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On the Existence of Positive Solutions for a Boundary Valued Problem of Fractional Order

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Abstract : In this paper, we investigate the problem of existence and nonexistence of positive solutions for the nonlinear boundary value problem of fractional order:

$$D^\alpha u(t) + \lambda\, a(t) f(u(t)) = 0, \;\; 0 < t < 1, \quad 2 < \alpha \leq 3,$$
$$u(0) = u''(0) = 0, \quad \gamma u'(1) + \beta u''(1) = 0,$$

where D^α is the Caputo's fractional derivative and λ is a positive parameter. By using Krasnoesel'skii's fixed-point theorem of cone preserving operators, we establish various results on the existence of positive solutions of the boundary value problem. Under various assumptions on $a(t)$ and $f(u(t))$, we give the intervals of the parameter λ which yield the existence of the positive solutions.

Keywords : Fractional differential equations, boundary value problems, Krasnoesel'skii's fixed point theorem, Green's function, Positive solution.
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1 Introduction

One of the most frequently used tools for proving the existence of positive solutions to the integral equations and boundary value problems is Krasnoselskii's theorem on cone expansion and compression and its norm-type version due to Guo and Lakshmikantham [3]. In 1994, Wang [9] applied Krasnoselskii's work to eigenvalue problems to establish intervals of the parameter for which there is at least one positive solution. Since this pioneering work a lot of research has been done in this area. Differential equations of fractional order, or fractional differential equations, in which an unknown function is contained under the operation of a derivative of fractional order, have been of great interest recently. Many papers and books on fractional calculus, fractional differential equations have appeared recently [1,3,5,6,9,10]. It should be noted that most of papers and books on fractional calculus are devoted to the solvability of linear initial fractional differential equations in terms of special functions. Recently, there are some papers deal with the existence and multiplicity of solution (or positive solution) of nonlinear fractional differential equation by the use of techniques of nonlinear analysis. Bai and Lü [1] studied the existence and multiplicity of positive solutions of nonlinear fractional differential equation boundary value problem

$$D^{\alpha}_{0+}u(t) = f(t, u(t)),\ \ 0 < t < 1,\ 1 < \alpha \leq 2,$$
$$u(0) = u(1) = 0,$$

where D^{α}_{0+} is the standard Riemann-Liouville fractional derivative. Zhang [10] considered the existence of solution of nonlinear fractional boundary value problems involving Caputo's derivative

$$D^{\alpha}u(t) = f(t, u(t)),\ \ 0 < t < 1,\ 1 < \alpha \leq 2,$$
$$u(0) = \nu \neq 0,\ u(1) = \rho \neq 0.$$

In another paper, by using fixed point theorem on cones, Zhang [11] studied the existence and multiplicity of positive solutions of nonlinear fractional boundary value problem

$$D^{\alpha}u(t) = f(t, u(t)),\ \ 0 < t < 1,\ 1 < \alpha \leq 2,$$
$$u(0) + u'(0) = 0,\ u(1) + u'(1) = 0,$$

where D^{α} is the Caputo's fractional derivative.

El-Shahed [2]considered the existence and non-existence of positive solutions to nonlinear fractional boundary value problem:

$$D^{\alpha}_{0+}u(t) + \lambda a(t)f(u(t)) = 0,\ \ 0 < t < 1,\ \ \ 2 < \alpha \leq 3, \tag{1.1}$$

$$u(0) = u'(0) = u'(1) = 0, \tag{1.2}$$

The purpose of this paper is to establish the existence and nonexistence of positive solutions to nonlinear fractional boundary value problem

$$D^{\alpha}u(t) + \lambda a(t)f(u(t)) = 0,\ \ 0 < t < 1,\ \ \ 2 < \alpha \leq 3, \tag{1.3}$$
$$u(0) = u''(0) = 0,\quad \gamma u'(1) + \beta u''(1) = 0, \tag{1.4}$$

where λ is a positive parameter, $a : (0, 1) \to [0, \infty)$ is continuous with $\int_0^1 a(t)\,dt > 0$ and $f : [0, \infty) \to [0, \infty)$ is continuous. Here, by a positive solution of the boundary value problem we mean a function which is positive on $(0, 1)$ and satisfies differential equation (1.3) and the boundary condition (1.4). The paper has been organized as follows. In Section 2, we give basic definitions and provide some properties of certain Greens functions which are needed later. We also state Krasnoselskiis fixed point theorem for cone preserving operators. In Section 3 we establish some results for the existence and non-existence of positive solutions to problem (1) and (2). In the end of this section, an example is also given to illustrate the main results.

2 Preliminaries

For the convenience of the reader, we present here some notations and lemmas that will be used to the proof of our main results.

Definition 2.1 *[5]. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : [0, \infty) \to R$ is given by:*

$$I^{\alpha} f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds.$$

Definition 2.2 *[6]. The Caputo's fractional derivative of order $\alpha > 0$ can be written as:*

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} f(s) ds, \ \ n = [\alpha] + 1.$$

Definition 2.3 *[2].Let E be a real Banach space. A nonempty closed convex set $K \subset E$ is called* ***cone*** *of E if it satisfies the following conditions:*

1. *$x \in K$, $\sigma \geq 0$ implies $\sigma\, x \in K$;*
2. *$x \in K$, $-x \in K$ implies $x = 0$.*

Definition 2.4 *[2].An operator is called* ***completely continuous*** *if it is continuous and maps bounded sets into precompact sets.*

All results are based on the following fixed point theorem of cone expansion-compression type due to Krasnoselskii's. See, for example, [2]and [4].

Theorem 1 *[2, 4].Let E be a Banach space and $K \subset E$ is a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$ be completely continuous operator. In addition suppose either:*

H1 : $\|Tu\| \leq \|u\|, \ \forall u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|, \ \forall u \in K \cap \partial\Omega_2$ *or*

H2 : $\|Tu\| \leq \|u\|, \ \forall u \in K \cap \partial\Omega_2$ *and* $\|Tu\| \geq \|u\|, \ \forall u \in K \cap \partial\Omega_1$.

holds. Then T has a fixed pint in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

$$G(t, s) \geq 0 \text{ and } G(1, s) \geq G(t, s), \ 0 \leq t, s \leq 1. \tag{2.1}$$

Lemma 2.5 *$G(t, s) \geq q(t) G(1, s)$ for $0 \leq t, s \leq 1$, where $q(t) = \frac{\beta(\alpha-2)t}{\gamma+\beta(\alpha-2)}$.*

Proof. If $t \geq s$, then

$$\frac{G(t,s)}{G(1,s)} = \frac{\gamma(\alpha-1)t(1-s)^{\alpha-2} + \beta t(\alpha-1)(\alpha-2)(1-s)^{\alpha-3} - \gamma(t-s)^{\alpha-1}}{\gamma(\alpha-1)(1-s)^{\alpha-2} + \beta(\alpha-1)(\alpha-2)(1-s)^{\alpha-3} - \gamma(1-s)^{\alpha-1}}$$

$$\geq \frac{\gamma(\alpha-1)t(1-s)^{\alpha-2}+\beta t(\alpha-1)(\alpha-2)(1-s)^{\alpha-3}-\gamma(t-s)^2(1-s)^{\alpha-3}}{\gamma(\alpha-1)(1-s)^{\alpha-2}+\beta(\alpha-1)(\alpha-2)(1-s)^{\alpha-3}-\gamma(1-s)^{\alpha-1}}$$

$$\geq \frac{\beta(\alpha-2)t}{\gamma+\beta(\alpha-2)}.$$

If $t \leq s$, then

$$\frac{G(t,s)}{G(1,s)} = \frac{\gamma t(1-s)^{\alpha-2}+\beta t(\alpha-2)(1-s)^{\alpha-3}}{\gamma(1-s)^{\alpha-2}+\beta(\alpha-2)(1-s)^{\alpha-3}} = t \geq \frac{\beta(\alpha-2)t}{\gamma+\beta(\alpha-2)}.$$

The proof is complete. □

3 Main results

In this section, we will apply Krasnoesel'skii's fixed point theorem to the eigenvalue problem (1.3) and (1.4). We note that $u(t)$ is a solution of (1.3) and (1.4) if and only if

$$u(t) = \lambda \int_0^1 G(t,s)\, a(s) f(u(s))\, ds, \qquad 0 \leq t \leq 1.$$

We shall consider the Banach space $X = C[0,1]$ equipped with standard norm $\|u\| = \max_{0\leq t\leq 1} |u(t)|,\ u \in X$. We define a cone P by

$$P = \{u \in X : u(t) \geq q(t)\, \|u\|, t \in [0,1]\}.$$

It is easy to see that if $u \in P$, then $\|u\| = u(1)$. Define an integral operator by:

$$Tu(t) = \lambda \int_0^1 G(t,s)\, a(s) f(u(s))\, ds, \qquad 0 \leq t \leq 1,\ u \in P. \tag{3.1}$$

It is well known that $T : P \longrightarrow X$ is a completely continuous operator.

Lemma 3.1 *$T(P) \subset P$.*

Proof. Notice from (7) and Lemma (2) that, for $u \in P$, $Tu(t) \geq 0$ on $[0,1]$ and

$$\begin{aligned} Tu(t) &= \lambda \int_0^1 G(t,s) a(s) f(u(s))\, ds \\ &\geq \lambda q(t) \int_0^1 G(1,s) a(s) f(u(s))\, ds \\ &\geq \lambda q(t) \max_{0\leq t\leq 1} \int_0^1 G(t,s)\, a(s) f(u(s))\, ds \\ &= q(t) \|Tu(t)\|, \quad \text{for all } t,s \in [0,1]. \end{aligned}$$

Thus, $T(P) \subset P$. □

We can write (8) in the form:

$$Tu(t) = \lambda \int_0^1 G(t,s)\, g(s,u(s))\, ds, \qquad 0 \le t \le 1,\ u \in P. \tag{3.2}$$

Lemma 3.2 *Assume that $g : [0,1] \times R \longrightarrow R$ is continuous function, then $Tu(t)$ is completely continuous operator.*

Proof. It is easy to see that $Tu(t)$ is continuous. For $u \in M = \{u \in X : \|u\| < l, l > 0\}$, we obtain

$$\begin{aligned}
\Big|Tu(t)\Big| &= \Big| \int_0^1 \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} g(s,u(s))ds + \frac{\beta}{\gamma}\int_0^1 \frac{t(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} g(s,u(s))ds \\
&\quad - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s,u(s))ds\Big| \\
&\le \int_0^1 \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \Big|g(s,u(s))\Big|ds + \frac{\beta}{\gamma}\int_0^1 \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} \Big|g(s,u(s))\Big|ds + \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big|g(s,u(s))\Big|ds \\
&\le \frac{L\,t}{\Gamma(\alpha)} + \frac{\beta L}{\Gamma(\alpha-1)} + \frac{L\,t^{\alpha}}{\Gamma(\alpha+1)} \\
&\le \frac{L}{\Gamma(\alpha)} + \frac{\beta L}{\gamma\Gamma(\alpha-1)} + \frac{L}{\Gamma(\alpha+1)},
\end{aligned}$$

where $L = \max_{0 \le t \le 1, 0 \le u \le l} \Big|g(t,u(t)\Big| + 1$, so $T(M)$ is bounded. Next we shall show the equicontinuity of $\overline{T(M)}$. $\forall u \in M, \epsilon > 0,\ t_1 < t_2$, we have

$$\Big|Tu(t_2) - Tu(t_1)\Big| =$$

$$\Big| \int_0^1 (t_2 - t_1)\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} g(s,u(s))ds + \frac{\beta}{\gamma}(t_2-t_1)\int_0^1 \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} g(s,u(s))ds$$

$$+ \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} g(s,u(s))ds - \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} g(s,u(s))ds\Big|$$

$$\le \frac{L}{\Gamma(\alpha)} + \frac{2L}{\Gamma(\alpha+1)}.$$

Thus $\overline{T(M)}$ is equicontinuous. The Arzela-Ascoli theorem implies that the operator T is completely continuous and the proof is complete. □

We define some important constants[8]

$$A=\int_0^1 G(1,s)\,a(s)q(s)\,ds, \qquad B=\int_0^1 G(1,s)\,a(s)\,ds,$$
$$F_0=\lim_{u\to 0^+}\sup\frac{f(u)}{u}, \qquad f_0=\lim_{u\to 0^+}\inf\frac{f(u)}{u},$$
$$F_\infty=\lim_{u\to +\infty}\sup\frac{f(u)}{u}, \qquad f_\infty=\lim_{u\to +\infty}\inf\frac{f(u)}{u}.$$

Here we assume that $\frac{1}{Af_\infty}=0$ if $f_\infty\to\infty$ and $\frac{1}{BF_0}=\infty$ if $F_0\to 0$ and $\frac{1}{Af_0}=0$ if $f_0\to\infty$ and $\frac{1}{BF_\infty}=\infty$ if $F_\infty\to 0$.

Theorem 2 *Suppose that $Af_\infty > BF_0$, then for each $\lambda\in\left(\frac{1}{Af_\infty},\frac{1}{BF_0}\right)$, the problem (1.3) and (1.4) has at least one positive solution.*

Proof. We choose $\epsilon>0$ sufficiently small such that $(F_0+\epsilon)\lambda B \le 1$. By definition of F_0, we can see that there exists an $l_1>0$, such that $f(u)\le (F_0+\epsilon)u$ for $0<u\le l_1$. If $u\in P$ with $\|u\|=l_1$, we have

$$\begin{aligned}\|Tu(t)\| = Tu(1) &= \lambda\int_0^1 G(1,s)\,a(s)f(u(s))\,ds\\
&\le \lambda\int_0^1 G(1,s)a(s)(F_0+\epsilon)u(s)\,ds\\
&\le \lambda\,(F_0+\epsilon)\|u\|\int_0^1 G(1,s)a(s)\,ds\\
&\le \lambda B\,(F_0+\epsilon)\|u\| \le \|u\|.\end{aligned}$$

Then we have $\|Tu\|\le\|u\|$. Thus if we let $\Omega_1=\{u\in X:\ \|u\|<l_1\}$, then $\|Tu\|\le\|u\|$ for $u\in P\cap\partial\Omega_1$. We choose $\delta>0$ and $c\in(0,\frac14)$, such that

$$\lambda\left((f_\infty-\delta)\int_c^1 G(1,s)a(s)q(s)\,ds\right)\ge 1.$$

There exists $l_3>0$, such that $f(u)\ge(f_\infty-\delta)\,u$ for $u>l_3$. Let $l_2=\max\{\frac{\gamma+\beta(\alpha-2)}{\beta(\alpha-2)c},2l_1\}$. If $u\in P$ with $\|u\|=l_2$, then we have

$$u(t)\ge q(t)l_2\ge\frac{\beta(\alpha-2)t}{\gamma+\beta(\alpha-2)}l_2\ge\frac{\beta(\alpha-2)c}{\gamma+\beta(\alpha-2)}l_2\ge l_3.$$

Therefore, for each $u\in P$ with $\|u\|=l_2$,we have

$$\begin{aligned}\|Tu(t)\| = (Tu)(1) &= \lambda\int_0^1 G(1,s)\,a(s)f(u(s))\,ds\\
&\ge \lambda\int_c^1 G(1,s)a(s)(f_\infty-\epsilon)u(s)\,ds\\
&\ge \lambda\,(f_\infty-\epsilon)\|u\|\int_c^1 G(1,s)a(s)\,q(s)\,ds \ge \|u\|.\end{aligned}$$

Thus if we let $\Omega_2 = \{u \in X : \|u\| < l_2\}$, then $\Omega_1 \subset \overline{\Omega}_2$ and $\|Tu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_2$. Condition (H1) of Krasnoesel'skii's fixed point theorem is satisfied. So there exists a fixed point of T in P. This completes the proof. □

Theorem 3 *Suppose that* C1 *and* C2 *hold. If* $A\,f_0 > B\,F_\infty$, *then for each* $\lambda \in \left(\frac{1}{Af_0}, \frac{1}{BF_\infty}\right)$ *the problem (1.3) and (1.4) has at least one positive solution.*

The proof of Theorem 3 is very similar to that of Theorem 2 and therefore omitted.

Theorem 4 *Suppose that* $\lambda Bf(u) < u$ *for* $u \in (0, \infty)$. *Then the problem (1.3) and (1.4) has no positive solution.*

Proof. Following Sun and Wen [8], assume to the contrary that u is a positive solution of (1.3) and (1.4). Then

$$\begin{aligned} u(1) = \lambda \int_0^1 G(1,s)\,a(s)f(u(s))\,ds &< \frac{1}{B}\int_0^1 G(1,s)a(s)u(s)\,ds \\ &\leq \frac{1}{B}\,u(1)\int_0^1 G(1,s)a(s)\,ds = u(1). \end{aligned}$$

This is a contradiction and completes the proof. □

Theorem 5 *Suppose that* $\lambda Af(u) > u$ *for* $u \in (0, \infty)$. *Then the problem (1.3) and (1.4) has no positive solution.*

Proof. Assume to the contrary that u is a positive solution of (1.3) and (1.4). Then

$$\begin{aligned} u(1) = \lambda \int_0^1 G(1,s)\,a(s)f(u(s))\,ds &> \frac{1}{A}\int_0^1 G(1,s)a(s)u(s)\,ds \\ &\geq \frac{u(1)}{A}\int_0^1 G(1,s)a(s)\,q(s)\,ds \geq u(1). \end{aligned}$$

This is a contradiction and completes the proof. □

Example 3.3 Consider the equation

$$u^{(2.5)}(t) + \lambda(\frac{t+2}{9})\,\frac{6u^2+u}{u+1}\,(3+\sin u) = 0, \quad 0 \leq t \leq 1, \tag{3.3}$$

$$u(0) = u''(0) = 0, u(1) + 2u'(1) = 0. \tag{3.4}$$

Then $F_0 = f_0 = 3$, $F_\infty = 24$, $f_\infty = 12$ and $3u < f(u) < 24u$. By direct calculations, we obtain that $A = 0.234034$ and $B = 0.668669$. From theorem 2 we see that if $\lambda \in (0.356073, 0.498503)$, then the problem (3.3)-(3.4) has a positive solution. From theorem 4 we have that if $\lambda < 0.0623128$, then the problem (3.3)-(3.4) has a positive solution. By theorem 5 we have that if $\lambda > 1.42429$, then the problem (3.3)-(3.4) has a positive solution.

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