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On the Diophantine Equation $4^x - p^y = 3z^2$ where p is a Prime

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Abstract : We find all solutions to $4^x - 7^y = z^2$ and $4^x - 11^y = z^2$ to complement the results found by Suvarnamani, et. al. in [1]. We also consider the two Diophantine equations $4^x - 7^y = 3z^2$ and $4^x - 19^y = 3z^2$ and show that these two equations have exactly two solutions (x, y, z) in non-negative integers, i.e. $(x, y, z) \in \{(0, 0, 0), (1, 0, 1)\}$. In fact, the Diophantine equation $4^x - p^y = 3z^2$ has the two solutions $(0, 0, 0)$ and $(1, 0, 1)$ under some additional assumption on p. These results were all obtained using elementary methods and Mihăilescu's Theorem. Finally, we end our paper with an open problem.

Keywords : exponential Diophantine equation; integer solutions.
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1 Introduction

Recently, there have been an increasing interest in finding solutions to exponential Diophantine equations of the form $p^x + q^y = z^2$, see e.g. [2–11], and the references therein.

In [1], A. Suvarnamani, A. Singta, and S. Chotchaisthit showed that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no solution in the set of non-negative integers. In fact, the Diophantine equation $4^x - 11^y = z^2$ also contain no solution in the set of non-negative integers (this set we denote by $\mathbb{N}_0$ throughout the paper) except possibly when $x = y = z = 0$, and the Diophantine equation $4^x - 7^y = z^2$ holds true in $\mathbb{N}_0$ for $(x, y, z) = (0, 0, 0)$ and $(2, 1, 3)$ only.

The results found in [1] were obtained using Mihăilescu's Theorem: $a^x - b^y = 1$ has the unique solution $(a, b, x, y) = (3, 2, 2, 3)$ in positive integers for $\min(a, b, x, y) > 1$. This remarkable result was first conjectured by E. Catalan in a one page note dated 1844 (see [12]) and was finally proven my P. Mihăilescu in 2002 (see [13]). B. Peker and S. I. Çenberci generalized the results found in [1] by considering the Diophantine equation $(4^n)^x + p^y = z^2$ where p is an odd prime, $n \in \mathbb{N}$, and x, y, and $z \in \mathbb{N}_0$ in [14]. On the other hand, in [15], the author and J. B. Bacani obtained all solutions to the Diophantine equation $p^x + q^y = z^2$ where p and q are twin primes under some additional assumptios on p and q. The paper [15] gives a correct set of solutions to $p^x + q^y = z^2$ (under some assumptions on p and q) in contrary to the main result presented in [16].

Another type of Diophantine equations of great interest are those of the form $a^x \pm b^y \pm c^z = w^n$. In [17] and [18], the authors studied exponential Diophantine equations of the form $p^x \pm q^y \pm r^z = c$ where p, q, r are primes, x, y and $z \in \mathbb{N}_0$, and c an integer have been studied. Particularly, J. Leitner [17] solved the equation $3^a + 5^b - 7^c = 1$ for $a, b, c \in \mathbb{N}_0$ and the equation $y^2 = 3^a + 2^b + 1$ for $a, b \in \mathbb{N}_0$ and integer y. R. Scott and R. Styer [18] studied, among other things, the Diophantine equation $p^x \pm q^y \pm 2^z = 0$ for primes p and q and positive integers x, y, and z. These authors used elementary methods to show that, with a few explicitly listed exceptions, there are at most two solutions (x, y) to $|p^x \pm q^y| = c$ (where c is a fixed positive integer) and at most two solutions (x, y, z) to $p^x \pm q^y \pm 2^z = 0$ in positive integers.

In an earlier paper, the author along with Bacani gave all solutions to the Diophantine equation $3^x + 5^y + 7^z = w^2$ in response to an open problem posed by B. Sroysang in [19].

In this note, we verify our claim that $4^x - 11^y = z^2$ contains no solution in $\mathbb{N}_0$ except possibly when $x = y = z = 0$, and the Diophantine equation $4^x - 7^y = z^2$ has exactly two solutions (x, y, z) in non-negative integers, i.e. $(x, y, z) \in \{(0, 0, 0), (2, 1, 3)\}$. We remark that our approach in proving these two claims can be applied to prove a general case of the problem. Also, we show that the two Diophantine equations $4^x - 7^y = 3z^2$ and $4^x - 19^y = 3z^2$ have exactly two solutions (x, y, z) in $\mathbb{N}_0$, i.e. $(x, y, z) \in \{(0, 0, 0), (1, 0, 1)\}$. Finally, we state and prove a generalization of these two previous results at the end of our paper. Most precisely, we show that $4^x - p^y = 3z^2$ has the two solutions $(x, y, z) = (0, 0, 0)$ and $(1, 0, 1)$ in $\mathbb{N}_0$ for prime $p \equiv 3 \pmod 4$.

2 Preliminaries

In this section we state some helpful results to prove our claims. First, it is known that the equation $X^2 - dY^2 = 1$ has a solution in positive integers X and Y for all positive, nonsquare integers d (see e.g. [20, Theorem 1, pg. 9]), and that if k is a perfect square, then the Pell Equation $X^2 - dY^2 = k$ is solvable in integers for all positive, nonsquare integers d (cf. [20, Theorem 6, pg. 16]). In relation to

Pell's equation the following lemma was proved by K. Matthews in [21, Section 3].

Lemma 2.1. *Let $N \geq 1$ be an odd integer, $D > 1$ and not a perfect square. Then, a necessary condition for the solvability of the equation $x^2 - Dy^2 = N$ with $\gcd(x, y) = 1$ is that the congruence $u^2 \equiv D \pmod{N}$ shall be soluble.*

The following results shall be used to show that the title equation has no solution in positive integers x, y, and z for prime $p \equiv 3 \pmod 4$.

Theorem 2.2 ([22], Theorem 2.9, pg. 32)**.** *Let $p \equiv 3 \pmod 4$ and $k = m^2 n$ with n square free. If $X^2 - pY^2 = k$ is solvable, then $n \equiv 1 \pmod 4$.*

Corollary 2.3 ([22], Corollary 2.10, pg. 33)**.** *Let $k = m^2 n$ with n square free. If $p \equiv n \equiv 3 \pmod 4$, then $X^2 - pY^2 = k$ is not solvable.*

Corollary 2.4 ([22], Corollary 2.11, pg. 33)**.** *If $p \equiv k \equiv 3 \pmod 4$ and $l \equiv 1 \pmod 4$, then $X^2 - pY^2 = kl$ is not solvable.*

Now we prove our results in the following section.

3 Main Results

We first consider the two Diophantine equations $4^x - 7^y = z^2$ and $4^x - 11^y = z^2$ and later in this section we study the equations $4^x - 7^y = 3z^2$ and $4^x - 19^y = 3z^2$.

Theorem 3.1. *The Diophantine equation $4^x - 7^y = z^2$ has exactly two solutions (x, y, z) in $\mathbb{N}_0$, namely (the trivial solution) $(0, 0, 0)$ and $(2, 1, 3)$.*

Proof. Evidently, the case when $z = 0$ will give us $(x, y, z) = (0, 0, 0)$, so we may assume that $z > 0$. For $z > 0$, we consider three cases.

Case 1. $x = 0$. This case is trivial.

Case 2. $y = 0$. If $y = 0$, then we have $(2^x)^2 - z^2 = 1$ which is impossible due to Mihăilescu's Theorem.

Case 3. $x, y > 0$. For this case we have $(2^x)^2 - z^2 = (2^x + z)(2^x - z) = 7^y$. It follows that $(2^x + z) + (2^x - z) = 2^{x+1} = 7^\beta + 7^\alpha$ for some $\alpha < \beta$, where $\alpha + \beta = y$. Hence, $2^{x+1} = 7^\alpha(7^{\beta-\alpha} + 1)$. Thus, $\alpha = 0$ and $2^{x+1} - 7^\beta = 1$, which is true when $x = 2$ and $y = 1$. These give us the value $z = 3$. Therefore, $(2, 1, 3)$ is a solution of $4^x - 7^y = z^2$. Now, if we assume $y > 1$, then we get $2^{x+1} - 7^\beta = 1$ which has no solution because of Mihăilescu's Theorem and this proves the theorem. □

Theorem 3.2. *The Diophantine equation $4^x - 11^y = z^2$ contains no solution in $\mathbb{N}_0$ except the trivial solution $x = y = z = 0$.*

Proof. The theorem can be shown easily by utilizing Mihăilescu's Theorem and is similar to the proof of the previous theorem. The case when $z = 0$ and $x = 0$ are both trivial. So we may assume without loss of generality that $\min(x, z) > 0$. If this is the case, then we have $(2^x)^2 - z^2 = (2^x + z)(2^x - z) = 11^y$. It follows that,

$(2^x+z)+(2^x-z)=2^{x+1}=11^\beta+11^\alpha$ for some $\alpha<\beta$, where $\alpha+\beta=y$. Hence, $2^{x+1}=11^\alpha(11^{\beta-\alpha}+1)$. Thus, $\alpha=0$ and $2^{x+1}-11^\beta=1$ and by Mihăilescu's Theorem, we can now conclude that this Diophantine equation has no solution. The theorem is now proved. □

In the following result we shall state and prove a more general case of Theorem 3.1 and Theorem 3.2.

Theorem 3.3. *The Diophantine equation* $4^x-p^y=z^2$ *has the set of all solutions* $\{(x,y,z)\}$ *given by*

$$\{(x,y,z)\}=\{(0,0,0)\}\cup\{(q-1,1,2^{q-1}-1)\},$$

for prime $p=2^q-1$ *(with* q *also a prime). For* $p\equiv 3 \pmod 4$ *not of the form* 2^q-1*, the Diophantine equation* $4^x-p^y=z^2$ *has only the trivial solution* $(x,y,z)=(0,0,0)$.

Proof. Consider the Diophantine equation $4^x-p^y=z^2$. We consider the following cases.

Case 1. $x=0$. If $x=0$, then $1-z^2=p^y$ which implies that $z=y=0$ and p is any prime number.

Case 2. $y=0$. If $y=0$, then $2^{2x}-z^2=1$ which is obviously impossible because of Mihăilescu's Theorem.

Case 3. $x,y>0$. If $\min(x,y)>0$, then $4^x-p^y=z^2$ is equivalent to $(2^x+z)(2^x-z)=p^y$. Hence, $2^{x+1}=(2x+z)+(2^x-z)=p^\alpha(p^{\beta-\alpha}-1)$ for some integers α and β such that $\alpha+\beta=y$ and $\beta>\alpha\geq 0$. Therefore, $\alpha=0$ and $2^{x+1}-p^y=1$ which has no solution for $\min(x,y)>1$ by Mihăilescu's Theorem. For $y=1$, we get $p=2^{x+1}-1$. Note that $2^{x+1}-1$ is a prime if and only if $x+1$ is also a prime. Thus, we get a family of solutions to $4^x-p^y=z^2$ given by $(x,y,z)=\{(q-1,1,2^q-1)\mid q \text{ is a prime}\}$ for $p=2^q-1$. On the other hand, if $p\equiv 1 \pmod 4$ not of the form 2^q-1 (with $y=1$), then we get $-1\equiv 1 \pmod 4$ and this a clear contradiction. Thus, we only have the trivial solution $(0,0,0)$ to $4^x-p^y=z^2$ for $p\equiv 3 \pmod 4$. Now, conclusion follows. □

Remark 3.4. Theorem 3.1 (respectively, Theorem 3.2) agrees with Theorem 3.3 since $7=2^3-1$ (respectively, $11\equiv 3 \pmod 4$)).

Theorem 3.5. *The Diophantine equation* $4^x-7^y=3z^2$ *has exactly two solutions* (x,y,z) *in* $\mathbb{N}_0$*. In particular, the solutions are* $(0,0,0)$ *and* $(1,0,1)$.

Proof. Evidently, for the case when $z=0$ we get $(x,y,z)=(0,0,0)$. So we let $z>0$ and consider the following three cases.

Case 1. $x=0$. This case is trivial.

Case 2. $y=0$. If $y=0$, then we have $4^x-3z^2=1$. It can be seen easily that the equation holds true for $x=z=1$. Here we get $(x,y,z)=(1,0,1)$. Now, it remains for us to show that there is no solution to $(2^x)^2-3z^2=1$ other than $(x,y,z)=(1,0,1)$ for $y=0$. Note that $(2^x)^2-1=(2^x+1)(2^x-1)=3z^2$.

So, $2^x + 1 = 3z$ and $2^x - 1 = z$. This claim is easily verified as follows: if x is odd, then there exists an integer $k > 0$ such that $2^x + 1 = 3k$. So $(2^x)^2 - 1 = (2^x+1)(2^x-1) = 3k(3k-2)$. Hence, we have $3k-2 = 2^x - 1$ but this is impossible since $3k - 2 \neq k$ for $k \neq 1$. Indeed, $2^x + 1 = 3z$ and $2^x - 1 = z$. It follows that, $z = 1$ and $2^x - 1 = z$. The latter equation is true only when $x = 1$. Thus, the case when $y = 0$ implies a unique solution $(x, y, z) = (1, 0, 1)$ to $4^x - 7^y = 3z^2$.

Case 3. $x, y > 0$. Suppose $4^x - 7^y = 3z^2$ has a solution in $\mathbb{N}$ for $\min(x, y) > 0$. We rewrite the equation into $(2^x)^2 - 3z^2 = 7^y$. First, suppose that y is even, say $y = 2m$ for some $m \in \mathbb{N}$. Then, by [20, Theorem 6, pg. 16], we could find $X_n = 2^{x_n}, Z_n = z_n$, and $Y_n = 7^{m_n}$ such that $X_n^2 - 3Z_n^2 = Y_n^2$. Furthermore, by [20, Theorem 1, pg. 9], there is a solution (u, v) such that $u^2 - 3v^2 = 1$. Let $u = 2^{x_j}$ and $v = z_j$ where $(2^{x_j})^2 - 3z_j^2 = 1$. Multiplying $(2^{x_j})^2 - 3z_j^2 = 1$ by 7^{2m_j} both sides with $m_j > 0$, we get $(2^{x_j}7^{m_j})^2 - 3(z_j7^{m_j})^2 = (7^{m_j})^2$. This is impossible since every solution X_n is a power of two. It follows that y is odd. Suppose now that y is odd. Then, we have $(2^x)^2 - 3z^2 = 7(49^m)$. Let $p = 3$ and $k = 7$ in Corollary 2.4. Obviously, $7 \equiv 3 \pmod 4$. Since 49 is of the form $4t + 1$ (with $t = 12$), then 49^m is of the form $4t' + 1$ for some $t' \in \mathbb{N}$. Thus, by Corollary 2.4, $(2^x)^2 - 3z^2 = 7^y$ for odd y is not solvable. This completes the proof of the theorem. ☐

Remark 3.6. The conclusion in Case 3 of the previous theorem can also be shown using Lemma 2.1. That is, if we rewrite $4^x - 7^y = 3z^2$ into $(X)^2 - 3z^2 = 7^y$ where $X = 2x$, then, by Lemma 2.1, this equation is soluble if and only if there is a natural number u such that $u^2 \equiv 3 \pmod{7^y}$. Note that Lemma 2.1 applies to $(X)^2 - 3z^2 = 7^y$ since $z^2 \equiv -3z^2 \equiv 7^y \pmod 4$ implies that z must be odd. Indeed, we have $\gcd(X, z) = 1$. Now, the equivalence relation $u^2 \equiv 3 \pmod{7^y}$ is soluble provided $u^2 = 3 \pmod 7$ has a solution. So we must find a solution to $u^2 \equiv 3 \pmod 7$. But, $3^3 \equiv 6 \pmod 7$. Hence, by Euler's Criterion, 3 is a quadratic nonresidue of 7. Thus, $u^2 \equiv 3 \pmod{7^y}$ is insoluble. Here we conclude that $(X)^2 - 3z^2 = (2^x)^2 - 3z^2 = 7^y$ has no solution in $\mathbb{N}_0$.

Theorem 3.7. *The Diophantine equation* $4^x - 19^y = 3z^2$ *has exactly two solutions* (x, y, z) *in* $\mathbb{N}_0$, *i.e.* $(x, y, z) \in \{(0, 0, 0), (1, 0, 1)\}$.

Proof. The proof is similar to the previous theorem. The case when $z = 0$ and $x = 0$ are equivalent and are both trivial. We only consider the following two cases.

Case 1. $y = 0$. If $y = 0$, then we have $4^x - 3z^2 = 1$ which has the unique solution $(x, y, z) = (1, 0, 1)$ by Case 2 of Theorem 3.5.

Case 2. $x, y > 0$. For $\min(x, y) > 0$, the Diophantine equation $4^x - 19^y = 3z^2$ is equivalent to $(2^x)^2 - 3z^2 = 19^y$. First, suppose that y is even, say $y = 2m$ for some $m \in \mathbb{N}$. Then, we could find $X_n = 2^{x_n}, Z_n = z_n$, and $Y_n = 19^{m_n}$ such that $X_n^2 - 3Z_n^2 = Y_n^2$. Moreover there is a solution (u, v) such that $u^2 - 3v^2 = 1$. Choose $u = 2^{x_j}$ and $v = z_j$ where $(2^{x_j})^2 - 3z_j^2 = 1$. Multiplying $(2^{x_j})^2 - 3z_j^2 = 1$ by 19^{2m_j} both sides (with $m_j > 0$), we obtain $(2^{x_j}19^{m_j})^2 - 3(z_j19^{m_j})^2 = (19^{m_j})^2$. This is clearly a contradiction since every solution X_n is a power of two. So y must be odd. Now, suppose that y is odd. Then, $(2^x)^2 - 3z^2 = 7(361^m)$. Let $p = 3$ and $k = 19$. Obviously, $19 \equiv 3 \pmod 4$. Since 361 is of the form $4t + 1$ (with $t = 90$), then 361^m is of the form $4t' + 1$ for some $t' \in \mathbb{N}$. Therefore, by Corollary 2.4, the Diophantine equation $(2^x)^2 - 3z^2 = 19^y$ for odd y is not solvable. This proves the theorem. □

Remark 3.8. Similar to what we remarked for Case 3 of Theorem 3.5, the conclusion obtained in Case 2 of Theorem 3.7 can be shown using Lemma 2.1 running along the same inductive line of argument in Remark 3.6.

We note that $4^x - 7^y = 3z^2$ and $4^x - 19^y = 3z^2$ are of the form $4^x - p^y = 3z^2$ where $p \equiv 3 \pmod 4$. This Diophantine equation has in fact the two solutions $(0, 0, 0)$ and $(1, 0, 1)$ in $\mathbb{N}_0$, and this result is the content of our last and final theorem.

Theorem 3.9. *Let $p \equiv 3 \pmod 4$ be a prime. Then, the Diophantine equation $4^x - p^y = 3z^2$ has exactly two solutions (x, y, z) in $\mathbb{N}_0$, i.e. $(x, y, z) \in \{(0, 0, 0), (1, 0, 1)\}$.*

Proof. Let $p \equiv 3 \pmod 4$ be a prime and consider the Diophantine equation $4^x - p^y = 3z^2$ where x, y, and z are non-negative integers. We first treat the case when $\min(x, y, z) = 0$. If $x = 0$, then we have $1 - p^y = 3z^2$. Note that $p^y \equiv 1 \pmod 4$ when y is even and $p^y \equiv -1 \pmod 4$ when y is odd. Also, note that z is odd. Hence, $1 - p^y \equiv 0, 2 \pmod 4$ whereas $3z^2 \equiv 3 \pmod 4$. Therefore, $4^x - p^y = 3z^2$ has no solution for $x = 0$.

If $y = 0$, then we get $4^x - 1 = 3z^2$ which has the unique solution $(x, y, z) = (1, 0, 1)$ by Case 2 of Theorem 3.5.

If $z = 0$, then it immediately follows that $x = y = 0$. Here we get $(x, y, z) = (0, 0, 0)$.

Now suppose $\min(x, y, z) > 0$. Note that the equivalence relation $4^x - p^y \equiv 3z^2 \equiv -1 \pmod 4$ implies that y and z are both odd. If y is odd, then $(2^x)^2 - 3z^2 = p(p^{2m})$. But, $p = 4t + 3$ for some $t \in \mathbb{N}_0$, hence p^2 is of the form $4t' + 1$ for some $t' \in \mathbb{N}$. Then, $p^{2m} \equiv 1 \pmod 4$. By virtue of Corollary 2.4, we conclude that $(2^x)^2 - 3z^2 = p^y$ is not solvable. This proves the theorem. □

4 Summary

In this work, we have exhibited all solutions to the Diophantine equation $4^x - p^y = z^2$ in the set of non-negative integers for prime number p. Also, we have given all solutions to the Diophantine equation $4^x - p^y = 3z^2$ under the assumption that $p \equiv 3 \pmod 4$. With this restriction on p, the case when $p \equiv 1 \pmod 4$ remains open and we leave this to the interested reader. Also, we leave the set of all solutions of the Diophantine equation $4^x - p^y = dz^2$ in $\mathbb{N}_0$ (where d is an integer) as an open problem. It is worth mentioning that the two Diophantine equations $2^x + 3y^2 = 4^z$ and $2^x + 7y^2 = 4^z$ were already been studied by the author in [23] in which a general solution to $2^x + dy^2 = 4^z$ in non-negative integers (with $d = (2k-1)/9$ and $k \equiv 0 \pmod 6$) was also also presented.

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