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New Hadamard's Inequality for (α_1, m_1)-(α_2, m_2)-Preinvex Functions on the Co-Ordinates

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Abstract : In this paper, we introduce some new classes of convex functions called (α, m)-preinvex, (α_1, m_1)-(α_2, m_2)-preinvex functions on the co-ordinates, and then we derive some new Hermite-Hadamard type inequalities whose modulus of their mixed derivatives lies in these novel classes of functions.

Keywords : Hermite-Hadamard inequality; preinvex functions; co-ordinates (α_1, m_1)-(α_2, m_2)-preinvex.

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1 Introduction

One of the most well-known inequalities in mathematics for convex functions is the so called Hermite-Hadamard integral inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where f is a real convex function on the finite interval $[a,b]$. If the function f is concave, then (1.1) holds in the reverse direction (see [1]).

The Hermite-Hadamard inequality plays an important role in nonlinear analysis and optimization. The double inequality above has attracted many researchers, various generalizations, refinements, extensions and variants of (1.1) have appeared in the literature, we can mention the works [2–18] and the references cited therein.

In recent years, lot of efforts have been made by many mathematicians to generalize the classical convexity. Hanson [19] introduced a new class of generalized convex functions, called invex functions. The authors in [20], defined the concept of preinvex function which is special case of invexity. Pini [21] gave the concept of prequasiinvex functions as a generalization of invex functions. Many authors have studied the basic properties of the preinvex functions and their role in optimization, variational inequalities and equilibrium problems we refer reader to [21–25]. Recently, Latif and Shoaib [26] introduced the notion of m-preinvex and (α,m)-preinvex functions. Matloka [12] defined a new class of functions which are (h_1,h_2)-preinvex on the co-ordinates.

Motivated by the results established in [12, 26], in this paper, we introduce two new classes of convex functions called (α,m)-preinvex on the co-ordinates and (α_1,m_1)-(α_2,m_2)-preinvex functions on the co-ordinates. Also we derive some new Hermite-Hadamard type inequalities for functions whose mixed partial derivatives in absolute value are (α,m)-preinvex on the co-ordinates and (α_1,m_1)-(α_2,m_2)-preinvex on the co-ordinates.

2 Preliminaries

Let us recall some known results. In what follows we assume that I is an interval of $\mathbb{R}$ and Δ is an bidimensional interval of $\mathbb{R}^2$ where $\Delta := [a,b]\times[c,d]$ with $a<b$ and $c<d$.

Definition 2.1. [1] A function $f : I \to \mathbb{R}$ is said to be *convex on* I if

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$$

holds for all $x,y\in I$ and all $t\in[0,1]$.

Definition 2.2. [9] A function $f : \Delta \to \mathbb{R}$ is said to be *convex on* Δ if

$$f(\lambda x+(1-\lambda)z, \lambda y+(1-\lambda)w) \leq \lambda f(x,y)+(1-\lambda)f(z,w)$$

holds for all $(x, y), (z, w) \in \Delta$ and $\lambda \in [0, 1]$.

Definition 2.3. [9] A function $f : \Delta \to \mathbb{R}$ is said to be *convex on the cordinates on* Δ if the following inequality

$$\begin{aligned} f(\lambda x + (1-\lambda)z, ty + (1-t)w) \quad \le \quad & \lambda t f(x, y) + \lambda(1-t)f(x, w) \\ & +(1-\lambda)tf(z, y) + (1-\lambda)(1-t)f(z, w) \end{aligned}$$

holds for all $(x, y), (z, w), (x, w), (z, y) \in \Delta$ and $\lambda, t \in [0, 1]$.

Let K be a subset in $\mathbb{R}^n$ and let $f : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}^n$ be continuous functions.

Definition 2.4. [24] A set K is said to be *invex at* x *with respect to* η if

$$x + t\eta (y, x) \in K$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

K is said to be an *invex set with respect to* η if K is invex at each $x \in K$.

Definition 2.5. [24]A function f on the invex set K is said to be *preinvex with respect to* η if

$$f (x + t\eta (y, x)) \le (1 - t) f (x) + tf(y)$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

Toader [27] introduced the notion of m-convexity as follows:

Definition 2.6. [27] A function $f : [0, b^*] \to \mathbb{R}$, $b^* > 0$ is said to be *m-convex* for some $m \in [0, 1]$, if

$$f (tx + m (1 - t) y) \le tf (x) + m (1 - t) f(y)$$

holds for all $x, y \in [0, b^*]$ and $t \in [0, 1]$.

Mihesan [28] generalized the above notion and introduced the (α, m)-convexity as follows:

Definition 2.7. [28] A function $f : [0, b^*] \to \mathbb{R}$, $b^* > 0$ is said to be *(α, m)-convex* for some $(\alpha, m) \in [0, 1]^2$, if

$$f (tx + m (1 - t) y) \le t^\alpha f (x) + m (1 - t^\alpha) f(y)$$

holds for all $x, y \in [0, b^*]$ and $t \in [0, 1]$.

Latif et al. [26] introduced the concept of m-preinvexity and (α, m)-preinvexity as follows:

Definition 2.8. [26] A function f on the invex set $K \subseteq [0, b^*]$, $b^* > 0$ is said to be *m-preinvex with respect to* η if

$$f(x + t\eta(y, x)) \leq (1 - t) f(x) + mtf(\frac{y}{m})$$

holds for all $x, y \in K$, $t \in [0, 1]$ and some fixed $m \in (0, 1]$.

Definition 2.9. [26] A function f on the invex set $K \subseteq [0, b^*]$, $b^* > 0$ is said to be (α, m)*-preinvex with respect to* η if

$$f(x + t\eta(y, x)) \leq (1 - t^\alpha) f(x) + mt^\alpha f(\frac{y}{m})$$

holds for all $x, y \in K$, $t \in [0, 1]$ and some $(\alpha, m) \in (0, 1]^2$.

Let K_1, K_2 be two nonempty closed subsets in $\mathbb{R}$, $f : K_1 \times K_2 \to \mathbb{R}$ and η_i : $K_i \times K_i \to \mathbb{R}$ be continuous functions, for $i = 1, 2$.

Definition 2.10. [12] Let $(u, v) \in K_1 \times K_2$. We say that $K_1 \times K_2$ is an *invex set at* (u, v) *with respect to* η_1 *and* η_2 if

$$(u + t\eta_1(x, u), v + s\eta_2(y, v)) \in K_1 \times K_2$$

holds for each $(x, y) \in K_1 \times K_2$ and $t, s \in [0, 1]$.

$K_1 \times K_2$ is said to be an *invex set with respect to* η_1 *and* η_2 if $K_1 \times K_2$ is invex at each $(u, v) \in K_1 \times K_2$.

Definition 2.11. [11] Let $K_1 \times K_2$ be an invex set with respect to $\eta_1 : K_1 \times K_1 \to \mathbb{R}$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}$. A function $f : K_1 \times K_2 \to \mathbb{R}$ is said to be *preinvex* if

$$f(u + t\eta_1(x, u), v + t\eta_2(y, v)) \leq (1 - t)f(u, v) + tf(x, y)$$

holds for all $(x, y), (u, v) \in K_1 \times K_2$ and $t \in [0, 1]$.

Definition 2.12. [11] Let $K_1 \times K_2$ be an invex set with respect to $\eta_1 : K_1 \times K_1 \to \mathbb{R}$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}$. A function $f : K_1 \times K_2 \to \mathbb{R}$ is said to be *preinvex on the co-ordinates* if

$$\begin{aligned} f(u + t\eta_1(x, u), v + s\eta_2(y, v)) \quad &\leq \quad (1 - t)(1 - s)f(u, v) + (1 - t)sf(u, y) \\ &\quad +(1 - s)tf(x, v) + tsf(x, y) \end{aligned}$$

holds for all $(x, y), (x, v), (u, y), (u, v) \in K_1 \times K_2$ and $t, s \in [0, 1]$.

Lemma 2.13. [11, Lemma 2] *Let* $K_1 \times K_2$ *be an open invex subset of* $\mathbb{R}^2$ *with respect to the mappings* $\eta_1 : K_1 \times K_1 \to \mathbb{R}$ *and* $\eta_2 : K_2 \times K_2 \to \mathbb{R}$. *Assume that* $f : K_1 \times K_2 \to \mathbb{R}$ *be a twice partially differentiable mapping such that* $\frac{\partial^2 f}{\partial t \partial s} \in$

$L([a, a+\eta_1(b,a)] \times [c, c+\eta_2(d,c)])$ with $\eta_1(b,a) \neq 0$, $\eta_2(d,c) \neq 0$ where $a, b \in K_1$ and $c, d \in K_2$, then the following equality holds

$$\begin{aligned} & \tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} \\ & + \tfrac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A \\ = & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4} \int_0^1 \int_0^1 (1-2t)(1-2s) \tfrac{\partial^2 f}{\partial t \partial s}(a+t\eta_1(b,a), c+s\eta_2(d,c))dtds, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} A = & \tfrac{1}{2\eta_1(b,a)} \int_a^{a+\eta_1(b,a)} [f(x,c) + f(x, c+\eta_2(d,c))]\,dx \\ & + \tfrac{1}{2\eta_2(d,c)} \int_c^{c+\eta_2(d,c)} [f(a,y) + f(a+\eta_1(b,a), y)]\,dy. \end{aligned}$$

Lemma 2.14. [29, Corollary 3.1] *Let $f(x,y)$ be a positive continuous function on $[a,b] \times [c,d] \subset \mathbb{R}^2$*

1. If $t \geq 1$ or $t < 0$ and $\int_a^b \int_c^d f(x,y)\,dxdy \geq [(b-a)(d-c)]^{t-1}$, then

$$\int_a^b \int_c^d f^t(x,y)\,dxdy \geq \left[\int_a^b \int_c^d f(x,y)\,dxdy \right]^{t-1}. \tag{2.2}$$

2. If $0 < t \leq 1$ and $\int_a^b \int_c^d f(x,y)\,dxdy \leq [(b-a)(d-c)]^{t-1}$, then

$$\int_a^b \int_c^d f^t(x,y)\,dxdy \leq \left[\int_a^b \int_c^d f(x,y)\,dxdy \right]^{t-1}. \tag{2.3}$$

3. If $t \notin [0,1)$ and $f(x,y) \geq [(b-a)(d-c)]^{t-2}$ for $(x,y) \in [a,b] \times [c,d]$, then the inequality (2.2) is valid.

4. If $0 < t \leq 1$ and $f(x,y) \leq [(b-a)(d-c)]^{t-2}$ for $(x,y) \in [a,b] \times [c,d]$, then the inequality (2.2) is reversed.

5. If $t \geq 2$ and $f(x,y) \geq (t-1)^2 [(x-a)(y-c)]^{t-2}$ for $(x,y) \in [a,b] \times [c,d]$, then the inequality (2.2) is valid.

We also recall that the *Euler Beta function* is defined as follows

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt.$$

3 Main Results

Firstly, we give the notion of (α,m)-preinvex on the co-ordinates and (α_1,m_1)-(α_2,m_2)-preinvex fonctions on the co-ordinates in both sens, then we derive some Hermite-Hadamard type inequalities for these new classes of functions.

Definition 3.1. A function f on the invex set $K_1\times K_2\subseteq[0,b^*]\times[0,d^*]$ with $b^*>0$ and $d^*>0$ is said to be (α,m)*-preinvex in the first sens with respect to* η_1 *and* η_2 where $\alpha,m\in(0,1]$ if

$$f\left(u+t\eta_1(x,u),v+t\eta_2(y,v)\right)\le(1-t^\alpha)f(u,v)+mt^\alpha f(\frac{x}{m},\frac{y}{m})$$

holds for all (x,y), $(u,v)\in K_1\times K_2$ and $t\in[0,1]$.

Definition 3.2. A function f on the invex set $K_1\times K_2\subseteq[0,b^*]\times[0,d^*]$ with $b^*>0$ and $d^*>0$ is said to be (α,m)*-preinvex in the second sens with respect to* η_1 *and* η_2 where $\alpha,m\in(0,1]$ if

$$f\left(u+t\eta_1(x,u),v+t\eta_2(y,v)\right)\le(1-t)^\alpha f(u,v)+mt^\alpha f(\frac{x}{m},\frac{y}{m})$$

holds for all (x,y), $(u,v)\in K_1\times K_2$ and $t\in[0,1]$.

Remark 3.3. *The Definition* 3.1 *and Definition* 3.2 *recapture the concept of preinvex functions if we take* $\alpha=m=1$. *Moreover, if we choose* $\eta_1(x,u)=\eta_2(x,u)=x-u$, *we obtain the definition of convexity on the co-ordinates.*

Definition 3.4. The function f on the invex set $K_1\times K_2\subseteq[0,b^*]\times[0,d^*]$ where $b^*>0$ and $d^*>0$ is said to be (α,m)*-preinvex in the first sens on the co-ordinates with respect to* η_1 *and* η_2 with $\alpha,m\in(0,1]$ if the partial mappings $f_y:K_1\to\mathbb{R}$, $f_y(x)=f(x,y)$ and $f_x:K_2\to\mathbb{R}$, $f_x(y)=f(x,y)$ are (α,m)-preinvex functions in the first sens with respect to η_1 and η_2 respectively for all $y\in K_2$ and $x\in K_1$.

Remark 3.5. *From the above definition it follows that if* f *is co-ordinated* (α,m)*-preinvex function in the first sens, then we have*

$$\begin{aligned} f\left(u+t\eta(x,u),v+s\eta(y,v)\right) &\le (1-t^\alpha)f\left(u,v+s\eta(y,v)\right)\\ &\quad +mt^\alpha f(\frac{x}{m},v+s\eta(y,v))\\ &\le (1-t^\alpha)(1-s^\alpha)f(u,v)+m(1-t^\alpha)s^\alpha f(u,\frac{y}{m})\\ &\quad +mt^\alpha(1-s^\alpha)f(\frac{x}{m},v)+m^2t^\alpha s^\alpha f(\frac{x}{m},\frac{y}{m}). \end{aligned}$$

Definition 3.6. A function f on the invex set $K_1 \times K_2 \subseteq [0, b^*] \times [0, d^*]$ where $b^* > 0$ and $d^* > 0$ is said to be (α, m)*-preinvex in the second sens on the co-ordinates with respect to* η_1 *and* η_2 with $\alpha, m \in (0, 1]$ if the partial mappings $f_y : K_1 \to \mathbb{R}$, $f_y(x) = f(x, y)$ and $f_x : K_2 \to \mathbb{R}$, $f_x(y) = f(x, y)$ are (α, m)-preinvex functions in the second sens with respect to η_1 and η_2 respectively for all $y \in K_2$ and $x \in K_1$.

Remark 3.7. *From Definition* 3.6 *it follows that if* f *is co-ordinated* (α, m)*-preinvex function in the second sens, we have*

$$\begin{aligned} f(u + t\eta(x, u), v + s\eta(y, v)) &\leq (1 - t)^{\alpha} f(u, v + s\eta(y, v)) \\ &\quad + mt^{\alpha} f(\frac{x}{m}, v + s\eta(y, v)) \\ &\leq (1 - t)^{\alpha} (1 - s)^{\alpha} f(u, v) + m(1 - t)^{\alpha} s^{\alpha} f(u, \frac{y}{m}) \\ &\quad + mt^{\alpha} (1 - s)^{\alpha} f(\frac{x}{m}, v) + m^2 t^{\alpha} s^{\alpha} f(\frac{x}{m}, \frac{y}{m}). \end{aligned}$$

Remark 3.8. *In Definition* 3.4 *and Definition* 3.6*, if we choose* $\alpha = m = 1$*, we obtain definition of preinvex function on the co-ordinates. Moreover, if we take* $\eta_1(x, u) = \eta_2(x, u) = x - u$*, then we get definition of convex function on the co-ordinates.*

Definition 3.9. A function f on the invex set $K_1 \times K_2 \subseteq [0, b^*] \times [0, d^*]$ with $b^* > 0$ and $d^* > 0$ is said to be (α_1, m_1)-(α_2, m_2)*-preinvex in the first sens on the co-ordinates with respect to* η_1 *and* η_2 if the partial mappings $f_y : K_1 \to \mathbb{R}$, $f_y(x) = f(x, y)$ and $f_x : K_2 \to \mathbb{R}$, $f_x(y) = f(x, y)$ are (α_1, m_1)-preinvex in the first sens with respect to η_1 and (α_2, m_2)-preinvex in the first sens with respect to η_2 respectively for all $y \in K_2$ and $x \in K_1$.

Remark 3.10. *From the above definition it follows that if* f *is co-ordinated* (α_1, m_1)-(α_2, m_2)*-preinvex function in the first sens, we have*

$$\begin{aligned} f(u + t\eta_1(x, u), v + s\eta_2(y, v)) &\leq (1 - t^{\alpha_1}) f(u, v + s\eta_2(y, v)) \\ &\quad + m_1 t^{\alpha_1} f(\frac{x}{m_1}, v + s\eta_2(y, v)) \\ &\leq (1 - t^{\alpha_1})(1 - s^{\alpha_2}) f(u, v) \\ &\quad + m_2 (1 - t^{\alpha_1}) s^{\alpha_2} f(u, \frac{y}{m_2}) \\ &\quad + m_1 t^{\alpha_1} (1 - s^{\alpha_2}) f(\frac{x}{m_1}, v) \\ &\quad + m_1 m_2 t^{\alpha_1} s^{\alpha_2} f(\frac{x}{m_1}, \frac{y}{m_2}). \end{aligned}$$

Definition 3.11. A function f on the invex set $K_1 \times K_2 \subseteq [0, b^*] \times [0, d^*]$ where $b^* > 0$ and $d^* > 0$ is said to be (α_1, m_1)-(α_2, m_2)*-preinvex in the second sens on the co-ordinates with respect to* η_1 *and* η_2 if the partial mappings $f_y : K_1 \to \mathbb{R}$, $f_y(x) = f(x, y)$ and $f_x : K_2 \to \mathbb{R}$, $f_x(y) = f(x, y)$ are (α_1, m_1)-preinvex in the second sens with respect to η_1 and (α_2, m_2)-preinvex in the second sens with respect to η_2 respectively for all $y \in K_2$ and $x \in K_1$.

Remark 3.12. *From the above definition it follows that if f is a co-ordinated (α_1, m_1)-(α_2, m_2)-preinvex function in the second sens, then*

$$\begin{aligned} f\left(u+t\eta_1\left(x,u\right),v+s\eta_2\left(y,v\right)\right) &\leq (1-t)^{\alpha_1} f\left(u, v+s\eta_2\left(y,v\right)\right) \\ &\quad +m_1 t^{\alpha_1} f(\frac{x}{m_1}, v+s\eta_2\left(y,v\right)) \\ &\leq (1-t)^{\alpha_1}(1-s)^{\alpha_2} f\left(u,v\right) \\ &\quad +m_2\left(1-t\right)^{\alpha_1} s^{\alpha_2} f(u,\frac{y}{m_2}) \\ &\quad +m_1 t^{\alpha_1}\left(1-s\right)^{\alpha_2} f(\frac{x}{m_1},v) \\ &\quad +m_1 m_2 t^{\alpha_1} s^{\alpha_2} f(\frac{x}{m_1},\frac{y}{m_2}). \end{aligned}$$

Now, we can state our results. From here we assume that K_1 and K_2 are two invex subset of $\mathbb{R}$ with $K_1 \times K_2 \subseteq [0, b^*] \times [0, d^*]$ such that $a, b \in K_1, c, d \in K_2$ and $b^* > \frac{b}{m_1}$ and $d^* > \frac{d}{m_2}$ with $m_1, m_2 \in (0,1]$ and $\eta_1 : K_1 \times K_1 \to \mathbb{R}$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}$ are two bifunction.

Theorem 3.13. *Let $f : K_1 \times K_2 \to \mathbb{R}$ be partially differentiable mapping such that $\frac{\partial^2 f}{\partial t \partial s} \in L\left([a, a+\eta_1(b,a)] \times [c, c+\eta_2(d,c)]\right)$ with $\eta_1(b,a) > 0$ and $\eta_2(d,c) > 0$. If $\left|\frac{\partial^2 f}{\partial t \partial s}\right|$ is (α_1, m_1)-(α_2, m_2)-preinvex in the first sens on the co-ordinates on $K_1 \times K_2$, then the following inequality holds*

$$\begin{aligned} &\Bigg|\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} \\ &\quad + \tfrac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f\left(x,y\right)dxdy - A\Bigg| \\ &\leq \tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\Bigg[\left(\tfrac{1}{2}-\theta_1\right)\left(\tfrac{1}{2}-\theta_2\right)\left|\tfrac{\partial^2 f}{\partial t\partial s}\left(a,c\right)\right| \\ &\quad +m_2\left(\tfrac{1}{2}-\theta_1\right)\theta_2\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,\frac{d}{m_2}))\right| + m_1\theta_1\left(\frac{1}{2}-\theta_2\right) \\ &\quad \times\left|\tfrac{\partial^2 f}{\partial t\partial s}(\frac{b}{m_1},c)\right| + m_1 m_2\theta_1\theta_2\left|\tfrac{\partial^2 f}{\partial t\partial s}(\frac{b}{m_1},\frac{d}{m_2})\right|\Bigg], \end{aligned} \tag{3.1}$$

where

$$\theta_1 = \tfrac{\left(\frac{1}{2}\right)^{\alpha_1}+\alpha_1}{(\alpha_1+1)(\alpha_1+2)} \text{ and } \theta_2 = \tfrac{\left(\frac{1}{2}\right)^{\alpha_2}+\alpha_2}{(\alpha_2+1)(\alpha_2+2)} \tag{3.2}$$

$\alpha_1, m_1, \alpha_2, m_2 \in (0,1]$ and A is defined as in Lemma 2.13.

Proof. From Lemma 2.13, we have

$$\begin{aligned}
& \left| \tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} \right. \\
& \left. + \tfrac{1}{\eta_1(b,a)\eta_2(d,c)} \int\limits_a^{a+\eta_1(b,a)} \int\limits_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A \right| \\
= & \left| \tfrac{\eta_1(b,a)\eta_2(d,c)}{4} \int\limits_0^1 \int\limits_0^1 (1-2t)(1-2s) \tfrac{\partial^2 f}{\partial t \partial s} (a + t\eta_1(b,a), c + s\eta_2(d,c)) dtds \right| \\
\leq & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4} \int\limits_0^1 \int\limits_0^1 |1-2t| |1-2s| \left| \tfrac{\partial^2 f}{\partial t \partial s} (a + t\eta_1(b,a), c + s\eta_2(d,c)) \right| dtds.
\end{aligned} \tag{3.3}$$

Using (α_1, m_1)-(α_2, m_2)-preinvexity in the first sens, we get

$$\begin{aligned}
& \left| \tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} \right. \\
& \left. + \tfrac{1}{\eta_1(b,a)\eta_2(d,c)} \int\limits_a^{a+\eta_1(b,a)} \int\limits_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A \right| \\
\leq & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4} \int\limits_0^1 \int\limits_0^1 |1-2t| |1-2s| \left[(1-t^{\alpha_1})(1-s^{\alpha_2}) \left| \tfrac{\partial^2 f}{\partial t \partial s}(a,c) \right| \right. \\
& + m_2 (1-t^{\alpha_1}) s^{\alpha_2} \left| \tfrac{\partial^2 f}{\partial t \partial s}(a, \frac{d}{m_2})) \right| + m_1 t^{\alpha_1} (1-s^{\alpha_2}) \left| \tfrac{\partial^2 f}{\partial t \partial s}(\frac{b}{m_1}, c) \right| \\
& \left. + m_1 m_2 t^{\alpha_1} s^{\alpha_2} \left| \tfrac{\partial^2 f}{\partial t \partial s}(\frac{b}{m_1}, \frac{d}{m_2}) \right| \right] dtds \\
= & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4} \left| \tfrac{\partial^2 f}{\partial t \partial s}(a,c) \right| \int\limits_0^1 \int\limits_0^1 |1-2t| |1-2s| (1-t^{\alpha_1})(1-s^{\alpha_2})\, dtds \\
& + \tfrac{\eta_1(b,a)\eta_2(d,c)m_2}{4} \left| \tfrac{\partial^2 f}{\partial t \partial s}(a, \frac{d}{m_2})) \right| \int\limits_0^1 \int\limits_0^1 |1-2t| |1-2s| (1-t^{\alpha_1}) s^{\alpha_2} dtds \\
& + \tfrac{\eta_1(b,a)\eta_2(d,c)m_1}{4} \left| \tfrac{\partial^2 f}{\partial t \partial s}(\frac{b}{m_1}, c) \right| \int\limits_0^1 \int\limits_0^1 |1-2t| |1-2s| t^{\alpha_1} (1-s^{\alpha_2})\, dtds \\
& + \tfrac{\eta_1(b,a)\eta_2(d,c)m_1 m_2}{4} \left| \tfrac{\partial^2 f}{\partial t \partial s}(\frac{b}{m_1}, \frac{d}{m_2}) \right| \int\limits_0^1 \int\limits_0^1 |1-2t| |1-2s| t^{\alpha_1} s^{\alpha_2} dtds.
\end{aligned} \tag{3.4}$$

Noting that

$$\int_0^1\int_0^1 |1-2t|\,|1-2s|\,(1-t^{\alpha_1})\,(1-s^{\alpha_2})\,dtds = \left(\tfrac{1}{2}-\theta_1\right)\left(\tfrac{1}{2}-\theta_2\right), \tag{3.5}$$

$$\int_0^1\int_0^1 |1-2t|\,|1-2s|\,(1-t^{\alpha_1})\,s^{\alpha_2}dtds = \left(\tfrac{1}{2}-\theta_1\right)\theta_2, \tag{3.6}$$

$$\int_0^1\int_0^1 |1-2t|\,|1-2s|\,t^{\alpha_1}\,(1-s^{\alpha_2})\,dtds = \theta_1\left(\tfrac{1}{2}-\theta_2\right), \tag{3.7}$$

$$\int_0^1\int_0^1 |1-2t|\,|1-2s|\,t^{\alpha_1}s^{\alpha_2}dtds = \theta_1\theta_2, \tag{3.8}$$

where θ_1 and θ_2 are defined in (3.2). Substituting (3.5)-(3.8) into (3.4), we obtain the desired inequality in (3.1). The proof is completed. □

Corollary 3.14. *Under the assumptions of Theorem* 3.13*, if* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ *is* (α,m)*-preinvex in the first sens on the co-ordinates on* $K_1\times K_2$*, then the following inequality holds*

$$\begin{aligned}&\Bigg|\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\\&+\tfrac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\Bigg|\\ \leq\;& \tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\Bigg[\left(\tfrac{1}{2}-\theta\right)^2\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,c)\right|\\&+m\left(\tfrac{1}{2}-\theta\right)\theta\left[\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,\frac{d}{m})\right|+\left|\tfrac{\partial^2 f}{\partial t\partial s}(\frac{b}{m},c)\right|\right]\\&+(m\theta)^2\left|\tfrac{\partial^2 f}{\partial t\partial s}(\frac{b}{m},\frac{d}{m})\right|\Bigg],\end{aligned} \tag{3.9}$$

where

$$\theta = \frac{\left(\frac{1}{2}\right)^{\alpha}+\alpha}{(\alpha+1)(\alpha+2)}, \tag{3.10}$$

$\alpha, m\in(0,1]$ *and* A *is defined as in Lemma* 2.13*.*

Corollary 3.15. *Under the assumptions of Theorem* 3.13 *if* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ *is* m_1*-preinvex with respect to* η_1 *and* m_2*-preinvex with respect to* η_2 *on* $K_1\times K_2$*, then the following* *inequality holds*

$$\begin{aligned}
&\left|\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\right. \\
&\left. + \tfrac{1}{\eta_1(b,a)\eta_2(d,c)} \int\limits_a^{a+\eta_1(b,a)} \int\limits_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\right| \\
\leq \quad & \tfrac{\eta_1(b,a)\eta_2(d,c)}{64}\left[\left|\tfrac{\partial^2 f}{\partial t \partial s}(a,c)\right| + m_2\left|\tfrac{\partial^2 f}{\partial t \partial s}(a,\frac{d}{m_2}))\right|\right. \\
&\left. + m_1\left|\tfrac{\partial^2 f}{\partial t \partial s}(\frac{b}{m_1},c)\right| + m_1 m_2\left|\tfrac{\partial^2 f}{\partial t \partial s}(\frac{b}{m_1},\frac{d}{m_2})\right|\right], \qquad (3.11)
\end{aligned}$$

where $m_1, m_2 \in (0,1]$ *and* A *is defined as in Lemma* 2.13.

Remark 3.16. *In Theorem* 3.13*, if we take* $m_1 = m_2 = \alpha_1 = \alpha_2 = 1$*, we obtain Theorem* 8 *from* [11]*. Moreover, if we take* $\eta_1(b,a) = \eta_2(b,a) = b - a$*, we obtain Theorem* 2 *from* [18]*.*

Theorem 3.17. *Let* $f : K_1 \times K_2 \to \mathbb{R}$ *be partially differentiable mapping such that* $\frac{\partial^2 f}{\partial t \partial s} \in L([a, a+\eta_1(b,a)] \times [c, c+\eta_2(d,c)])$ *with* $\eta_1(b,a) > 0$ *and* $\eta_2(d,c) > 0$*. If* $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$ *is* (α_1, m_1)-(α_2, m_2)*-preinvex in the first sens on the co-ordinates on* $K_1 \times K_2$ *where* $q \in (1, \infty)$*, then the following inequality holds*

$$\begin{aligned}
&\left|\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\right. \\
&\left. + \tfrac{1}{\eta_1(b,a)\eta_2(d,c)} \int\limits_a^{a+\eta_1(b,a)} \int\limits_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\right| \\
\leq \quad & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4(p+1)^{\frac{2}{p}}\lambda}\left[\alpha_1\alpha_2\left|\tfrac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q + m_2\alpha_1\left|\tfrac{\partial^2 f}{\partial t \partial s}(a,\frac{d}{m_2}))\right|^q\right. \\
&\left. + m_1\alpha_2\left|\tfrac{\partial^2 f}{\partial t \partial s}(\frac{b}{m_1},c)\right|^q + m_1 m_2\left|\tfrac{\partial^2 f}{\partial t \partial s}(\frac{b}{m_1},\frac{d}{m_2})\right|^q\right]^{\frac{1}{q}}, \qquad (3.12)
\end{aligned}$$

where A *is defined as in Lemma* 2.13*,* $\alpha_1, m_1, \alpha_2, m_2 \in (0,1]$*,* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\lambda = [(1+\alpha_1)(1+\alpha_2)]^{\frac{1}{q}}$.

Proof. From Lemma 2.13, and Hölder inequality, we have

$$\begin{aligned}
&\left|\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\right. \\
&\left.+\tfrac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\right| \\
\leq\; & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\left(\int_0^1\int_0^1 |1-2t|^p\,|1-2s|^p\,dtds\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1\int_0^1 \left|\tfrac{\partial^2 f}{\partial t\partial s}\left(a+t\eta_1(b,a), c+s\eta_2(d,c)\right)\right|^q dtds\right)^{\frac{1}{q}}. \qquad (3.13)
\end{aligned}$$

Using (α_1, m_1)-(α_2, m_2)-preinvexity in the first sens, we get

$$\begin{aligned}
&\left|\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\right. \\
&\left.+\tfrac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\right| \\
\leq\; & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\left(\int_0^1\int_0^1 |1-2t|^p\,|1-2s|^p\,dtds\right)^{\frac{1}{p}} \\
&\times\left(\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\int_0^1\int_0^1 (1-t^{\alpha_1})(1-s^{\alpha_2})\,dtds\right. \\
&+m_2\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,\frac{d}{m_2}))\right|^q\int_0^1\int_0^1 (1-t^{\alpha_1})\,s^{\alpha_2}dtds \\
&+m_1\left|\tfrac{\partial^2 f}{\partial t\partial s}(\frac{b}{m_1},c)\right|^q\int_0^1\int_0^1 t^{\alpha_1}(1-s^{\alpha_2})\,dtds \\
&\left.+m_1m_2\left|\tfrac{\partial^2 f}{\partial t\partial s}(\frac{b}{m_1},\frac{d}{m_2})\right|^q\int_0^1\int_0^1 t^{\alpha_1}s^{\alpha_2}dtds\right)^{\frac{1}{q}}. \qquad (3.14)
\end{aligned}$$

A simple computation gives

$$\begin{aligned}
\int_0^1\int_0^1 |1-2t|^p\,|1-2s|^p\,dtds &= \tfrac{1}{(p+1)^2},\\
\int_0^1\int_0^1 (1-t^{\alpha_1})\,(1-s^{\alpha_2})\,dtds &= \tfrac{\alpha_1\alpha_2}{(1+\alpha_1)(1+\alpha_2)},\\
\int_0^1\int_0^1 (1-t^{\alpha_1})\,s^{\alpha_2}dtds &= \tfrac{\alpha_1}{(1+\alpha_1)(1+\alpha_2)},\\
\int_0^1\int_0^1 t^{\alpha_1}\,(1-s^{\alpha_2})\,dtds &= \tfrac{\alpha_2}{(1+\alpha_1)(1+\alpha_2)},\\
\int_0^1\int_0^1 t^{\alpha_1}s^{\alpha_2}dtds &= \tfrac{1}{(1+\alpha_1)(1+\alpha_2)}. \qquad (3.15)
\end{aligned}$$

Substituting (3.15) into (3.14), we obtain the desired inequality in (3.12). The proof is completed. □

Corollary 3.18. *Under the assumptions of Theorem* 3.17*, and if* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ *is* (α, m)*-preinvex in the first sens on the co-ordinates on* $K_1\times K_2$ *where* $q\in(1,\infty)$*, then the following inequality holds*

$$\begin{aligned}
&\Bigg|\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\\
&\quad+\tfrac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy-A\Bigg|\\
\le\ &\tfrac{\eta_1(b,a)\eta_2(d,c)}{4(p+1)^{\frac{2}{p}}(1+\alpha)^{\frac{2}{q}}}\ \Bigg[\alpha^2\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+m\alpha\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,\frac{d}{m}))\right|^q\\
&+m\alpha\left|\tfrac{\partial^2 f}{\partial t\partial s}(\frac{b}{m},c)\right|^q+m^2\left|\tfrac{\partial^2 f}{\partial t\partial s}(\frac{b}{m},\frac{d}{m})\right|^q\Bigg]^{\frac{1}{q}}, \qquad (3.16)
\end{aligned}$$

where A *is defined as in Lemma* 2.13*,* $\alpha, m\in(0,1]\times\,]0,1]$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 3.19. *Under the assumptions of Theorem* 3.17*, if* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ *is* m_1*-preinvex with respect to* η_1 *and* m_2*-preinvex with respect to* η_2 *on* $K_1\times K_2$ *with* $q\in(1,\infty)$, *then the following inequality holds*

$$\left|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\right.$$
$$\left.+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\right|$$

$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{4^{1+\frac{1}{q}}(p+1)^{\frac{2}{p}}}\left[\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q + m_2\left|\frac{\partial^2 f}{\partial t\partial s}(a,\frac{d}{m_2}))\right|^q\right.$$
$$\left.+ m_1\left|\frac{\partial^2 f}{\partial t\partial s}(\frac{b}{m_1},c)\right|^q + m_1 m_2\left|\frac{\partial^2 f}{\partial t\partial s}(\frac{b}{m_1},\frac{d}{m_2})\right|^q\right]^{\frac{1}{q}}, \tag{3.17}$$

where A is defined as in Lemma 2.13, $m_1, m_2 \in (0,1]$ and $\frac{1}{p}+\frac{1}{q}=1$.

Remark 3.20. *Theorem 3.17 will be reduced to Theorem 9 from [11], if we choose $m_1 = m_2 = \alpha_1 = \alpha_2 = 1$. Moreover, if we take $\eta_1(b,a) = \eta_2(b,a) = b-a$, we obtain Theorem 3 from [18].*

Theorem 3.21. *Let $f : K_1 \times K_2 \to \mathbb{R}$ be partially differentiable mapping such that $\frac{\partial^2 f}{\partial t\partial s} \in L([a, a+\eta_1(b,a)] \times [c, c+\eta_2(d,c)])$ with $\eta_1(b,a) > 0$ and $\eta_2(d,c) > 0$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is (α_1, m_1)-(α_2, m_2)-preinvex in the first sens on the co-ordinates with respect to η_1 and η_2 on $K_1 \times K_2$, $q \in [1,\infty)$, then the following inequality holds*

$$\left|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\right.$$
$$\left.+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\right|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{4^{2-\frac{1}{q}}}\left[\left(\tfrac{1}{2}-\theta_1\right)\left(\tfrac{1}{2}-\theta_2\right)\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\right.$$
$$+m_2\left(\tfrac{1}{2}-\theta_1\right)\theta_2\left|\frac{\partial^2 f}{\partial t\partial s}(a,\frac{d}{m_2}))\right|^q + m_1\left(\frac{1}{2}-\theta_2\right)\theta_1\left|\frac{\partial^2 f}{\partial t\partial s}(\frac{b}{m_1},c)\right|^q$$
$$\left.+ m_1 m_2\theta_1\theta_2\left|\frac{\partial^2 f}{\partial t\partial s}(\frac{b}{m_1},\frac{d}{m_2})\right|^q\right]^{\frac{1}{q}}, \tag{3.18}$$

where A is defined as in Lemma 2.13, θ_1, θ_2 are defined in Theorem 3.13 and $\alpha_1, m_1, \alpha_2, m_2 \in (0,1]$.

Proof. In the case where $q = 1$, the proof is similar to that of Theorem 3.13, now we will treat the case where $q > 1$.

From Lemma 2.13, and power mean inequality, we have

$$\begin{aligned}
&\Bigg| \tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} \\
&+ \tfrac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A \Bigg| \\
\leq\ & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4} \left(\int_0^1 \int_0^1 |1-2t|\,|1-2s|\,dtds \right)^{1-\frac{1}{q}} \\
&\times \left(\int_0^1 \int_0^1 |1-2t|\,|1-2s| \left| \tfrac{\partial^2 f}{\partial t \partial s} \left(a + t\eta_1(b,a), c + s\eta_2(d,c)\right) \right|^q dtds \right)^{\frac{1}{q}}.
\end{aligned} \tag{3.19}$$

Using (α_1, m_1)-(α_2, m_2)-preinvexity in the first sens, we get

$$\begin{aligned}
&\Bigg| \tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} \\
&+ \tfrac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A \Bigg| \\
\leq\ & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4} \left(\int_0^1 \int_0^1 |1-2t|\,|1-2s|\,dtds \right)^{1-\frac{1}{q}} \\
&\times \Bigg(\left| \tfrac{\partial^2 f}{\partial t \partial s}(a,c) \right|^q \int_0^1 \int_0^1 |1-2t|\,|1-2s|\,(1-t^{\alpha_1})(1-s^{\alpha_2})\,dtds \\
&+ m_2 \left| \tfrac{\partial^2 f}{\partial t \partial s}(a, \frac{d}{m_2})) \right|^q \int_0^1 \int_0^1 |1-2t|\,|1-2s|\,(1-t^{\alpha_1})\, s^{\alpha_2} dtds \\
&+ m_1 \left| \tfrac{\partial^2 f}{\partial t \partial s}(\frac{b}{m_1}, c) \right|^q \int_0^1 \int_0^1 |1-2t|\,|1-2s|\, t^{\alpha_1} (1-s^{\alpha_2})\,dtds \\
&+ m_1 m_2 \left| \tfrac{\partial^2 f}{\partial t \partial s}(\frac{b}{m_1}, \frac{d}{m_2}) \right|^q \int_0^1 \int_0^1 |1-2t|\,|1-2s|\, t^{\alpha_1} s^{\alpha_2} dtds \Bigg)^{\frac{1}{q}}.
\end{aligned} \tag{3.20}$$

Substituting (3.5)-(3.8) into (3.20), and taking into account that

$$\int_0^1 \int_0^1 |1-2t|\,|1-2s|\,dtds = \tfrac{1}{4},$$

we get the required inequality in (3.18). The proof is completed. □

Corollary 3.22. *Under the assumptions of Theorem 3.21, If* $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$ *is* (α, m)*-preinvex in the first sens on the co-ordinates with respect to* η_1 *and* η_2 *on* $K_1 \times K_2$, $q \in [1, \infty)$, *then the following inequality holds*

$$\begin{aligned}
&\Bigg|\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} \\
&+\tfrac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\Bigg| \\
\leq\;& \tfrac{\eta_1(b,a)\eta_2(d,c)}{4^{2-\frac{1}{q}}}\Bigg[\left(\tfrac{1}{2}-\theta\right)^2\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q \\
&+m\left(\tfrac{1}{2}-\theta\right)\theta\left[\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,\frac{d}{m}))\right|^q+\left|\tfrac{\partial^2 f}{\partial t\partial s}(\frac{b}{m},c)\right|^q\right] \\
&+(m\theta)^2\left|\tfrac{\partial^2 f}{\partial t\partial s}(\frac{b}{m},\frac{d}{m})\right|^q\Bigg]^{\frac{1}{q}},
\end{aligned} \tag{3.21}$$

where A is defined as in Lemma 2.13, θ *is defined as in Corollary 3.14 and* $\alpha, m \in (0,1]$.

Corollary 3.23. *Under the assumptions of Theorem 3.21, if* $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$ *is* m_1*-preinvex with respect to* η_1 *and* m_2*-preinvex with respect to* η_2 *on* $K_1 \times K_2$, $q \in [1, \infty)$, *then the following inequality holds*

$$\begin{aligned}
&\Bigg|\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} \\
&+\tfrac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\Bigg| \\
\leq\;& \tfrac{\eta_1(b,a)\eta_2(d,c)}{4^{2+\frac{1}{q}}}\Bigg[\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+m_2\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,\frac{d}{m_2}))\right|^q \\
&+m_1\left|\tfrac{\partial^2 f}{\partial t\partial s}(\frac{b}{m_1},c)\right|^q+m_1m_2\left|\tfrac{\partial^2 f}{\partial t\partial s}(\frac{b}{m_1},\frac{d}{m_2})\right|^q\Bigg]^{\frac{1}{q}},
\end{aligned} \tag{3.22}$$

where A is defined as in Lemma 2.13 and $m_1, m_2 \in (0,1]$.

Remark 3.24. *In Theorem 3.21, if we take* $m_1 = m_2 = \alpha_1 = \alpha_2 = 1$, *we obtain Theorem 10 from [11]. Moreover, if we take* $\eta_1(b,a) = \eta_2(b,a) = b - a$, *we obtain Theorem 4 from [18].*

Theorem 3.25. *Let $f : K_1 \times K_2 \to \mathbb{R}$ be partially differentiable mapping such that $\frac{\partial^2 f}{\partial t \partial s} \in L([a, a+\eta_1(b,a)] \times [c, c+\eta_2(d,c)])$ with $\eta_1(b,a) > 0$ and $\eta_2(d,c) > 0$. If $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$ is (α_1, m_1)-(α_2, m_2)-preinvex in the first sens on the co-ordinates on $K_1 \times K_2$ with $q > 1$ and*

$$\varphi \geq [\eta_1(b,a)\, \eta_2(d,c)]^{q-1}, \tag{3.23}$$

then the following inequality holds

$$\begin{aligned}
&\Bigg| \tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} \\
&+ \tfrac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f(x,y)\, dx dy - A \Bigg| \\
\leq\ & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4} \\
&\times \Bigg[\left| \tfrac{\partial^2 f}{\partial t \partial s}(a,c) \right|^q \left[\tfrac{1+q+2\alpha_1}{2(1+q+\alpha_1)(q+1)} - \tfrac{1}{2^{\alpha_1}} \beta(1+\alpha_1, 1+q) \right] \\
&\times \left[\tfrac{1+q+2\alpha_2}{2(1+q+\alpha_2)(q+1)} - \tfrac{1}{2^{\alpha_2}} \beta(1+\alpha_2, 1+q) \right] \\
&+ m_2 \left| \tfrac{\partial^2 f}{\partial t \partial s}(a, \frac{d}{m_2})) \right|^q \left[\tfrac{1+q+2\alpha_1}{2(1+q+\alpha_1)(q+1)} - \tfrac{1}{2^{\alpha_1}} \beta(1+\alpha_1, 1+q) \right] \\
&\times \left[\tfrac{1}{(1+q+\alpha_2)} \left(1 - \tfrac{1}{2^{1+q+\alpha_2}}\right) + \tfrac{1}{2^{\alpha_2}} \beta(1+\alpha_2, 1+q) \right] \\
&+ m_1 \left| \tfrac{\partial^2 f}{\partial t \partial s}(\frac{b}{m_1}, c) \right|^q \left[\tfrac{1+q+2\alpha_2}{2(1+q+\alpha_2)(q+1)} - \tfrac{1}{2^{\alpha_2}} \beta(1+\alpha_2, 1+q) \right] \\
&\times \left[\tfrac{1}{(1+q+\alpha_1)} \left(1 - \tfrac{1}{2^{1+q+\alpha_1}}\right) + \tfrac{1}{2^{\alpha_1}} \beta(1+\alpha_1, 1+q) \right] \\
&+ m_1 m_2 \left| \tfrac{\partial^2 f}{\partial t \partial s}(\frac{b}{m_1}, \frac{d}{m_2}) \right|^q \left[\tfrac{1}{(1+q+\alpha_1)} \left(1 - \frac{1}{2^{1+q+\alpha_1}}\right) + \tfrac{1}{2^{\alpha_1}} \beta(1+\alpha_1, 1+q) \right] \\
&\times \left[\tfrac{1}{(1+q+\alpha_2)} \left(1 - \frac{1}{2^{1+q+\alpha_2}}\right) + \tfrac{1}{2^{\alpha_2}} \beta(1+\alpha_2, 1+q) \right] \Bigg]^{\frac{1}{q-1}},
\end{aligned} \tag{3.24}$$

where A is defined as in Lemma 2.13, $\alpha_1, m_1, \alpha_2, m_2 \in (0,1]$,

and $\varphi = \inf\limits_{(t,s) \in [a, a+\eta_1(b,a)] \times [c, c+\eta_2(d,c)]} \left| \frac{\partial^2 f}{\partial t \partial s}(t,s) \right|.$

Proof. From Lemma 2.13, we have

$$\begin{aligned}
&\left|\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\right.\\
&\left.+\tfrac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\right|^{q-1}\\
\leq\ & \left[\tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\right]^{q-1}\\
&\times\left[\int_0^1\int_0^1 |1-2t|\,|1-2s|\left|\tfrac{\partial^2 f}{\partial t\partial s}\left(a+t\eta_1(b,a),c+s\eta_2(d,c)\right)\right|dtds\right]^{q-1}.
\end{aligned} \tag{3.25}$$

On the other hand, from (3.23), we have

$$\begin{aligned}
&\int_0^1\int_0^1 |1-2t|\,|1-2s|\left|\tfrac{\partial^2 f}{\partial t\partial s}\left(a+t\eta_1(b,a),c+s\eta_2(d,c)\right)\right|dtds\\
=\ &\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} \tfrac{1}{\eta_1(b,a)\eta_2(d,c)}\left|1-2\tfrac{x-a}{\eta_1(b,a)}\right|\left|1-2\tfrac{y-c}{\eta_2(d,c)}\right|\left|\tfrac{\partial^2 f}{\partial y\partial x}(x,y)\right|dxdy\\
\geq\ & \varphi \geq \left[\eta_1(b,a)\,\eta_2(d,c)\right]^{q-1}.
\end{aligned}$$

The above inequality allows us to apply the first assertion of Lemma 2.14, thus (3.25) becomes

$$\begin{aligned}
&\left|\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\right.\\
&\left.+\tfrac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\right|^{q-1}\\
\leq\ & \left[\tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\right]^{q-1}\\
&\times\int_0^1\int_0^1 |1-2t|^q\,|1-2s|^q\left|\tfrac{\partial^2 f}{\partial t\partial s}\left(a+t\eta_1(b,a),c+s\eta_2(d,c)\right)\right|^q dtds.
\end{aligned} \tag{3.26}$$

Using (α_1, m_1)-(α_2, m_2)-preinvexity in the first sens, we get

$$\begin{aligned}
&\left| \tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} \right. \\
&\left. + \tfrac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A \right|^{q-1} \\
\leq\ & \left[\tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\right]^{q-1} \\
&\times \left(\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q \int_0^1\int_0^1 |1-2t|^q |1-2s|^q (1-t^{\alpha_1})(1-s^{\alpha_2})\,dtds \right. \\
&+ m_2 \left|\tfrac{\partial^2 f}{\partial t\partial s}(a, \frac{d}{m_2}))\right|^q \int_0^1\int_0^1 |1-2t|^q |1-2s|^q (1-t^{\alpha_1}) s^{\alpha_2}\,dtds \\
&+ m_1 \left|\tfrac{\partial^2 f}{\partial t\partial s}(\frac{b}{m_1}, c)\right|^q \int_0^1\int_0^1 |1-2t|^q |1-2s|^q t^{\alpha_1}(1-s^{\alpha_2})\,dtds \\
&\left. + m_1 m_2 \left|\tfrac{\partial^2 f}{\partial t\partial s}(\frac{b}{m_1}, \frac{d}{m_2})\right|^q \int_0^1\int_0^1 |1-2t|^q |1-2s|^q t^{\alpha_1} s^{\alpha_2}\,dtds \right). \qquad (3.27)
\end{aligned}$$

Clearly, we have

$$\int_{\frac{1}{2}}^1 (2t-1)^{q+\alpha_1}\,dt = \tfrac{1}{2(1+q+\alpha_1)}, \qquad (3.28)$$

$$\int_{\frac{1}{2}}^1 t^{q+\alpha_1} dt = \tfrac{1}{(1+q+\alpha_1)}\left(1 - \tfrac{1}{2^{1+q+\alpha_1}}\right), \qquad (3.29)$$

$$\begin{aligned}
\int_0^{\frac{1}{2}} (1-2t)^q t^{\alpha_1} dt &= \tfrac{1}{2^{\alpha_1}} \int_0^1 (1-\tau)^q \tau^{\alpha_1} d\tau \\
&= \tfrac{1}{2^{\alpha_1}} \beta(1+\alpha_1, 1+q), \qquad (3.30)
\end{aligned}$$

$$\int_0^1 |1-2t|^q\,dt = \tfrac{1}{1+q}. \qquad (3.31)$$

Since $2t-1 \leq t$ is true for all $t \leq 1$, then we have

$$\int_{\frac{1}{2}}^1 (2t-1)^{q+\alpha_1}\,dt \leq \int_{\frac{1}{2}}^1 (2t-1)^q t^{\alpha_1} dt \leq \int_{\frac{1}{2}}^1 t^{q+\alpha_1} dt. \qquad (3.32)$$

Using (3.28) and (3.29) into (3.32), we get

$$\frac{1}{2(1+q+\alpha_1)} \leq \int_{\frac{1}{2}}^{1} (2t-1)^q \, t^{\alpha_1} dt \leq \frac{1}{(1+q+\alpha_1)} \left(1 - \frac{1}{2^{1+q+\alpha_1}}\right). \tag{3.33}$$

From (3.30) and (3.33), we obtain

$$\begin{aligned} & \frac{1}{2^{\alpha_1}} \beta(1+\alpha_1, 1+q) + \frac{1}{2(1+q+\alpha_1)} \\ \leq \; & \int_0^1 |1-2t|^q \, t^{\alpha_1} dt \\ \leq \; & \frac{1}{(1+q+\alpha_1)} \left(1 - \frac{1}{2^{1+q+\alpha_1}}\right) + \frac{1}{2^{\alpha_1}} \beta(1+\alpha_1, 1+q). \end{aligned} \tag{3.34}$$

Since

$$\int_0^1 |1-2t|^q \,(1-t^{\alpha_1})\, dt = \int_0^1 |1-2t|^q \, dt - \int_0^1 |1-2t|^q \, t^{\alpha_1} dt,$$

from (3.31) and (3.34), we get

$$\begin{aligned} \int_0^1 |1-2t|^q \,(1-t^{\alpha_1})\, dt \quad & \leq \quad \frac{1}{q+1} - \frac{\beta(1+\alpha_1, 1+q)}{2^{\alpha_1}} - \frac{1}{2(1+q+\alpha_1)} \\ & = \quad \frac{1+q+2\alpha_1}{2(1+q+\alpha_1)(q+1)} - \frac{\beta(1+\alpha_1, 1+q)}{2^{\alpha_1}}. \end{aligned} \tag{3.35}$$

Thus, we have

$$\begin{aligned} & \int_0^1 \int_0^1 |1-2t|^q \, |1-2s|^q \,(1-t^{\alpha_1})\,(1-s^{\alpha_2})\, dtds \\ \leq \; & \left[\frac{1+q+2\alpha_1}{2(1+q+\alpha_1)(q+1)} - \frac{1}{2^{\alpha_1}} \beta(1+\alpha_1, 1+q)\right] \\ & \times \left[\frac{1+q+2\alpha_2}{2(1+q+\alpha_2)(q+1)} - \frac{1}{2^{\alpha_2}} \beta(1+\alpha_2, 1+q)\right], \end{aligned} \tag{3.36}$$

$$\begin{aligned} & \int_0^1 \int_0^1 |1-2t|^q \, |1-2s|^q \,(1-t^{\alpha_1})\, s^{\alpha_2} dtds \\ \leq \; & \left[\frac{1+q+2\alpha_1}{2(1+q+\alpha_1)(q+1)} - \frac{1}{2^{\alpha_1}} \beta(1+\alpha_1, 1+q)\right] \\ & \times \left[\frac{1}{(1+q+\alpha_2)} \left(1 - \frac{1}{2^{1+q+\alpha_2}}\right) + \frac{1}{2^{\alpha_2}} \beta(1+\alpha_2, 1+q)\right], \end{aligned} \tag{3.37}$$

$$\begin{aligned}
&\int_0^1\int_0^1 |1-2t|^q |1-2s|^q t^{\alpha_1}(1-s^{\alpha_2})\,dtds \\
\leq\ & \left[\tfrac{1+q+2\alpha_2}{2(1+q+\alpha_2)(q+1)} - \frac{1}{2^{\alpha_2}}\beta(1+\alpha_2, 1+q)\right] \\
&\times\left[\tfrac{1}{(1+q+\alpha_1)}\left(1-\tfrac{1}{2^{1+q+\alpha_1}}\right) + \frac{1}{2^{\alpha_1}}\beta(1+\alpha_1, 1+q)\right], \qquad (3.38)
\end{aligned}$$

$$\begin{aligned}
&\int_0^1\int_0^1 |1-2t|^q |1-2s|^q t^{\alpha_1}s^{\alpha_2}\,dtds \\
\leq\ & \left[\tfrac{1}{(1+q+\alpha_1)}\left(1-\frac{1}{2^{1+q+\alpha_1}}\right) + \frac{1}{2^{\alpha_1}}\beta(1+\alpha_1, 1+q)\right] \\
&\times\left[\tfrac{1}{(1+q+\alpha_2)}\left(1-\frac{1}{2^{1+q+\alpha_2}}\right) + \frac{1}{2^{\alpha_2}}\beta(1+\alpha_2, 1+q)\right]. \qquad (3.39)
\end{aligned}$$

Substituting (3.36)-(3.39) into (3.27), we obtain the desired inequality in (3.24). The proof is completed. □

Corollary 3.26. *Under the assumptions of Theorem* 3.25*, If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ *is* (α, m)*-preinvex in the first sens on the co-ordinates with respect to* η_1 *and* η_2 *on* $K_1 \times K_2$ *with* $q \in [1, \infty)$*, then the following inequality holds*

$$\begin{aligned}
&\left|\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\right. \\
&\left. + \tfrac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\right| \\
\leq\ & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\left[\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\left[\tfrac{1+q+2\alpha}{2(1+q+\alpha)(q+1)} - \tfrac{1}{2^\alpha}\beta(1+\alpha, 1+q)\right]^2\right. \\
&+m\left[\left|\tfrac{\partial^2 f}{\partial t\partial s}(a, \frac{d}{m}))\right|^q + \left|\tfrac{\partial^2 f}{\partial t\partial s}(\frac{b}{m}, c)\right|^q\right]\left[\tfrac{1+q+2\alpha}{2(1+q+\alpha)(q+1)} - \tfrac{1}{2^\alpha}\beta(1+\alpha, 1+q)\right] \\
&\times\left[\tfrac{1}{(1+q+\alpha)}\left(1-\tfrac{1}{2^{1+q+\alpha}}\right) + \tfrac{1}{2^\alpha}\beta(1+\alpha, 1+q)\right] \\
&\left.+ m^2\left|\tfrac{\partial^2 f}{\partial t\partial s}(\frac{b}{m}, \frac{d}{m})\right|^q\left[\tfrac{1}{(1+q+\alpha)}\left(1-\tfrac{1}{2^{1+q+\alpha}}\right) + \tfrac{1}{2^\alpha}\beta(1+\alpha, 1+q)\right]^2\right]^{\frac{1}{q-1}}, \qquad (3.40)
\end{aligned}$$

where A *is defined as in Theorem* 3.25 *and* $\alpha, m \in (0, 1]$.

Corollary 3.27. *Under the assumptions of Theorem* 3.25*, if* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ *is* m_1*-preinvex with respect to* η_1 *and* m_2*-preinvex with respect to* η_2 *on* $K_1 \times K_2$ *with* $q \in [1, \infty)$

then the following inequality holds

$$\begin{aligned}
&\left|\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\right.\\
&\left.+\tfrac{1}{\eta_1(b,a)\eta_2(d,c)}\int\limits_a^{a+\eta_1(b,a)}\int\limits_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\right|\\
\leq\ & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\left[\tfrac{1}{4(q+1)^2}\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\right.\\
&+\tfrac{m_2}{(2+q)^2}\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,\frac{d}{m_2}))\right|^q\left[\tfrac{q}{2(q+1)}\right]\left[\left(1-\tfrac{1}{2^{2+q}}\right)+\tfrac{1}{2(q+1)}\right]\\
&+\left|\tfrac{\partial^2 f}{\partial t\partial s}(\frac{b}{m_1},c)\right|^q\left[\tfrac{m_1}{2(q+1)(2+q)}\right]\left[\left(1-\tfrac{1}{2^{2+q}}\right)+\tfrac{1}{2(q+1)}\right]\\
&\left.+\tfrac{m_1 m_2}{(2+q)}\left|\tfrac{\partial^2 f}{\partial t\partial s}(\frac{b}{m_1},\frac{d}{m_2})\right|^q\left[\left(1-\tfrac{1}{2^{2+q}}\right)+\tfrac{1}{2(q+1)}\right]\right]^{\frac{1}{q}-1}. \qquad (3.41)
\end{aligned}$$

Corollary 3.28. *Under the assumptions of Theorem* 3.25, *if* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ *is preinvex on the co-ordinates with respect to* η_1 *and* η_2 *on* $K_1 \times K_2$ *with* $q \in [1,\infty)$, *then the following inequality holds*

$$\begin{aligned}
&\left|\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\right.\\
&\left.+\tfrac{1}{\eta_1(b,a)\eta_2(d,c)}\int\limits_a^{a+\eta_1(b,a)}\int\limits_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\right|\\
\leq\ & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\left(\tfrac{1}{4(q+1)^2}\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\right.\\
&+\tfrac{1}{(2+q)^2}\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,d))\right|^q\left[\tfrac{q}{2(q+1)}\right]\left[\left(1-\tfrac{1}{2^{2+q}}\right)+\tfrac{1}{2(q+1)}\right]\\
&+\left|\tfrac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q\left[\tfrac{1}{2(q+1)(2+q)}\right]\left[\left(1-\tfrac{1}{2^{2+q}}\right)+\tfrac{1}{2(q+1)}\right]\\
&\left.+\tfrac{1}{(2+q)}\left|\tfrac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\left[\left(1-\tfrac{1}{2^{2+q}}\right)+\tfrac{1}{2(q+1)}\right]\right)^{\frac{1}{q-1}}. \qquad (3.42)
\end{aligned}$$

Remark 3.29. *In Corollary* 3.28, *if we choose* $\eta_1(b,a) = \eta_2(b,a) = b - a$, *we* *obtain the following inequality*

$$\begin{aligned}
&\left| \tfrac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \tfrac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dxdy - A \right| \\
\leq \quad & \tfrac{(b-a)(d-c)}{4} \left[\tfrac{1}{4(q+1)^2} \left| \tfrac{\partial^2 f}{\partial t \partial s}(a,c) \right|^q \right. \\
& + \tfrac{1}{(2+q)^2} \left| \tfrac{\partial^2 f}{\partial t \partial s}(a,d)) \right|^q \left[\tfrac{q}{2(q+1)} \right] \left[\left(1 - \tfrac{1}{2^{2+q}}\right) + \tfrac{1}{2(q+1)} \right] \\
& + \left| \tfrac{\partial^2 f}{\partial t \partial s}(b,c) \right|^q \left[\tfrac{1}{2(q+1)(2+q)} \right] \left[\left(1 - \tfrac{1}{2^{2+q}}\right) + \tfrac{1}{2(q+1)} \right] \\
& \left. + \tfrac{1}{(2+q)} \left| \tfrac{\partial^2 f}{\partial t \partial s}(b,d) \right|^q \left[\left(1 - \tfrac{1}{2^{2+q}}\right) + \tfrac{1}{2(q+1)} \right] \right]^{\frac{1}{q-1}}. \qquad (3.43)
\end{aligned}$$

Theorem 3.30. *Let $f : K_1 \times K_2 \to \mathbb{R}$ be partially differentiable mapping such that $\frac{\partial^2 f}{\partial t \partial s} \in L([a, a+\eta_1(b,a)] \times [c, c+\eta_2(d,c)])$ with $\eta_1(b,a) > 0$ and $\eta_2(d,c) > 0$. If $\left| \frac{\partial^2 f}{\partial t \partial s} \right|^q$ is (α_1, m_1)-(α_2, m_2)-preinvex in the first sens on the co-ordinates on $K_1 \times K_2$, $q \geq 1$ and*

$$M \leq [\eta_1(b,a)\,\eta_2(d,c)]^{\frac{1-q}{q^2}}, \qquad (3.44)$$

then the following inequality holds

$$\begin{aligned}
&\left| \tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} \right. \\
&\left. + \tfrac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A \right| \\
\leq \quad & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4} \left[\left| \tfrac{\partial^2 f}{\partial t \partial s}(a,c) \right|^q \left[\tfrac{1+q+2\alpha_1}{2(1+q+\alpha_1)(q+1)} - \tfrac{1}{2^{\alpha_1}} \beta(1+\alpha_1, 1+q) \right] \right. \\
& \times \left[\tfrac{1+q+2\alpha_2}{2(1+q+\alpha_2)(q+1)} - \tfrac{1}{2^{\alpha_2}} \beta(1+\alpha_2, 1+q) \right] \\
& + m_2 \left| \tfrac{\partial^2 f}{\partial t \partial s}(a, \frac{d}{m_2})) \right|^q \left[\tfrac{1+q+2\alpha_1}{2(1+q+\alpha_1)(q+1)} - \tfrac{1}{2^{\alpha_1}} \beta(1+\alpha_1, 1+q) \right] \\
& \times \left[\tfrac{1}{(1+q+\alpha_2)} \left(1 - \tfrac{1}{2^{1+q+\alpha_2}}\right) + \tfrac{1}{2^{\alpha_2}} \beta(1+\alpha_2, 1+q) \right] \\
& + m_1 \left| \tfrac{\partial^2 f}{\partial t \partial s}(\frac{b}{m_1}, c) \right|^q \left[\tfrac{1+q+2\alpha_2}{2(1+q+\alpha_2)(q+1)} - \tfrac{1}{2^{\alpha_2}} \beta(1+\alpha_2, 1+q) \right] \\
& \times \left[\tfrac{1}{(1+q+\alpha_1)} \left(1 - \tfrac{1}{2^{1+q+\alpha_1}}\right) + \tfrac{1}{2^{\alpha_1}} \beta(1+\alpha_1, 1+q) \right] \\
& + m_1 m_2 \left| \tfrac{\partial^2 f}{\partial t \partial s}(\frac{b}{m_1}, \frac{d}{m_2}) \right|^q \left[\tfrac{1}{(1+q+\alpha_1)} \left(1 - \tfrac{1}{2^{1+q+\alpha_1}}\right) + \tfrac{1}{2^{\alpha_1}} \beta(1+\alpha_1, 1+q) \right] \\
& \left. \times \left[\tfrac{1}{(1+q+\alpha_2)} \left(1 - \tfrac{1}{2^{1+q+\alpha_2}}\right) + \tfrac{1}{2^{\alpha_2}} \beta(1+\alpha_2, 1+q) \right] \right]^{\frac{1}{q}-1},
\end{aligned} \qquad (3.45)$$

where A *is defined as in Lemma* 2.13*,* $\alpha_1, m_1, \alpha_2, m_2 \in (0,1]$*,*

and $M = \sup_{(t,s)\in[a,a+\eta_1(b,a)]\times[c,c+\eta_2(d,c)]} \left|\frac{\partial^2 f}{\partial t \partial s}(t,s)\right|^q$.

Proof. In the case where $q = 1$, the proof is similar to that of Theorem 3.13, now we will treat the case where $q > 1$.

From Lemma 2.13, we have

$$\begin{aligned} &\Bigg|\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} \\ &+\tfrac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\Bigg| \\ \leq\ & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4} \\ &\times\int_0^1\int_0^1 \left[|1-2t|^q|1-2s|^q\left|\tfrac{\partial^2 f}{\partial t\partial s}(a+t\eta_1(b,a),c+s\eta_2(d,c))\right|^q\right]^{\frac{1}{q}} dtds, \end{aligned} \tag{3.46}$$

on the other hand from (3.44), we have

$$\begin{aligned} &\int_0^1\int_0^1 |1-2t|^q|1-2s|^q\left|\tfrac{\partial^2 f}{\partial t\partial s}(a+t\eta_1(b,a),c+s\eta_2(d,c))\right|^q dtds \\ =\ &\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} \tfrac{1}{\eta_1(b,a)\eta_2(d,c)}\left|1-2\tfrac{x-a}{\eta_1(b,a)}\right|^q\left|1-2\tfrac{y-c}{\eta_2(d,c)}\right|^q\left|\tfrac{\partial^2 f}{\partial y\partial x}(x,y)\right|^q dxdy \\ \leq\ & M^q \leq [\eta_1(b,a)\eta_2(d,c)]^{\frac{1}{q}-1}. \end{aligned}$$

Thus, the above inequality allows us to apply the second assertion of Lemma 2.14 and (3.46) becomes

$$\begin{aligned} &\Bigg|\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} \\ &+\tfrac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\Bigg| \\ \leq\ & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4} \\ &\times\left(\int_0^1\int_0^1 |1-2t|^q|1-2s|^q\left|\tfrac{\partial^2 f}{\partial t\partial s}(a+t\eta_1(b,a),c+s\eta_2(d,c))\right|^q dtds\right)^{\frac{1}{q}-1}. \end{aligned} \tag{3.47}$$

Using (α_1, m_1)-(α_2, m_2)-preinvexity in the first sens, we have

$$\begin{aligned}
&\left|\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\right. \\
&\left.+\tfrac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\right| \\
\leq\ &\tfrac{\eta_1(b,a)\eta_2(d,c)}{4} \\
&\times\left[\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q \int_0^1\int_0^1 |1-2t|^q\,|1-2s|^q\,(1-t^{\alpha_1})(1-s^{\alpha_2})\,dtds\right. \\
&+m_2\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,\frac{d}{m_2}))\right|^q \int_0^1\int_0^1 |1-2t|^q\,|1-2s|^q\,(1-t^{\alpha_1})\,s^{\alpha_2}dtds \\
&+m_1\left|\tfrac{\partial^2 f}{\partial t\partial s}(\frac{b}{m_1},c)\right|^q \int_0^1\int_0^1 |1-2t|^q\,|1-2s|^q\,t^{\alpha_1}(1-s^{\alpha_2})\,dtds \\
&\left.+\,m_1m_2\left|\tfrac{\partial^2 f}{\partial t\partial s}(\frac{b}{m_1},\frac{d}{m_2})\right|^q \int_0^1\int_0^1 |1-2t|^q\,|1-2s|^q\,t^{\alpha_1}s^{\alpha_2}dtds\right]^{\frac{1}{q}-1}.
\end{aligned} \tag{3.48}$$

Using (3.36)-(3.39) into (3.48), we obtain the required inequality in (3.45). The proof is completed. □

Corollary 3.31. *Under the assumptions of Theorem* 3.30*, If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ *is* (α, m)*-preinvex on the co-ordinates on* $K_1 \times K_2$*,* $q \in [1, \infty)$*, then the following inequality holds*

$$\begin{aligned}
&\left|\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\right. \\
&\left.+\tfrac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\right| \\
\leq\ &\tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\left[\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\left[\tfrac{1+q+2\alpha}{2(1+q+\alpha)(q+1)}-\tfrac{1}{2^\alpha}\beta(1+\alpha,1+q)\right]^2\right. \\
&+m\left[\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,\frac{d}{m}))\right|^q+\left|\tfrac{\partial^2 f}{\partial t\partial s}(\frac{b}{m},c)\right|^q\right] \\
&\times\left[\tfrac{1+q+2\alpha}{2(1+q+\alpha)(q+1)}-\tfrac{1}{2^\alpha}\beta(1+\alpha,1+q)\right]
\end{aligned}$$

$$\times\left[\tfrac{1}{(1+q+\alpha)}\left(1-\tfrac{1}{2^{1+q+\alpha}}\right)+\tfrac{1}{2^{\alpha}}\beta\left(1+\alpha,1+q\right)\right]$$
$$+\,m^{2}\left|\tfrac{\partial^{2}f}{\partial t\partial s}(\frac{b}{m},\frac{d}{m})\right|^{q}\left[\tfrac{1}{(1+q+\alpha)}\left(1-\tfrac{1}{2^{1+q+\alpha}}\right)+\tfrac{1}{2^{\alpha}}\beta\left(1+\alpha,1+q\right)\right]^{2}\Bigg]^{\frac{1}{q}-1},\tag{3.49}$$

where A is defined as in Theorem 3.25 and $\alpha, m \in (0,1]$.

Corollary 3.32. *Under the assumptions of Theorem 3.30, if $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is m_1-preinvex with respect to η_1 and m_2-preinvex with respect to η_2 on $K_1\times K_2$ with $q\in[1,\infty)$, then the following inequality holds*

$$\begin{aligned}&\Bigg|\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\\&+\tfrac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)}f(x,y)\,dxdy-A\Bigg|\\ \le\ &\tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\Bigg[\tfrac{1}{4(q+1)^2}\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\\&+\tfrac{m_2}{(2+q)^2}\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,\frac{d}{m_2}))\right|^q\left[\tfrac{q}{2(q+1)}\right]\left[\left(1-\tfrac{1}{2^{2+q}}\right)+\tfrac{1}{2(q+1)}\right]\\&+\left|\tfrac{\partial^2 f}{\partial t\partial s}(\frac{b}{m_1},c)\right|^q\left[\tfrac{m_1}{2(q+1)(2+q)}\right]\left[\left(1-\tfrac{1}{2^{2+q}}\right)+\tfrac{1}{2(q+1)}\right]\\&+\tfrac{m_1m_2}{(2+q)}\left|\tfrac{\partial^2 f}{\partial t\partial s}(\frac{b}{m_1},\frac{d}{m_2})\right|^q\left[\left(1-\tfrac{1}{2^{2+q}}\right)+\tfrac{1}{2(q+1)}\right]\Bigg]^{\frac{1}{q}-1}.\end{aligned}\tag{3.50}$$

Corollary 3.33. *Under the assumptions of Theorem 3.30, if $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is preinvex on the co-ordinates with respect to η_1 and η_2 on $K_1\times K_2$ with $q\in[1,\infty)$, then the following inequality holds*

$$\begin{aligned}&\Bigg|\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\\&+\tfrac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)}f(x,y)\,dxdy-A\Bigg|\\ \le\ &\tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\Bigg[\tfrac{1}{4(q+1)^2}\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\\&+\tfrac{1}{(2+q)^2}\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,d))\right|^q\left[\tfrac{q}{2(q+1)}\right]\left[\left(1-\tfrac{1}{2^{2+q}}\right)+\tfrac{1}{2(q+1)}\right]\\&+\left|\tfrac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q\left[\tfrac{1}{2(q+1)(2+q)}\right]\left[\left(1-\tfrac{1}{2^{2+q}}\right)+\tfrac{1}{2(q+1)}\right]\\&+\tfrac{1}{(2+q)}\left|\tfrac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\left[\left(1-\tfrac{1}{2^{2+q}}\right)+\tfrac{1}{2(q+1)}\right]\Bigg]^{\frac{1}{q}-1}.\end{aligned}\tag{3.51}$$

Remark 3.34. *If we take in Corollary* 3.33, $\eta_1(b,a) = \eta_2(b,a) = b - a$, *we obtain the following inequality*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy - A\right|$$
$$\leq \frac{(b-a)(d-c)}{4}\left[\frac{1}{4(q+1)^2}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\right.$$
$$+\frac{1}{(2+q)^2}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d))\right|^q\left[\frac{q}{2(q+1)}\right]\left[\left(1-\frac{1}{2^{2+q}}\right)+\frac{1}{2(q+1)}\right]$$
$$+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q\left[\frac{1}{2(q+1)(2+q)}\right]\left[\left(1-\frac{1}{2^{2+q}}\right)+\frac{1}{2(q+1)}\right]$$
$$\left.+\frac{1}{(2+q)}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\left[\left(1-\frac{1}{2^{2+q}}\right)+\frac{1}{2(q+1)}\right]\right]^{\frac{1}{q}-1}. \tag{3.52}$$

Remark 3.35. *Since* $\frac{1}{q} - 1 < \frac{1}{q-1}$ *then if* $A > 1$, *the estimation given in Theorem* 3.30 *is better than the one given in Theorem* 3.25, *otherwise the opposite is true, where*

$$A = \left[\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\left[\frac{1+q+2\alpha_1}{2(1+q+\alpha_1)(q+1)} - \frac{1}{2^{\alpha_1}}\beta(1+\alpha_1, 1+q)\right]\right.$$
$$\times\left[\frac{1+q+2\alpha_2}{2(1+q+\alpha_2)(q+1)} - \frac{1}{2^{\alpha_2}}\beta(1+\alpha_2, 1+q)\right]$$
$$+m_2\left|\frac{\partial^2 f}{\partial t\partial s}(a,\frac{d}{m_2}))\right|^q\left[\frac{1+q+2\alpha_1}{2(1+q+\alpha_1)(q+1)} - \frac{1}{2^{\alpha_1}}\beta(1+\alpha_1, 1+q)\right]$$
$$\times\left[\frac{1}{(1+q+\alpha_2)}\left(1-\frac{1}{2^{1+q+\alpha_2}}\right) + \frac{1}{2^{\alpha_2}}\beta(1+\alpha_2, 1+q)\right]$$
$$+m_1\left|\frac{\partial^2 f}{\partial t\partial s}(\frac{b}{m_1}, c)\right|^q\left[\frac{1+q+2\alpha_2}{2(1+q+\alpha_2)(q+1)} - \frac{1}{2^{\alpha_2}}\beta(1+\alpha_2, 1+q)\right]$$
$$\times\left[\frac{1}{(1+q+\alpha_1)}\left(1-\frac{1}{2^{1+q+\alpha_1}}\right) + \frac{1}{2^{\alpha_1}}\beta(1+\alpha_1, 1+q)\right]$$
$$+m_1m_2\left|\frac{\partial^2 f}{\partial t\partial s}(\frac{b}{m_1}, \frac{d}{m_2})\right|^q\left[\frac{1}{(1+q+\alpha_1)}\left(1-\frac{1}{2^{1+q+\alpha_1}}\right) + \frac{1}{2^{\alpha_1}}\beta(1+\alpha_1, 1+q)\right]$$
$$\left.\times\left[\frac{1}{(1+q+\alpha_2)}\left(1-\frac{1}{2^{1+q+\alpha_2}}\right) + \frac{1}{2^{\alpha_2}}\beta(1+\alpha_2, 1+q)\right]\right],$$

and $\frac{\partial^2 f}{\partial x\partial y}(x,y)$ *satisfies* (3.23) *and* (3.44).

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