Thai Journal of Mathematics
Volume 16 (2018) Number 3 : 651–663
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Fine Spectrum of the Generalized Difference Operator Δ_{uv} on the Sequence Space c_0

Parmeshwary Dayal Srivastava and Sudhanshu Kumar[1]

Department of Mathematics, Indian Institute of Technology, Kharagpur
Kharagpur - 721302, India
e-mail : pds@maths.iitkgp.ernet.in (P. D. Srivastava)
sudhanshu_tomar@yahoo.com (S. Kumar)

Abstract : The purpose of this paper is to determine spectrum and fine spectrum of the operator Δ_{uv} on the sequence space c_0. The operator Δ_{uv} on sequence space c_0 is defined as $\Delta_{uv}x = (u_n x_n + v_{n-1}x_{n-1})_{n=0}^{\infty}$ satisfying certain conditions, where $x_{-1} = 0$ and $x = (x_n) \in c_0$. In this paper we have obtained the results on the spectrum and point spectrum for the operator Δ_{uv} on the sequence space c_0. Further, the results on continuous spectrum, residual spectrum and fine spectrum of the operator Δ_{uv} on sequence space c_0 are also derived.

Keywords : spectrum of an operator; generalized difference operator; sequence space.
2010 Mathematics Subject Classification : 41A17; 47H09.

1 Introduction

Let $u = (u_k)$ and $v = (v_k)$ be sequences such that
(i) u is either a constant sequence or sequence of distinct real numbers with $U = \lim\limits_{k\to\infty} u_k$,
(ii) v is a sequence of nonzero real numbers with $V = \lim\limits_{k\to\infty} v_k \neq 0$, and
(iii) $|U - u_k| < |V|$ for each $k \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$.

We define the operator Δ_{uv} on the sequence space c_0 as follows:

$$\Delta_{uv}x = (u_n x_n + v_{n-1}x_{n-1})_{n=0}^{\infty} \text{ with } x_{-1} = 0, \text{ where } x = (x_n) \in c_0. \tag{1.1}$$

[1]Corresponding author.

It is easy to verify that the operator Δ_{uv} can be represented by the matrix

$$\Delta_{uv} = \begin{pmatrix} u_0 & 0 & 0 & \dots \\ v_0 & u_1 & 0 & \dots \\ 0 & v_1 & u_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.2}$$

The spectrum of the Cesaro operator on the sequence space c_0 is investigated by Reade [1], Akhmedov and Basar [2]. Spectrum of the Cesaro operator on sequence spaces bv_0 and bv is obtained by Okutoyi [3] and Okutoyi [4], respectively. Furthermore, Coskun [5] studied the spectrum and fine spectrum for p-Cesaro operator acting on the space c_0. Yildirim [6] and [7] examined fine spectrum of the Rhaly operator on sequence spaces c_o and c. The spectrum and fine spectrum of the difference operator Δ over the sequence spaces c_0 and c is determined by Altay and Basar [8], where $\Delta x = (x_n - x_{n-1})$. The fine spectrum of the Zweier matrix Z^s on sequence spaces l_1 and bv is obtained by Altay and Karakus [9], where s is a real number with $s \neq 0, 1$ and $Z^s x = (s x_n + (1-s)x_{n-1})$. Altay and Basar [10] determined fine spectrum of the operator $B(r,s)$ over sequence spaces c_0 and c, where $B(r,s)x = (rx_n + sx_{n-1})$. Recently, spectrum and fine spectrum of the operator $B(r,s,t)$ on sequence spaces c_0 and c is studied by Furkan, Bilgic and Altay [11], where $B(r,s,t)x = (rx_n + sx_{n-1} + tx_{n-2})$.

In this paper we determine spectrum, point spectrum, continuous spectrum and residual spectrum of the operator Δ_{uv} on the sequence space c_0. It is easy to verify that by choosing suitably u and v sequences, one can get easily the operators such as $B(r,s)$, Z^s etc. Choosing $u = (r)$, $v = (s)$ and $u = (s)$, $v = (1-s)$, then the operator Δ_{uv} reduces to $B(r,s)$ and Z^s, respectively. Similarly, if $u = (1)$, $v = (-1)$ and $u = (0)$, $v = (1)$, then the operator Δ_{uv} reduces to Δ and right-shift operator, respectively. Thus, the results of this paper generalizes the corresponding results of many operator whose matrix representation has diagonal and post-diagonal elements studied by earlier authors.

2 Preliminaries and Notation

Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. The set of all bounded linear operators on X into itself is denoted by $B(X)$. The adjoint $T^{\times} : X^{\star} \to X^{\star}$ of T is defined by

$$\left(T^{\times}\phi\right)(x) = \phi\,(Tx) \text{ for all } \phi \in X^{\star} \text{ and } x \in X.$$

Clearly, $T^{\times}$ is a bounded linear operator on the dual space $X^{\star}$.

Let $X \neq \{\mathbf{0}\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. With T, we associate the operator $T_\alpha = (T - \alpha I)$, where α is a complex number and I is the identity operator on $D(T)$. The inverse of T_α (if exists) is denoted by T_α^{-1} and known as the resolvent operator of T.

Since the spectral theory is concerned with many properties of T_α and T_α^{-1}, which depend on α, so we are interested the set of those α in the complex plane for which T_α^{-1} exists or T_α^{-1} is bounded or domain of T_α^{-1} is dense in X.

Definition 2.1. ([12], pp. 371) Let $X \neq \{\mathbf{0}\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. A *regular* value of T is a complex number α such that
(R1) T_α^{-1} exists,
(R2) T_α^{-1} is bounded,
(R3) T_α^{-1} is defined on a set which is dense in X.

Resolvent set $\rho(T, X)$ of T is the set of all regular values α of T. Its complement $\sigma(T, X) = \mathbb{C} \setminus \rho(T, X)$ in the complex plane $\mathbb{C}$ is called *spectrum* of T. The spectrum $\sigma(T, X)$ is further partitioned into three disjoint sets namely point spectrum, continuous spectrum and residual continuous as follows:

Point spectrum $\sigma_p(T, X)$ is the set of all $\alpha \in \mathbb{C}$ such that T_α^{-1} does not exist, i.e., condition (R1) fails. The element of $\sigma_p(T, X)$ is called *eigenvalue* of T.

Continuous spectrum $\sigma_c(T, X)$ is the set of all $\alpha \in \mathbb{C}$ such that conditions (R1) and (R3) hold but condition (R2) fails, i.e., T_α^{-1} exists, domain of T_α^{-1} is dense in X but T_α^{-1} is unbounded.

Residual spectrum $\sigma_r(T, X)$ is the set of all $\alpha \in \mathbb{C}$ such that T_α^{-1} exists but do not satisfy condition (R3), i.e., domain of T_α^{-1} is not dense in X. The condition (R2) may or may not holds good.

Goldberg's classification of operator T_α ([13], pp. 58): Let X be a Banach space and $T_\alpha \in B(X)$, where α is a complex number. Again, let $R(T_\alpha)$ and T_α^{-1} denote the range and inverse of the operator T_α, respectively. Then the following possibilities may occur;
(A) $R(T_\alpha) = X$,
(B) $R(T_\alpha) \neq \overline{R(T_\alpha)} = X$,
(C) $\overline{R(T_\alpha)} \neq X$,
and
(1) T_α is injective and T_α^{-1} is continuous,
(2) T_α is injective and T_α^{-1} is discontinuous,
(3) T_α is not injective.

Remark 2.2. *Combining (A), (B), (C) and (1), (2), (3); we get nine different cases. These are labeled by A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2 and C_3. The notation $\alpha \in A_2\sigma(T, X)$ means the operator $T_\alpha \in A_2$, i.e., $R(T_\alpha) = X$ and T_α is injective but T_α^{-1} is discontinuous. Similarly others.*

Remark 2.3. *If α is a complex number such that $T_\alpha \in A_1$ or $T_\alpha \in B_1$, then α belongs to the resolvent set $\rho(T, X)$ of T on X. The other classification gives rise to the fine spectrum of T.*

Lemma 2.4. ([14], pp. 129) *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(c_0)$ from c_0 to itself if and only if*

(i) *the rows of A in l_1 and their l_1 norms are bounded, and*
(ii) *the columns of A are in c_0.*

Note: The operator norm of T is the *supremum* of the l_1 norms of the rows.

Lemma 2.5. ([13], pp. 59) *T has a dense range if and only if $T^{\times}$ is one to one, where $T^{\times}$ denotes the adjoint operator of the operator T.*

Lemma 2.6. ([13], pp. 60) *The adjoint operator $T^{\times}$ of T is onto if and only if T has a bounded inverse.*

3 Main Results

3.1 Spectrum and Point Spectrum of the Operator Δ_{uv} on the Sequence Space c_0

In this section we obtain spectrum and point spectrum of the operator Δ_{uv} on c_0.

Theorem 3.1. *The operator $\Delta_{uv} : c_0 \to c_0$ is a bounded linear operator and*

$$\|\Delta_{uv}\|_{B(c_0)} = \sup_k (|u_k| + |v_{k-1}|) .$$

Proof. Proof is simple. So we omit. □

Theorem 3.2. *Spectrum of the operator Δ_{uv} on the sequence space c_0 is given by*

$$\sigma(\Delta_{uv}, c_0) = \{\alpha \in \mathbb{C} : |U - \alpha| \leqslant |V|\} .$$

Proof. The proof of this theorem is divided into two parts. In the first part, we show that $\sigma(\Delta_{uv}, c_0) \subseteq \{\alpha \in \mathbb{C} : |U - \alpha| \leqslant |V|\}$, which is equivalent to

$$\alpha \in \mathbb{C} \text{ with } |U - \alpha| > |V| \text{ implies } \alpha \notin \sigma(\Delta_{uv}, c_0), \text{ i.e., } \alpha \in \rho(\Delta_{uv}, c_0).$$

In the second part, we establish the reverse inclusion, i.e.,

$$\{\alpha \in \mathbb{C} : |U - \alpha| \leqslant |V|\} \subseteq \sigma(\Delta_{uv}, c_0).$$

Part I : Let $\alpha \in \mathbb{C}$ with $|U - \alpha| > |V|$. Clearly, $\alpha \neq U$ and $\alpha \neq u_k$ for each $k \in \mathbb{N}_0$ as it does not satisfy this condition. Further, $(\Delta_{uv} - \alpha I) = (a_{nk})$ reduces to a triangle and hence has an inverse $(\Delta_{uv} - \alpha I)^{-1} = (b_{nk})$, where

$$(b_{nk}) = \begin{pmatrix} \frac{1}{(u_0 - \alpha)} & 0 & 0 & \cdots \\ \frac{-v_0}{(u_0 - \alpha)(u_1 - \alpha)} & \frac{1}{(u_1 - \alpha)} & 0 & \cdots \\ \frac{v_0 v_1}{(u_0 - \alpha)(u_1 - \alpha)(u_2 - \alpha)} & \frac{-v_1}{(u_1 - \alpha)(u_2 - \alpha)} & \frac{1}{(u_2 - \alpha)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} . \quad (3.1)$$

By Lemma 2.4, the operator $(\Delta_{uv} - \alpha I)^{-1} \in B(c_0)$ if

(i) series $\sum_{k=0}^{\infty} |b_{nk}|$ is convergent for each $n \in \mathbb{N}_0$ and $\sup_n \sum_{k=0}^{\infty} |b_{nk}| < \infty$, and

(ii) $\lim_{n\to\infty} |b_{nk}| = 0$ for each $k \in \mathbb{N}_0$.

In order to show $\sup_n \sum_{k=0}^{\infty} |b_{nk}| < \infty$, first we prove that the series $\sum_{k=0}^{\infty} |b_{nk}|$ is convergent for each $n \in \mathbb{N}_0$. For this consider $S_n = \sum_{k=0}^{\infty} |b_{nk}|$. Clearly, the series

$$S_n = \left|\frac{v_0 v_1 \cdots v_{n-1}}{(u_0-\alpha)(u_1-\alpha)\cdots(u_n-\alpha)}\right| + \ldots + \left|\frac{v_{n-1}}{(u_{n-1}-\alpha)(u_n-\alpha)}\right| + \left|\frac{1}{(u_n-\alpha)}\right| \quad (3.2)$$

is convergent for each $n \in \mathbb{N}_0$. Now we claim that $\sup_n S_n$ is finite. For this, suppose

$$\beta = \lim_{n\to\infty} \left|\frac{v_{n-1}}{u_n - \alpha}\right|, \text{ which is equal to } \left|\frac{V}{U-\alpha}\right|.$$

So, $0 < \beta < 1$. We choose $\epsilon > 0$ such that $\beta + \epsilon < 1$. Since $\lim_{n\to\infty} \left|\frac{v_{n-1}}{u_n-\alpha}\right| = \beta$, so there exists a positive integer n_0 such that

$$\left|\frac{v_{n-1}}{u_n-\alpha}\right| < \beta + \epsilon \quad \text{and} \quad \left|\frac{1}{u_n-\alpha}\right| < \frac{\beta+\epsilon}{m} \text{ for all } n \geqslant n_0, \quad (3.3)$$

where m is a lower bound of bounded sequence $v = (v_k)$.

For $n \geqslant n_0$, S_n can be write as

$$\begin{aligned} S_n &= \left|\frac{v_0 v_1 \cdots v_{n_0-2} v_{n_0-1} \cdots v_{n-1}}{(u_0-\alpha)(u_1-\alpha)(u_2-\alpha)\cdots(u_{n_0-1}-\alpha)(u_{n_0}-\alpha)\cdots(u_n-\alpha)}\right| + \cdots \\ &+ \left|\frac{v_{n_0-1}\cdots v_{n-1}}{(u_{n_0-1}-\alpha)(u_{n_0}-\alpha)\cdots(u_n-\alpha)}\right| + \left|\frac{v_{n_0}\cdots v_{n-1}}{(u_{n_0}-\alpha)(u_{n_0+1}-\alpha)\cdots(u_n-\alpha)}\right| \\ &+ \cdots + \left|\frac{1}{u_n-\alpha}\right|. \end{aligned}$$

Take

$$M = \max\left\{\left|\frac{1}{u_0-\alpha}\right|, \cdots, \left|\frac{1}{u_{n_0-1}-\alpha}\right|, \left|\frac{v_0}{u_1-\alpha}\right|, \cdots, \left|\frac{v_{n_0-2}}{u_{n_0-1}-\alpha}\right|\right\}.$$

Using inequalities in (3.3), we have

$$\begin{aligned} S_n &< M^{n_0}(\beta+\epsilon)^{n-n_0+1} + \cdots + M(\beta+\epsilon)^{n-n_0+1} + \frac{(\beta+\epsilon)^{n-n_0+1}}{m} + \cdots + \frac{(\beta+\epsilon)}{m} \\ &= (\beta+\epsilon)^{n-n_0+1}\left[M^{n_0} + \cdots + M\right] + \frac{(\beta+\epsilon)}{m}\left[1 + \cdots + (\beta+\epsilon)^{n-n_0}\right] \\ &< \left[M^{n_0} + \cdots + M\right] + \frac{1}{m}\left[\frac{1}{1-(\beta+\epsilon)}\right] < \infty. \end{aligned}$$

Thus, $S_n < \infty$ for each $n \in \mathbb{N}$ and hence $\sup_n S_n < \infty$.
Again, since $\beta < 1$, therefore $\left|\dfrac{v_{n-1}}{u_n - \alpha}\right| < 1$ for large n and consequently,

$$\lim_{n\to\infty} |b_{n0}| = \lim_{n\to\infty} \left|\frac{v_0 v_1 \cdots v_{n-1}}{(u_0-\alpha)(u_1-\alpha)\cdots(u_n-\alpha)}\right| = 0.$$

Similarly, we can show that $\lim_{n\to\infty} |b_{nk}| = 0$ for all $k = 1, 2, 3, \cdots$.
Thus,

$$(\Delta_{uv} - \alpha I)^{-1} \in B(c_0) \text{ for } \alpha \in \mathbb{C} \text{ with } |U - \alpha| > |V|. \tag{3.4}$$

Next, we show that domain of the operator $(\Delta_{uv} - \alpha I)^{-1}$ is dense in c_0, which follows if the operator $(\Delta_{uv} - \alpha I)$ is onto. Suppose $(\Delta_{uv} - \alpha I)\, x = y$, which gives

$$x = (\Delta_{uv} - \alpha I)^{-1} y, \text{ i.e., } x_n = \left((\Delta_{uv} - \alpha I)^{-1} y\right)_n, \ n \in \mathbb{N}_0.$$

Thus for every $y \in c_0$, we can find $x \in c_0$ such that $(\Delta_{uv} - \alpha I)x = y$.
Hence we have

$$\sigma(\Delta_{uv}, c_0) \subseteq \{\alpha \in \mathbb{C} : |U - \alpha| \leqslant |V|\}. \tag{3.5}$$

Part II: Conversely it is required to show

$$\{\alpha \in \mathbb{C} : |U - \alpha| \leqslant |V|\} \subseteq \sigma(\Delta_{uv}, c_o). \tag{3.6}$$

We first prove inclusion (3.6) under the assumption $\alpha \neq U$ and $\alpha \neq u_k$ for each $k \in \mathbb{N}_0$. Let $\alpha \in \mathbb{C}$ with $|U - \alpha| \leqslant |V|$. Clearly, $(\Delta_{uv} - \alpha I)$ is a triangle and hence $(\Delta_{uv} - \alpha I)^{-1}$ exists. So condition (R1) is satisfied but condition (R2) fails as can be seen below:

Suppose $\alpha \in \mathbb{C}$ with $|U - \alpha| < |V|$. Then $\beta > 1$. This means that $\left|\dfrac{v_{n-1}}{u_n - \alpha}\right| > 1$ for large n and consequently, $\lim_{n\to\infty} |b_{n0}| \neq 0$. Hence

$$(\Delta_{uv} - \alpha I)^{-1} \notin B(c_0) \text{ for } \alpha \in \mathbb{C} \text{ with } |U - \alpha| < |V|. \tag{3.7}$$

Next, we consider $\alpha \in \mathbb{C}$ with $|U - \alpha| = |V|$. Proof is by contradiction. Equality (3.2) can be write as

$$S_n = \left|\frac{v_{n-1}}{u_n - \alpha}\right| S_{n-1} + \left|\frac{1}{u_n - \alpha}\right|. \tag{3.8}$$

Taking limit both sides of equality (3.8) and using condition $|U - \alpha| = |V|$, we get $\left|\dfrac{1}{V}\right| = 0$, which is not possible. Thus, $\lim_{n\to\infty} S_n$ does not exist and consequently, $\sup_n S_n$ is unbounded. Hence

$$(\Delta_{uv} - \alpha I)^{-1} \notin B(c_0) \text{ for } \alpha \in \mathbb{C} \text{ with } |U - \alpha| = |V|. \tag{3.9}$$

Finally, we prove the inclusion (3.6) under the assumption $\alpha = U$ and $\alpha = u_k$ for all $k \in \mathbb{N}_0$. For this, we consider

$$(\Delta_{uv} - \alpha I)\, x = \begin{pmatrix} (u_0-\alpha)x_0 \\ v_0x_0 + (u_1-\alpha)x_1 \\ \vdots \\ -v_{k-1}x_{k-1} + (u_k-\alpha)x_k \\ \vdots \end{pmatrix}.$$

Case(i): If (u_k) is a constant sequence, say $u_k = U$ for all $k \in \mathbb{N}_0$, then for $\alpha = U$

$$(\Delta_{uv} - UI)x = \mathbf{0} \quad \Rightarrow x_0 = 0,\ x_1 = 0,\ x_2 = 0, \cdots.$$

This shows that the operator $(\Delta_{uv} - UI)$ is one to one, but $R(\Delta_{uv} - UI)$ is not dense in c_0. So condition (R3) fails. Hence $U \in \sigma(\Delta_{uv}, c_0)$.

Case(ii): If (u_k) is a sequence of distinct real numbers, then the series S_k is divergent for each $\alpha = u_k$ from equality (3.2) and consequently, $\sup_n S_n$ is unbounded. Hence

$$(\Delta_{uv} - \alpha I)^{-1} \notin B(c_0) \text{ for } \alpha = u_k. \tag{3.10}$$

So condition (R2) fails. Hence $u_k \in \sigma\left(\Delta_{uv}, c_0\right)$ for all $k \in \mathbb{N}_0$.

Again, taking limit both sides of equality (3.9), we see that $\lim_{n\to\infty} S_n$ does not exist for $\alpha = U$. So $\sup_n S_n$ is unbounded. Hence

$$(\Delta_{uv} - \alpha I)^{-1} \notin B(c_0) \text{ for } \alpha = U. \tag{3.11}$$

So condition (R2) fails. Hence $U \in \sigma(\Delta_{uv}, c_0)$. Thus, in this case also $u_k \in \sigma(\Delta_{uv}, c_0)$ for all $k \in \mathbb{N}_0$ and $U \in \sigma(\Delta_{uv}, c_0)$. Hence we have

$$\{\alpha \in \mathbb{C} : |U - \alpha| \leqslant |V|\} \subseteq \sigma\left(\Delta_{uv}, c_0\right). \tag{3.12}$$

From inclusions (3.5) and (3.12), we get

$$\sigma(\Delta_{uv}, c_0) = \{\alpha \in \mathbb{C} : |U - \alpha| \leqslant |V|\}.$$

This completes the proof. □

Theorem 3.3. *Point spectrum of the operator Δ_{uv} on the sequence space c_0 is*

$$\sigma_p(\Delta_{uv}, c_0) = \emptyset.$$

Proof. For the point spectrum of the operator Δ_{uv}, we find those α in $\mathbb{C}$ such that the matrix equation $\Delta_{uv}x = \alpha x$ is satisfy for non-zero vector $x = (x_k)$ in c_0.

Consider $\Delta_{uv}x = \alpha x$ for $x \neq \mathbf{0} = (0, 0, \cdots)$ in c_0, which gives system of equations

$$\left.\begin{aligned} u_0x_0 &= \alpha x_0 \\ v_0x_0 + u_1x_1 &= \alpha x_1 \\ &\vdots \\ v_{k-1}x_{k-1} + u_kx_k &= \alpha x_k \\ &\vdots \end{aligned}\right\} \tag{3.13}$$

The proof of this Theorem is divided into two cases.
Case(i): Suppose (u_k) is a constant sequence, say $u_k = U$ for all $k \in \mathbb{N}_0$. Let x_t be the first nonzero entry of the sequence $x = (x_n)$. Then equation $v_{t-1}x_{t-1} + Ux_t = \alpha x_t$ gives $\alpha = U$, and from the equation $v_t x_t + Ux_{t+1} = \alpha x_{t+1}$, we get $x_t = 0$, which is a contradiction to our assumption. Hence $\sigma_p(\Delta_{uv}, c_0) = \emptyset$.
Case(ii): Suppose (u_k) is a sequence of distinct real numbers. Clearly,

$$x_k = \left(\frac{v_{k-1}}{\alpha - u_k}\right) x_{k-1} \quad \text{for all } k \geqslant 1.$$

If $\alpha = u_0$, then $\lim_{k\to\infty} \left|\frac{x_k}{x_{k-1}}\right| > 1$ because $|U - u_0| < |V|$.
So $x \notin l_1$ and hence $x \notin c_0$ for $x_0 \neq 0$.
Similarly, if $\alpha = u_k$ for all $k \geqslant 1$, then $x_{k-1} = 0,\ x_{k-2} = 0,\ \cdots,\ x_0 = 0$ and

$$x_{n+1} = \left(\frac{v_n}{u_k - u_{n+1}}\right) x_n \text{ for all } n \geqslant k.$$

This implies $\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| > 1$ because $|U - u_k| < |V|$ for all $k \geqslant 1$.
So $x \notin l_1$ and hence $x \notin c_0$ for $x_0 \neq 0$. If $x_0 = 0$, then $x_k = 0$ for all $k \geqslant 1$. Only possibility is $x = \mathbf{0} = (0, 0, \cdots)$. Hence $\sigma_p(\Delta_{uv}, c_0) = \emptyset$. □

3.2 Residual and Continuous Spectrum of the Operator Δ_{uv} on the Sequence Space c_0

Let $T : X \to X$ be a bounded linear operator having matrix representation A and the dual space of X denoted by $X^\star$. Again, let $T^\times$ be its adjoint operator on $X^\star$. Then the matrix representation of $T^\times$ is the transpose of the matrix A.

Theorem 3.4. *Point spectrum of the adjoint operator $\Delta_{uv}^\times$ on $c_o^\star$ is*

$$\sigma_p(\Delta_{uv}^\times, c_o^\star) = \{\alpha \in \mathbb{C} : |U - \alpha| < |V|\}.$$

Proof. For the point spectrum of the operator $\Delta_{uv}^\times$, we find those α in $\mathbb{C}$ such that the matrix equation $\Delta_{uv}^\times f = \alpha f$ is satisfy for non-zero vector $f = (f_k)$ in $c_0^\star \cong l_1$. Consider $\Delta_{uv}^\times f = \alpha f$, which gives system of equations

$$\begin{aligned} u_0 f_0 + v_0 f_1 &= \alpha f_0 \\ u_1 f_1 + v_1 f_2 &= \alpha f_1 \\ &\vdots \\ u_{k-1} f_{k-1} + v_{k-1} f_k &= \alpha f_{k-1} \\ &\vdots \end{aligned}$$

This gives

$$|f_k| = \left|\frac{\alpha - u_{k-1}}{v_{k-1}}\right| |f_{k-1}| \quad \text{for all } k \geqslant 1. \tag{3.14}$$

Now, we take those $\alpha \in \mathbb{C}$ which satisfy the condition $|U - \alpha| < |V|$.
From equality (3.14), $\lim_{k\to\infty} \frac{|f_k|}{|f_{k-1}|} < 1$. So, series $\sum_{k=0}^{\infty} |f_k|$ converges and hence $f \in l_1$.
Thus, $\alpha \in \mathbb{C}$ satisfying the condition $|U - \alpha| < |V|$ implies $f \in l_1$.

Conversely, we show that

$$\sum_{k=0}^{\infty} |f_k| < \infty \text{ implies } \alpha \in \mathbb{C} \text{ satisfy the condition } |U - \alpha| < |V|$$

or equivalently for $\alpha \in \mathbb{C}$ satisfy the condition $|U - \alpha| \geqslant |V|$ implies $\sum_{k=0}^{\infty} |f_k|$ diverges. We first consider $\alpha \in \mathbb{C}$ which satisfy the condition $|U - \alpha| > |V|$. From equality (3.14), $\lim_{k\to\infty} \frac{|f_k|}{|f_{k-1}|} > 1$. So, series $\sum_{k=0}^{\infty} |f_k|$ diverges.
Next, we consider $\alpha \in \mathbb{C}$ such that $|U - \alpha| = |V|$, i.e., $\lim_{k\to\infty} \left| \frac{u_k - \alpha}{v_k} \right| = 1$. So for each $\epsilon > 0$, there exists a positive integer k_0 such that

$$1 - \epsilon < \left| \frac{u_k - \alpha}{v_k} \right| < 1 + \epsilon \quad \text{for all } k \geqslant k_0. \tag{3.15}$$

Take

$$m = \min \left\{ \left| \frac{u_0 - \alpha}{v_0} \right|, \left| \frac{u_1 - \alpha}{v_1} \right|, \cdots, \left| \frac{u_{k_0-1} - \alpha}{v_{k_0-1}} \right| \right\}. \tag{3.16}$$

Using equality (3.14), the series $\sum_{k=0}^{\infty} |f_k|$ can be write as

$$\begin{aligned}
\sum_{k=0}^{\infty} |f_k| &= |f_0| + \left| \frac{u_0 - \alpha}{v_0} \right| |f_0| + \cdots + \left| \frac{u_0 - \alpha}{v_0} \right| \cdots \left| \frac{u_{k_0-1} - \alpha}{v_{k_0-1}} \right| |f_0| \\
&\quad + \left| \frac{u_0 - \alpha}{v_0} \right| \cdots \left| \frac{u_{k_0} - \alpha}{v_{k_0}} \right| |f_0| + \left| \frac{u_0 - \alpha}{v_0} \right| \cdots \left| \frac{u_{k_0+1} - \alpha}{v_{k_0+1}} \right| |f_0| + \cdots \\
&> |f_0| + m|f_0| + \cdots + m^{k_0}|f_0| + m^{k_0}(1 - \epsilon)|f_0| \\
&\quad + m^{k_0}(1 - \epsilon)^2|f_0| + \cdots, \qquad \text{(using (3.15) and (3.16))} \\
&= \left(1 + m + \cdots + m^{k_0-1}\right) |f_0| + \frac{m^{k_0}|f_0|}{\epsilon} \to \infty \text{ as } \epsilon \to 0.
\end{aligned}$$

So, in this case also series $\sum_{k=0}^{\infty} |f_k|$ diverges. Thus, $f \in l_1$ implies $\alpha \in \mathbb{C}$ satisfying the condition $|U - \alpha| < |V|$.
This means that $f \in c_0^\star$ if and only if $f_0 \neq 0$ and $\alpha \in \mathbb{C}$ such that $|U - \alpha| < |V|$.
Hence

$$\sigma_p(\Delta_{uv}^{\times}, c_0^{\star}) = \{\alpha \in \mathbb{C} : |U - \alpha| < |V|\}. \qquad \square$$

Theorem 3.5. *Residual spectrum of the operator Δ_{uv} on the sequence space c_0 is*

$$\sigma_r(\Delta_{uv}, c_0) = \{\alpha \in \mathbb{C} : |U - \alpha| < |V|\}.$$

Proof. The proof of this theorem is divided into two cases.
Case(i): Suppose (u_k) is a constant sequence, say $u_k = U$ for all $k \in \mathbb{N}_0$. For $\alpha \in \mathbb{C}$ with $|U - \alpha| < |V|$, the operator $(\Delta_{uv} - \alpha I)$ is a triangle except $\alpha = U$ and consequently, the operator $(\Delta_{uv} - \alpha I)$ has an inverse. Further by Theorem 3.3, the operator $(\Delta_{uv} - \alpha I)$ is one to one for $\alpha = U$ and hence has an inverse.

But by Theorem 3.4, the operator $(\Delta_{uv} - \alpha I)^{\times}$ is not one to one for $\alpha \in \mathbb{C}$ with $|U - \alpha| < |V|$. Hence by Lemma 2.5, the range of the operator $(\Delta_{uv} - \alpha I)$ is not dense in c_0. Thus, $\sigma_r(\Delta_{uv}, c_0) = \{\alpha \in \mathbb{C} : |U - \alpha| < |V|\}$.

Case(ii): Suppose (u_k) is a sequence of distinct real numbers. For $\alpha \in \mathbb{C}$ such that $|U - \alpha| < |V|$, the operator $(\Delta_{uv} - \alpha I)$ is a triangle except $\alpha = u_k$ for all $k \in \mathbb{N}_0$ and consequently, the operator $(\Delta_{uv} - \alpha I)$ has an inverse. Further by Theorem 3.3, the operator $(\Delta_{uv} - u_k I)$ is one to one and hence $(\Delta_{uv} - u_k I)^{-1}$ exists for all $k \in \mathbb{N}_0$.

On the basis of argument as given in Case(i), it is easy to verify that the range of the operator $(\Delta_{uv} - \alpha I)$ is not dense in c_0. Thus,

$$\sigma_r(\Delta_{uv}, c_0) = \{\alpha \in \mathbb{C} : |U - \alpha| < |V|\}. \qquad \square$$

Theorem 3.6. *Continuous spectrum of the operator Δ_{uv} on the sequence space c_0 is*

$$\sigma_c(\Delta_{uv}, c_0) = \{\alpha \in \mathbb{C} : |U - \alpha| = |V|\}.$$

Proof. The proof of this theorem is divided into two cases.
Case(i): Suppose (u_k) is a constant sequence. For $\alpha \in \mathbb{C}$ with $|U - \alpha| = |V|$, the operator $(\Delta_{uv} - \alpha I)$ is a triangle because $\alpha \neq U$ and has an inverse. The operator $(\Delta_{uv} - \alpha I)^{-1}$ is discontinuous by condition (3.9). Therefore, the operator $(\Delta_{uv} - \alpha I)$ has an unbounded inverse.

As the operator $(\Delta_{uv} - \alpha I)^{\times}$ is one to one for $\alpha \in \mathbb{C}$ satisfying $|U - \alpha| = |V|$ follows from Theorem 3.4. So, the range of the operator $(\Delta_{uv} - \alpha I)$ is dense in c_0 by Lemma 2.5. Hence

$$\sigma_c(\Delta_{uv}, c_0) = \{\alpha \in \mathbb{C} : |U - \alpha| = |V|\}.$$

Case(ii): Suppose (u_k) is a sequence of distinct real numbers. For $\alpha \in \mathbb{C}$ with $|U - \alpha| = |V|$, the operator $(\Delta_{uv} - \alpha I)$ is a triangle because $\alpha \neq u_k$ for each $k \in \mathbb{N}$ and consequently, the operator $(\Delta_{uv} - \alpha I)$ has an inverse. The operator $(\Delta_{uv} - \alpha I)^{-1}$ is discontinuous by condition (3.9). Therefore, $(\Delta_{uv} - \alpha I)$ has an unbounded inverse.

On the basis of argument as given in Case (i), it is easy to verify that the range of the operator $(\Delta_{uv} - \alpha I)$ is dense in c_0. Hence

$$\sigma_c(\Delta_{uv}, c_0) = \{\alpha \in \mathbb{C} : |U - \alpha| = |V|\}. \qquad \square$$

3.3 Fine Spectrum of the Operator Δ_{uv} on the Sequence Space c_0

Theorem 3.7. *If α satisfies $|U - \alpha| > |V|$, then $(\Delta_{uv} - \alpha I) \in A_1$.*

Proof. It is required to show that the operator $(\Delta_{uv} - \alpha I)$ is bijective and has a continuous inverse for $\alpha \in \mathbb{C}$ with $|U - \alpha| > |V|$. Since $\alpha \neq U$ and $\alpha \neq u_k$ for each $k \in \mathbb{N}_0$, therefore the operator $(\Delta_{uv} - \alpha I)$ is a triangle. Hence it has an inverse. The operator $(\Delta_{uv} - \alpha I)^{-1}$ is continuous for $\alpha \in \mathbb{C}$ with $|U - \alpha| > |V|$ by statement (3.4). Also the equation

$$(\Delta_{uv} - \alpha I)x = y \ \text{ gives } \ x = (\Delta_{uv} - \alpha I)^{-1}y, \text{ i.e.,}$$
$$x_n = \left((\Delta_{uv} - \alpha I)^{-1}y\right)_n, \ n \in \mathbb{N}_0.$$

Thus, for every $y \in c_0$, we can find $x \in c_0$ such that

$$(\Delta_{uv} - \alpha I)x = y, \ \text{ since } \ (\Delta_{uv} - \alpha I)^{-1} \in B(c_0).$$

This shows that the operator $(\Delta_{uv} - \alpha I)$ is onto and hence $(\Delta_{uv} - \alpha I) \in A_1$. $\square$

Theorem 3.8. *Let u be constant sequence, say $u_k = U$ for all $k \in \mathbb{N}_0$. Then $U \in C_1\sigma(\Delta_{uv}, c_0)$.*

Proof. We have $\sigma_r(\Delta_{uv}, c_0) = \{\alpha \in \mathbb{C} : |U - \alpha| < |V|\}$. Clearly, $U \in \sigma_r(\Delta_{uv}, c_0)$. It is sufficient to show that the operator $(\Delta_{uv} - UI)^{-1}$ is continuous. By Lemma 2.6, it is enough to show that $(\Delta_{uv} - UI)^{\times}$ is onto, i.e., for given $y = (y_n) \in c_0^{\star}$, we have to find $x = (x_n) \in c_0^{\star}$ such that $(\Delta_{uv} - UI)^{\times} x = y$. Now $(\Delta_{uv} - UI)^{\times} x = y$, i.e.,

$$\begin{aligned} v_0 x_1 &= y_0 \\ v_1 x_2 &= y_1 \\ &\vdots \\ v_{i-1} x_i &= y_{i-1} \\ &\vdots \end{aligned}$$

Thus, $v_{n-1}x_n = y_{n-1}$ for all $n \geqslant 1$ which implies $\sum_{n=0}^{\infty} |x_n| < \infty$, since $y \in l_1$ and $v = (v_k)$ is a convergent sequence. This shows that operator $(\Delta_{uv} - UI)^{\times}$ is onto and hence $U \in C_1\sigma(\Delta_{uv}, c_0)$. $\square$

Theorem 3.9. *Let u be constant sequence, say $u_k = U$ for all $k \in \mathbb{N}_0$ and $\alpha \neq U$ but $\alpha \in \sigma_r(\Delta_{uv}, c_0)$. Then $\alpha \in C_2\sigma(\Delta_{uv}, c_0)$.*

Proof. It is sufficient to show that the operator $(\Delta_{uv} - \alpha I)^{-1}$ is discontinuous for $\alpha \neq U$ and $\alpha \in \sigma_r(\Delta_{uv}, c_0)$. The operator $(\Delta_{uv} - \alpha I)^{-1}$ is discontinuous by statement (3.7) for $U \neq \alpha \in \mathbb{C}$ with $|U - \alpha| < |V|$. □

Theorem 3.10. *Let u be a sequence of distinct real numbers and $\alpha \in \sigma_r(\Delta_{uv}, c_0)$. Then $\alpha \in C_2\sigma(\Delta_{uv}, c_0)$.*

Proof. It is sufficient to show that the operator $(\Delta_{uv} - \alpha I)^{-1}$ is discontinuous for $\alpha \in \sigma_r(\Delta_{uv}, c_0)$. The operator $(\Delta_{uv} - \alpha I)^{-1}$ is discontinuous by statements (3.7), (3.10) and (3.11) for $\alpha \in \mathbb{C}$ with $|U - \alpha| < |V|$. □

Theorem 3.11. *Let u and v be constant sequences and $\alpha \in \sigma_c(\Delta_{uv}, c_0)$. Then $\alpha \in B_2\sigma(\Delta_{uv}, c_0)$.*

Proof. It is sufficient to show that the operator $(\Delta_{uv} - \alpha I)$ is not onto, i.e., there is no sequence $x = (x_n)$ in c_0 such that $(\Delta_{uv} - \alpha I)x = y$ for some $y \in c_0$. Clearly, $y = (1, 0, 0, \cdots) \in c_0$. We have

$$(\Delta_{uv} - \alpha I)x = y \;\Rightarrow\; x_n = (-1)^n \frac{V^n}{(U-\alpha)^{n+1}} \text{ for each n} \geqslant 0.$$

Therefore, $|x_n| = \left|\frac{1}{V}\right|$ for each $n \geqslant 0$ because $|U - \alpha| = |V|$. Consequently, $\lim_{n\to\infty} |x_n| = \left|\frac{1}{V}\right| > 0$. This shows that $x \notin c_0$ and hence the operator $(\Delta_{uv} - \alpha I)$ is not onto. □

Acknowledgements : I would like to thank the referee(s) for his comments and suggestions on the manuscript. This work was supported by the Indian Institute of Technology, Kharagpur.

References

[1] J.B. Reade, On the spectrum of the Cesaro operator, Bull. London Math. Soc. 17 (3) (1985) 263-267.

[2] A.M. Akhmedov, F. Basar, On the fine spectrum of the Cesaro operator in c_0, Math. J. Ibaraki Univ. 36 (2004) 25-32.

[3] J.I. Okutoyi, On the spectrum of C_1 as an operator on bv_0, J. Austral. Math. Soc. Ser. A 48 (1990) 79-86.

[4] J.T. Okutoyi, On the spectrum of C_1 as an operator on bv, Comm. Fac. sci. univ. Ankara ser. A_1 Math. Statist. 41 (1-2) (1992) 197-207.

[5] C. Coskun, The spectra and fine spectra for p-Cesaro operators, Turkish J. Math. 21 (2) (1997) 207-212.

[6] M. Yildirim, On the spectrum and fine spectrum of the compact Rhaly operators, Indian J. Pure Appl. Math. 27 (8) (1996) 779-784.

[7] M. Yildirim, The fine spectra of the Rhaly operators on c_0, Turkish J. Math. 26 (2002) 273-282.

[8] B. Altay, F. Basar, On the fine spectrum of the difference operator Δ on c_0 and c, Inform. Sci. 168 (2004) 217-224.

[9] B. Altay, M. Karakus, On the spectrum and the fine spectrum of the Zweier matrix as an operator on some sequence spaces, Thai J. Math. 3 (2) (2005) 153-162.

[10] B. Altay, F. Basar, On the fine spectrum of the generalized difference operator $B(r, s)$ over the sequence spaces c_0 and c, Int. J. Math. Math. Sci. 18 (2005) 3005-3013.

[11] H. Furkan, H. Bilgic, B. Altay, On the fine spectrum of the operator $B(r, s, t)$ over c_0 and c, Comput. Math. Appl. 53 (2007) 989-998.

[12] E. Kreyszig, Introductory Functional Analysis with Applications, John Wiley and Sons, Inc. New York-Chichester-Brisbane-Toronato, 1978.

[13] S. Goldberg, Unbounded Linear Operators, Dover Publications, Inc. New York, 1985.

[14] A. Wilansky, Summability Through Functional Analysis, North-Holland Mathematics Studies, North-Holland, Amsterdam, 85, 1984.

(Received 12 July 2012)
(Accepted 12 September 2016)