{\emptyset, \{a, b, d, e\}, X\}$. Let $g : (X, \mathfrak{T}') \to (X, \mathfrak{T}'')$ be the identity function. It is easily observed that g is also ϵ-g-continuous as the only super set of $\{c\}$ is X. But the composition function $g \circ f : (X, \mathfrak{T}) \to (X, \mathfrak{T}'')$ is not ϵ-g-continuous since $\{c\}$ is closed in $(X, \mathfrak{T}'')$, but not ϵ-g-closed in $(X, \mathfrak{T})$.

We end this section by giving a necessary condition for ϵ-g-continuous function to be ϵ-g-irresolute.

Theorem 4.5. *If $f : (X, \mathfrak{T}) \to (Y, \mathfrak{T}')$ is bijective, open and ϵ-g-continuous, then f is ϵ-g-irresolute.*

Proof. Let V be a ϵ-g-closed subset of Y and let $f^{-1}(V) \subseteq O$, where $O \in \mathfrak{T}$. Clearly, $V \subseteq f(O)$. Since $f(O) \in \mathfrak{T}'$ and since V is ϵ-g-closed, $Cl_\epsilon(V) \subseteq f(O)$ and thus $f^{-1}(Cl_\epsilon(V)) \subseteq O$. Since f is ϵ-generalized continuous and since $Cl_\epsilon(V)$ is closed in Y, $f^{-1}(Cl_\epsilon(V)$ is ϵ-g-closed. $f^{-1}(Cl_\epsilon(V) \subseteq Cl_\epsilon(f^{-1}(Cl_\epsilon(V))) = f^{-1}(Cl_\epsilon(V)) \subseteq O$. Therefore, $f^{-1}(V)$ is ϵ-g-closed and hence, f is ϵ-g-irresolute. □

Acknowledgement : I would like to thank the referee for his comments and suggestions on the manuscript.

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(Received 15 April 2013)
(Accepted 10 November 2016)