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Global Behavior of a Fourth Order Rational Difference Equation

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Abstract : In this paper, we investigate the global stability, periodic nature, and the oscillation of solutions of the difference equation

$$x_{n+1} = \frac{Ax_{n-3}}{B + Cx_{n-2}^2}, \qquad n = 0, 1, 2, \ldots$$

where $A, C, B > 0$ and the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0$ are nonnegative real numbers. We show that under certain conditions unbounded solutions will be obtained.

Keywords : difference equation; periodic solution; globally asymptotically stable.
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1 Introduction

Difference equations, although their forms look very simple, it is extremely difficult to understand thoroughly the global behaviors of their solutions. One can refer to [1, 2].

The study of nonlinear rational difference equations of higher order is of paramount importance, since we still know so little about such equations. It is

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worthwhile to point out that although several approaches have been developed for finding the global character of difference equations [2–5], relatively a large number of difference equations have not been thoroughly understood yet [6–9]. Hence a great challenge and reward for further investigations are remained and are still at their infancy.

In this paper, we study the global asymptotic stability of the difference equation

$$x_{n+1} = \frac{Ax_{n-3}}{B + Cx_{n-2}^2}, \qquad n = 0, 1, 2, \ldots \tag{1.1}$$

where $A, C, B > 0$ and the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0$ are nonnegative real numbers.

Here we recall some results which will be useful in the sequel.

Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}) \qquad , n = 0, 1, \ldots \tag{1.2}$$

where $f : R^{k+1} \to R$.

Definition 1.1. [2] An equilibrium point for equation (1.2) is a point $\bar{x} \in R$ such that $\bar{x} = f(\bar{x}, \bar{x}, \ldots, \bar{x})$.

Definition 1.2. [2]

1. An equilibrium point $\bar{x}$ for equation (1.2) is called *locally stable* if for every $\epsilon > 0$, there exists a $\delta > 0$ such that every solution $\{x_n\}$ with initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in]\bar{x} - \delta, \bar{x} + \delta[$ is such that $x_n \in]\bar{x} - \epsilon, \bar{x} + \epsilon[$ for all $n \in \mathbb{N}$. Otherwise $\bar{x}$ is said to be *unstable*.
2. The equilibrium point $\bar{x}$ of equation (1.2) is called *locally asymptotically stable* if it is locally stable and there exists $\gamma > 0$ such that for any initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in]\bar{x} - \gamma, \bar{x} + \gamma[$, the corresponding solution $\{x_n\}$ tends to $\bar{x}$.
3. An equilibrium point $\bar{x}$ for equation (1.2) is called a *global attractor* if every solution $\{x_n\}$ converges to $\bar{x}$ as $n \to \infty$.
4. The equilibrium point $\bar{x}$ for equation (1.2) is called *globally asymptotically stable* if it is locally asymptotically stable and global attractor.

Suppose that f is continuously differentiable in some open neighborhood of $\bar{x}$. Let

$$a_i = \frac{\partial f}{\partial x_{n-i}}(\bar{x}, \ldots, \bar{x}), \quad \text{for} \quad i = 0, 1, \ldots, k$$

denote the partial derivatives of $f(x_n, x_{n-1}, \ldots, x_{n-k})$ with respect to x_{n-i} evaluated at the equilibrium point $\bar{x}$ of equation (1.2). Then the equation

$$y_{n+1} = \sum_{i=0}^{k} a_i y_{n-i} \qquad , n = 0, 1, \ldots \tag{1.3}$$

is called the linearized equation associated with equation (1.2) about the equilibrium point $\bar{x}$, and the equation

$$\lambda^{k+1} - \sum_{i=0}^{k} a_i \lambda^{k-i} = 0 \tag{1.4}$$

is called the characteristic equation associated with equation (1.3) about the equilibrium point $\bar{x}$.

Theorem 1.3. [2] *Assume that f is a C^1 function and let $\bar{x}$ be an equilibrium point of equation* (1.2). *Then the following statements are true:*

1. *If all roots of equation* (1.4) *lie in the open disk $|\lambda| < 1$, then $\bar{x}$ is locally asymptotically stable.*
2. *If at least one root of equation* (1.4) *has absolute value greater than one, then $\bar{x}$ is unstable.*

Now we give the definitions for the positive and negative semicycle of a solution of equation (1.2) relative to an equilibrium point $\bar{x}$.

Definition 1.4. [8] A positive semicycle of a solution $\{x_n\}_{n=-1}^{\infty}$ of equation (1.2) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all greater than or equal to the equilibrium $\bar{x}$, with $l \geq -1$ and $m \leq \infty$ and such that

either $l = -1$, or $l > -1$ and $x_{l-1} < \bar{x}$

and

either $m = \infty$, or $m < \infty$ and $x_{m+1} < \bar{x}$.

Definition 1.5. [8] A negative semicycle of a solution $\{x_n\}_{n=-1}^{\infty}$ of equation (1.2) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all less than or equal to the equilibrium $\bar{x}$, with $l \geq -1$ and $m \leq \infty$ and such that

either $l = -1$, or $l > -1$ and $x_{l-1} \geq \bar{x}$

and

either $m = \infty$, or $m < \infty$ and $x_{m+1} \geq \bar{x}$.

Theorem 1.6. [8] *Assume that $f \in C[(0,\infty) \times (0,\infty), (0,\infty)]$ is such that: $f(x,y)$ is decreasing in x for each fixed y, and $f(x,y)$ is increasing in y for each fixed x. Let $\bar{x}$ be a positive equilibrium of equation* (1.2). *Then except possibly for the first semicycle, every solution of equation* (1.2) *has semicycles of length one.*

The change of variables $x_n = \sqrt{\frac{A}{C}}\, y_n$ reduces equation (1.1) to the difference equation

$$y_{n+1} = \frac{y_{n-3}}{\gamma + y_{n-2}^2}, \qquad n = 0, 1, 2, \cdots \tag{1.5}$$

where $\gamma = \frac{B}{A}$.

2 Local Asymptotic Stability of the Equilibrium Points

Now we examine the equilibrium points of equation (1.5) and their local asymptotic behavior. Clearly equation (1.5) has two nonnegative equilibrium points $\bar{y} = 0$ and $\bar{y} = \sqrt{1-\gamma}$ when $\gamma < 1$ and $\bar{y} = 0$ only when $\gamma \geq 1$.

The linearized equation associated with equation (1.5) about $\bar{y}$ is

$$z_{n+1} + \frac{2\bar{y}^2}{(\gamma+\bar{y}^2)^2} z_{n-2} - \frac{1}{\gamma+\bar{y}^2} z_{n-3} = 0. \tag{2.1}$$

The characteristic equation associated with this equation is

$$\lambda^4 + \frac{2\bar{y}^2}{(\gamma+\bar{y}^2)^2}\lambda - \frac{1}{\gamma+\bar{y}^2} = 0. \tag{2.2}$$

We summarize the results of this section in the following theorem.

Theorem 2.1. *1. If $\gamma > 1$, then the zero equilibrium point is locally asymptotically stable.*

2. If $\gamma < 1$, then the equilibrium point $\bar{y} = 0$ is unstable (repeller) and the equilibrium point $\bar{y} = \sqrt{1-\gamma}$ is unstable (saddle point).

Proof. The linearized equation (2.1) about $\bar{y} = 0$ is $z_{n+1} - \frac{1}{\gamma} z_{n-3} = 0$. The characteristic equation associated with this equation is $\lambda^4 - \frac{1}{\gamma} = 0$.

1. If $\gamma > 1$, then $|\lambda| < 1$ and $\bar{y} = 0$ is locally asymptotically stable.
2. If $\gamma < 1$, then $\bar{y} = 0$ is unstable (repeller).
   Now, the characteristic equation (2.2) about $\bar{y} = \sqrt{1-\gamma}$ is

   $$\lambda^4 + 2(1-\gamma)\lambda - 1 = 0.$$

   It is clear that this equation has a root in the interval $(0, 1)$ and another root in the interval $(-\infty, -1)$, from which the result follows. □

3 Global Behavior of Equation (1.5)

Assume that $\gamma > 1$. Our main result is the following theorem.

Theorem 3.1. *If $\gamma > 1$, then the zero equilibrium point is globally asymptotically stable.*

Proof. Let $\{y_n\}_{n=-3}^{\infty}$ be a solution of equation (1.5). Hence

$$y_{4m+i} = \frac{y_{4(m-1)+i}}{\gamma + y^2_{4(m-1)+i+1}} < \frac{y_{4(m-1)+i}}{\gamma}, \quad i = 1, 2, 3, 4.$$

This implies that

$$\lim_{m\to\infty} y_{(4m+i)} = 0, \qquad i = 1, 2, 3, 4.$$

Therefore, $\lim_{m\to\infty} y_n = 0$.
In view of Theorem (3.1), $\bar{y} = 0$ is globally asymptotically stable. □

Example 3.2. Figure 1. ($\gamma > 1$) shows that the solution $\{y_n\}_{n=-3}^{\infty}$ of the equation

$$y_{n+1} = \frac{y_{n-3}}{1.2 + y_{n-2}^2}, \quad n = 0, 1, \ldots$$

with initial conditions $y_{-3} = 0.2$, $y_{-2} = 1$, $y_{-1} = 3$ and $y_0 = 0.1$ converges to zero.

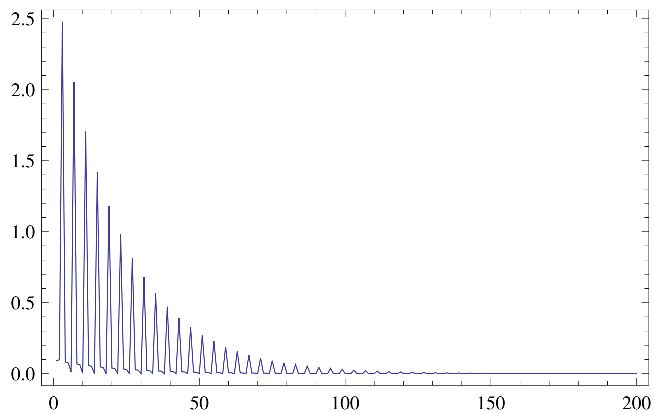

Figure 1: The difference equation $y_{n+1} = \frac{y_{n-3}}{1.2+y_{n-2}^2}$

4 Periodic Nature

Theorem 4.1. *Suppose that $\gamma = 1$. Then every solution of equation* (1.5) *converges to a period* 4 *solution and there exist periodic solutions of equation* (1.5) *with prime period* 4.

Proof. Assume that $\gamma = 1$ and let $\{y_n\}_{n=-3}^{\infty}$ be a solution of equation (1.5). Then the subsequences $\{y_{4n+i}\}_{n=-1}^{\infty}$ are decreasing for each $1 \le i \le 4$. Let

$$\lim_{n\to\infty} y_{4n+i} = \rho_i, \qquad i = 1, 2, 3, 4.$$

It is clear that $\{\ldots, \rho_1, \rho_2, \rho_3, \rho_4, \rho_1, \rho_2, \rho_3, \rho_4, \ldots\}$ is a period 4 solution of equation (1.5).

Now let φ_1, φ_2 be distinct positive real numbers. It follows that the sequence

$$\{\ldots, \varphi_1, 0, \varphi_2, 0, \varphi_1, 0, \varphi_2, 0, \ldots\}$$

is a periodic solution of equation (1.5) with prime period 4. This completes the proof. □

5 Oscillation and Unbounded Solutions

In this section, we study the semicycle analysis and the existence of unbounded solutions for equation (1.5).

Theorem 5.1. *Assume that $\gamma < 1$ and let $\{y_n\}_{n=-3}^{\infty}$ be a nontrivial solution of equation (1.5). If $\bar{y}$ is the unique positive equilibrium of equation (1.5), then the following statements are true.*

1. *Let the initial conditions be such that either*
   (C_1) $y_{-2}, y_0 > \bar{y}$ *and* $y_{-3}, y_{-1} < \bar{y}$
   or
   (C_2) $y_{-2}, y_0 < \bar{y}$ *and* $y_{-3}, y_{-1} > \bar{y}$
   is satisfied. Then $\{y_n\}_{n=-3}^{\infty}$ oscillates about $\bar{y}$ with semicycles of length one.
2. *There exist solutions of equation (1.5) which are neither bounded nor persist.*

Proof. 1. The proof follows immediately from Theorem 1.6.

2. Let $\{y_n\}_{n=-3}^{\infty}$ be a solution of equation (1.5) with initial conditions $y_{-3}, y_{-2}, y_{-1}, y_0$ such that $y_{-3}, y_{-1} < \bar{y} < y_{-2}, y_0$. Then

$$y_1 = \frac{y_{-3}}{\gamma + y_{-2}^2} < y_{-3}, \qquad y_2 = \frac{y_{-2}}{\gamma + y_{-1}^2} > y_{-2},$$

$$y_3 = \frac{y_{-1}}{\gamma + y_0^2} < y_{-1}, \quad \text{and} \quad y_4 = \frac{y_0}{\gamma + y_1^2} > y_0.$$

By induction we get,

$$y_{4m+i} < y_{4(m-1)+i}, \quad i = 1, 3,$$

and

$$y_{4m+i} > y_{4(m-1)+i}, \quad i = 2, 4.$$

It follows that, for each $j = 1, 2$ we have that $\lim_{m\to\infty} y_{4m+2j} = L_{2j} \in (\sqrt{1-\gamma}, \infty]$ and $\lim_{m\to\infty} y_{4m+2j+1} = L_{2j+1} \in [0, \sqrt{1-\gamma})$.
We show that for each $j = 1, 2$, $L_{2j+1} = 0$. For the sake of contradiction, suppose that there exists $j \in \{1, 2\}$ with $L_{2j+1} \in (0, \sqrt{1-\gamma})$. Then

$$L_{2j+1} = \lim_{m\to\infty} y_{4(m+1)+2j+1} = \lim_{m\to\infty} \frac{y_{4m+2j+1}}{\gamma + y_{4m+2j+2}^2} = \frac{L_{2j+1}}{\gamma + L_{2j+2}^2}.$$

As $\lim_{m\to\infty} y_{4m+2j+1} = L_{2j+1} \in (0, \sqrt{1-\gamma})$, we have

$$1 = \gamma + L_{2j+2}^2 > 1,$$

which is a contradiction. It follows that, for each $j = 1, 2$ we have that $L_{2j+1} = 0$, and so $\lim_{n\to\infty} y_{2n+1} = 0$.

Now we show that $L_{2j} = \infty$ for each $j = 1, 2$. For the sake of contradiction, suppose that there exists $j \in \{1, 2\}$ with $L_{2j} \in (\sqrt{1-\gamma}, \infty)$. Then

$$L_{2j} = \lim_{m\to\infty} y_{4(m+1)+2j} = \lim_{m\to\infty} \frac{y_{4m+2j}}{\gamma + y^2_{4(m)+2j+1}} = \frac{L_{2j}}{\gamma}.$$

This implies that $\gamma = 1$, which is a contradiction. Therefore, $\lim_{n\to\infty} y_{2n} = \infty$, and the proof is complete. □

Example 5.2. Figure 2. ($\gamma = 1$) shows that the solution $\{y_n\}_{n=-3}^{\infty}$ of the equation

$$y_{n+1} = \frac{y_{n-3}}{1 + y^2_{n-2}}, \quad n = 0, 1, \ldots$$

with initial conditions $y_{-3} = 1.9$, $y_{-2} = 1$, $y_{-1} = 0.4$ and $y_0 = 0.1$ converges to a period-4 solution.

Example 5.3. Figure 3. ($\gamma < \frac{1}{2}$) shows that the solution $\{y_n\}_{n=-3}^{\infty}$ of the equation

$$y_{n+1} = \frac{y_{n-3}}{0.75 + y^2_{n-2}}, \quad n = 0, 1, \ldots$$

with initial conditions $y_{-3} = 0.4$, $y_{-2} = 0.6$, $y_{-1} = 0.1$ and $y_0 = 0.7$ is nether bounded nor persist.

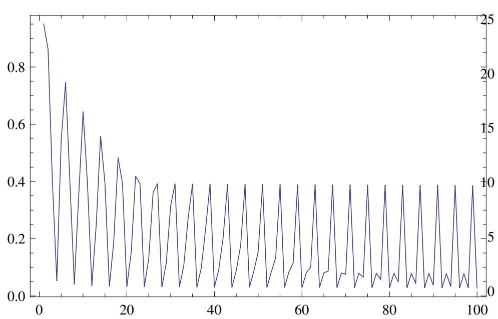

Figure 2: $y_{n+1} = \frac{y_{n-3}}{1+y^2_{n-2}}$

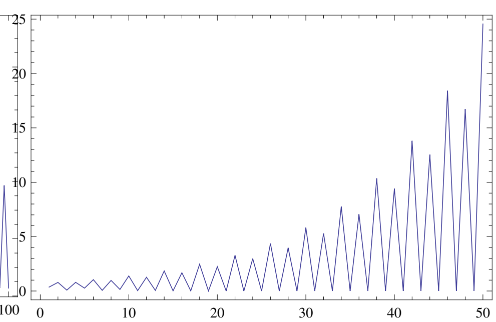

Figure 3: $y_{n+1} = \frac{y_{n-3}}{0.75+y^2_{n-2}}$

Theorem 5.4. *Assume that $\gamma < 1/2$ and let $\{y_n\}_{n=-3}^{\infty}$ be a nontrivial solution of equation* (1.5)*. If $\bar{y}$ is the unique positive equilibrium of equation* (1.5)*, then the following statements are true.*

1. *If there exists a positive integer K, $y_{n-3} \geq \bar{y}$ for every $n \geq K$, then $\{y_n\}_{n=-3}^{\infty}$ converges monotonically to the equilibrium $\bar{y}$.*

2. *If the initial conditions satisfy* $0 < y_{-3}, y_{-2}, y_{-1}, y_0 < \sqrt{1-\gamma}$ *with* $y_{-3} > y_{-2}, y_{-1} > y_0$ *and* $y_{-2} < \sqrt{1-\gamma}(\gamma + y_{-1}^2)$*, then* $\{y_n\}_{n=1}^{\infty}$ *oscillates about* $\bar{y}$ *with semicycles of length one. Moreover, the subsequences* $\{y_{4n+i}\}_{n=0}^{\infty}$ *are increasing when* $i = 1, 3$ *and decreasing when* $i = 2, 4$.
3. *If the initial conditions satisfy* $\sqrt{1-\gamma} < y_{-3}, y_{-2}, y_{-1}, y_0$ *with* $y_{-3} < y_{-2}, y_{-1} < y_0$ *and* $y_{-2} > \sqrt{1-\gamma}(\gamma + y_{-1}^2)$*, then* $\{y_n\}_{n=1}^{\infty}$ *oscillates about* $\bar{y}$ *with semicycles of length one. Moreover, the subsequences* $\{y_{4n+i}\}_{n=0}^{\infty}$ *are decreasing when* $i = 1, 3$ *and increasing when* $i = 2, 4$.

Proof. 1. Let $\{y_n\}$ be a solution of equation (1.5). We will assume that there exists a positive integer K such that $y_{n-3} \geq \bar{y}$ for every $n \geq K$.

It is sufficient to show that $\{y_n\}$ is a decreasing sequence for $n \geq K$.
For, assume for the sake of contradiction that for some $n_0 \geq K$, $y_{n_0} > y_{n_0-1}$. Then there exist $m_0 \in \mathbb{N}$ and $i_0 \in \{1, 2, 3, 4\}$ such that $n_0 = 4m_0 + i_0$. Clearly the condition $\gamma < \frac{1}{2}$ implies that the function

$$f(x) = \frac{x}{\gamma + x^2}$$

is decreasing. Then

$$y_{4(m_0+1)+i-1} = \frac{y_{4m_0+i-1}}{\gamma + y_{4m_0+i}^2} < \frac{y_{4m_0+i}}{\gamma + y_{4m_0+i}^2} = f(y_{4m_0+i}) < f(\bar{y}) = \bar{y},$$

which is a contradiction.

2. Assume that the initial conditions satisfy $0 < y_{-3}, y_{-2}, y_{-1}, y_0 < \sqrt{1-\gamma}$ with $y_{-3} > y_{-2}, y_{-1} > y_0$ and $y_{-2} < \sqrt{1-\gamma}(\gamma + y_{-1}^2)$. Then

$$y_1 = \frac{y_{-3}}{\gamma + y_{-2}^2} > \frac{y_{-2}}{\gamma + y_{-2}^2} > \bar{y},$$

$$y_2 = \frac{y_{-2}}{\gamma + y_{-1}^2} < \sqrt{1-\gamma} = \bar{y},$$

$$y_3 = \frac{y_{-1}}{\gamma + y_0^2} > \frac{y_0}{\gamma + y_0^2} > \bar{y},$$

and

$$y_4 = \frac{y_0}{\gamma + y_1^2} < \bar{y}.$$

Using Theorem 1.6, we get the result. It follows by induction that that

$$y_{4m+i} > \bar{y}, \quad i = 1, 3,$$

and

$$y_{4m+i} < \bar{y}, \quad i = 2, 4.$$

But as

$$y_{4(m+1)+i} = \frac{y_{4m+i}}{\gamma + y_{4m+i+1}^2},$$

we have for $i = 1, 3$ that

$$y_{4(m+1)+i} = \frac{y_{4m+i}}{\gamma + y_{4m+i+1}^2} > \frac{y_{4m+i}}{\gamma + \bar{y}^2} = y_{4m+i}$$

and for $i = 2, 4$, we have

$$y_{4(m+1)+i} = \frac{y_{4m+i}}{\gamma + y_{4m+i+1}^2} < \frac{y_{4m+i}}{\gamma + \bar{y}^2} = y_{4m+i}.$$

This completes the proof.

3. The proof is similar to (2) and will be omitted. □

Example 5.5. Figure 4. ($\gamma < \frac{1}{2}$) shows that for the solution $\{y_n\}_{n=-3}^{\infty}$ of the equation

$$y_{n+1} = \frac{y_{n-3}}{0.25 + y_{n-2}^2}, \quad n = 0, 1, \ldots$$

with initial conditions $y_{-3} = 0.7$, $y_{-2} = 0.6$, $y_{-1} = 0.8$ and $y_0 = 0.78$, the solution oscillates about $\bar{y} \simeq 0.866$) with semycycles of length one. Moreover, the subsequences $\{y_{4n+i}\}_{n=0}^{\infty}$ are increasing when $i = 1, 3$ and decreasing when $i = 2, 4$.

Example 5.6. Figure 5. ($\gamma < \frac{1}{2}$) shows that for the solution $\{y_n\}_{n=-3}^{\infty}$ of the equation

$$y_{n+1} = \frac{y_{n-3}}{0.25 + y_{n-2}^2}, \quad n = 0, 1, \ldots$$

with initial conditions $y_{-3} = 0.82$, $y_{-2} = 0.85$, $y_{-1} = 0.8$ and $y_0 = 0.81$, the solution oscillates about $\bar{y} \simeq 0.866$) with semycycles of length one. Moreover, the subsequences $\{y_{4n+i}\}_{n=0}^{\infty}$ are decreasing when $i = 1, 3$ and increasing when $i = 2, 4$.

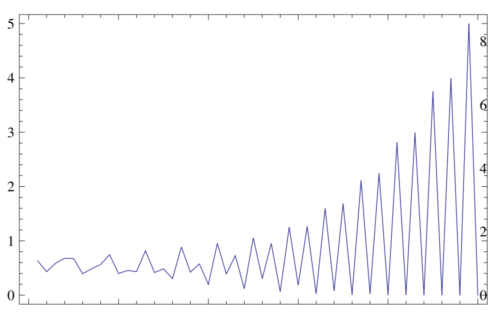

Figure 4: $y_{n+1} = \frac{y_{n-3}}{0.25+y_{n-2}^2}$

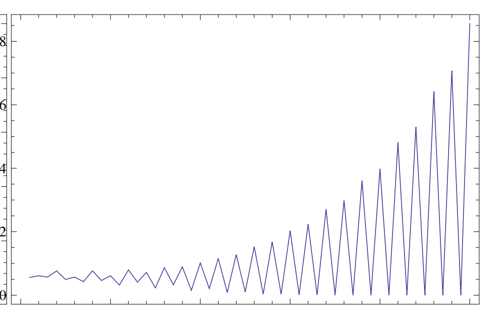

Figure 5: $y_{n+1} = \frac{y_{n-3}}{0.25+y_{n-2}^2}$

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