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On Decompositions of Intra-Regular and Left Regular Ordered $\star$-Semigroups

Chong-Yih Wu

Department of Applied Mathematics, National Pingtung University, Taiwan
e-mail : cwu@mail.nptu.edu.tw

Abstract : Let S be an intra-regular or left (resp. right) regular ordered $\star$-semigroup with order preserving involution $\star$. Structure theorems referring to the decompositions of such semigroups into their simples subsemigroups are developed. Also in view of semilattice congruences we characterize intra-regular ordered $\star$-semigroups in which any two ideals are comparable under inclusion relation $\subseteq$.

Keywords : ordered $\star$-semigroup; intra-regular; left(right) regular; (complete) semilattice congruence.
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1 Introduction and Preliminaries

Nordall defined $\star$ semigroup in terms of unary operation $\star$ [1], then a regular $\star$ semigroup was introduced. In order to let the results which Kehayopulu did in ordered semigroups be more flexible, Wu imposed unary operation $\star$ on ordered semigroups with additional property that $\star$ preserves orders [2]. The word "flexible" refers to the fact that if $\star$ is an identity mapping, then the results in ordered $\star$-semigroups will be the same as those in ordered semigroups. Ordered $\star$-semigroups, in which all ideals are (weakly) prime, was characterized. And after the analogue of the definition of filters was made, he created a characterization on intra-regular ordered $\star$-semigroups in terms of the least filter [2]. In this paper we will develop further analogous results which provided in [3, 4].

An ordered semigroup S is a partial ordering set at the same semigroup such that for any $a, b, x \in S$, $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$. An ordered semigroup

S with a unary operation $\star : S \longrightarrow S$ is called an ordered $\star$-semigroup if it satisfies $(x^\star)^\star = x$ and $(xy)^\star = y^\star x^\star$ for any $x, y \in S$. Such a unary operation $\star$ is called an involution [1]. If for any a, b with $a \geq b$, we have $a^\star \geq b^\star$, then $\star$ is called an order preserving involution [2]. Let S be an ordered $\star$-semigroup, we denote $(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}$ for $H \subseteq S$ [2].

Let S be a $\star$-semigroup (or an ordered $\star$-semigroup) and T be a subset of S. Then $S = S^\star$, but $T = T^\star$ is not necessary. However if $T = T^\star$, then $a^\star \in T$ if and only if $a \in T$. Let S be an ordered $\star$-semigroup with order preserving involution and I be an ideal of S. Then $I^\star$ will be an ideal ([2] Proposition 2.2).

Definition 1.1. ([2]; Definition 2.3) Let S be an ordered $\star$-semigroup. A subset T of S is called *prime* if $ab \in T$ implies $a^\star \in T$ or $b^\star \in T$.

Definition 1.2. ([2]; Definition 2.5) Let S be an ordered $\star$-semigroup. A subset T of S is called *semiprime* if $aa \in T$ implies $a^\star \in T$.

Definition 1.3. ([2]; Definition 2.9) An ordered $\star$-semigroup S is called *intra-regular* if $a \in (Sa^\star a^\star S]$ for any $a \in S$.

Definition 1.4. ([2]; Definition 3.1) Let S be an ordered $\star$-semigroup. A subsemigroup F of S is called a *filter* if

1. for any $a, b \in S$, $ab \in F$ implies $a^\star \in F$ and $b^\star \in F$,
2. for any $a \in F$, $c \in S$, $c \geq a$ implies $c \in F$.

Let $N(x)$ be the least filter of S containing x. Let $\mathcal{N}$ defined by $\mathcal{N} := \{(x, y) \in S \times S \mid N(x) = N(y)\}$. A congruence on ordered $\star$-semigroup S is an equivalence relation σ on S which preserves both $\cdot$ and $\star$. In other words, if $(a, b) \in \sigma$, then $(a^\star, b^\star) \in \sigma$ [1].

Definition 1.5. ([2]; Definition 3.2) A congruence σ on ordered $\star$-semigroup S is called *semilattice congruence* if $(a^\star a^\star, a) \in \sigma$ and $(ab, ba) \in \sigma$ for all $a, b \in S$. A semilattice congruence σ on S is called *complete* if $a \leq b$ implies $(a, ab) \in \sigma$.

Proposition 1.6. ([2]; Proposition 3.3) *Let S be an ordered $\star$-semigroup. Then the relation $\mathcal{N}$ is a complete semilattice congruence on S.*

Let $a \in S$. We denote the $\mathcal{N}$-class of S containing a by $(a)_\mathcal{N}$.

2 Decomposition of Intra-Regular Ordered $\star$-Semigroup

Let S be an ordered $\star$-semigroup. In this paragraph, we first introduce an equivalence relation $\mathcal{N}$ relating least filter of S. Then it is shown that $\mathcal{N}$ is a semilattice congruence. If S is intra-regular, then each equivalence class of $\mathcal{N}$ is a simple subsemigroup of S. From this we can induce that S is a union of these simple subsemigroups satisfying some special properties. On the other hand if S

is a union of these simple subsemigroups satisfying some special properties, then S can be proved to be intra-regular. Furthermore we find a characterization of intra-regular ordered $\star$-semigroup in which any two ideals are comparable under the inclusion relation $\subseteq$.

A non-empty subset L (resp. R) of S is called a left (resp. right) ideal of ordered $\star$-semigroup S if (1) $SL \subseteq L$ (resp. $RS \subseteq R$), and (2) $a \in L$ (resp. R), $S \ni b \leq a$ implies $b \in L$ (resp. R). I is called an ideal of S if it is both a left and a right ideal of S [2].

Proposition 2.1. *Let S be an ordered $\star$-semigroup with order preserving involution $\star$. Then $(a)_{\mathcal{N}}$ is a subsemigroup of S for any $a \in S$.*

Proof. We claim that $(x, x^\star) \in \mathcal{N}$ for any $x \in S$. Since $x^\star \in N(x^\star)$, we have $x^\star x^\star \in N(x^\star)$ because $N(x^\star)$ is a subsemigroup. So $x \in N(x^\star)$ by Definition 1.4. Thus $N(x) \subseteq N(x^\star)$. Similarly $N(x^\star) \subseteq N(x)$. Therefore $N(x) = N(x^\star)$, i.e. $(x, x^\star) \in \mathcal{N}$.

Let $a \in S$. To show that $(a)_{\mathcal{N}}$ is a subsemigroup it suffices to prove that $bc \in (a)_{\mathcal{N}}$ for any $b, c \in (a)_{\mathcal{N}}$. Since $b, c \in (a)_{\mathcal{N}}$, we have $(b, c) \in \mathcal{N}$, hence $(bc, cc) \in \mathcal{N}$ by Proposition 1.6. Also since $(cc, c^\star) \in \mathcal{N}$ by Proposition 1.6 and $(c^\star, c) \in \mathcal{N}$ as claimed above, we have $(bc, c) \in \mathcal{N}$. Consequently, $(bc, a) \in \mathcal{N}$ because $(c, a) \in \mathcal{N}$. This means that $bc \in (a)_{\mathcal{N}}$, hence $(a)_{\mathcal{N}}$ is a subsemigroup of S. □

Proposition 2.2. ([2]; Proposition 3.4) *Let S be an ordered $\star$-semigroup with order preserving involution $\star$. Then S is intra-regular if and only if $N(x) = \{y \in S \mid x \in (Sy^\star S]\}$.*

Theorem 2.3. ([2]; Theorem 3.5) *Let S be an ordered $\star$-semigroup with order preserving involution $\star$. Then S is intra-regular if and only if $\mathcal{N} = \mathcal{I}$.*

A subsemigroup T of ordered $\star$-semigroup S is called simple if for every ideal I of T we have $I = T$. Also as in ordered semigroups [3,5], if σ is a congruence on ordered $\star$-semigroup S, then the multiplication "." on the set $S/\sigma := \{(x)_\sigma | x \in S\}$ is defined by $(x)_\sigma \cdot (y)_\sigma := (xy)_\sigma$ for any $x, y \in S$, and $(S/\sigma, \cdot)$ is a semigroup. Then since $\mathcal{N}$ is a complete semilattice congruence and $(x, x^\star) \in \mathcal{N}$, it is easy to see that $(x)_{\mathcal{N}}(y)_{\mathcal{N}} := (xy)_{\mathcal{N}} = (yx)_{\mathcal{N}} = (y)_{\mathcal{N}}(x)_{\mathcal{N}}$ and $(x, x^2) \in \mathcal{N}$. These facts will be used in the proof of Proposition 2.4 below.

Proposition 2.4. *Let S be an intra-regular ordered $\star$-semigroup with order preserving involution $\star$. Then*

1. *$(a)_{\mathcal{N}}$ is a simple subsemigroup of S for any $a \in S$,*
2. *$(a)_{\mathcal{N}} = ((a)_{\mathcal{N}})^\star$ for any $a \in S$.*

Proof. (1) By Proposition 2.1, we need only show that $(a)_{\mathcal{N}}$ is simple.

We first claim that $(b^\star)_{\mathcal{N}} \subseteq (Sb^3S]$ for any $b \in S$. Since $b^\star \leq b^\star$, $(b^\star)^2 \leq (b^\star)^2$, we get that $(b^\star, (b^\star)^2) \in \mathcal{N}$ and $((b^\star)^2, (b^\star)^4) \in \mathcal{N}$ by Proposition 1.6. Hence

$(b^\star)_{\mathcal{N}} = ((b^\star)^4)_{\mathcal{N}}$. Let $x \in (b^\star)_{\mathcal{N}}$. Then $x \in ((b^\star)^4)_{\mathcal{N}}$. So $N(x) = N((b^\star)^4)$, thus $(b^\star)^4 \in N(x)$. This implies $x \in (Sb^4S]$ by Proposition 2.2. Consequently, $(b^\star)_{\mathcal{N}} \subseteq (Sb^3S]$ since $(Sb^4S] \subseteq (Sb^3S]$.

To show that $(a)_{\mathcal{N}}$ is simple it suffices to prove that $(a)_{\mathcal{N}} \subseteq (Sb^3S]$ for any $b \in I$ and I is any ideal $(a)_{\mathcal{N}}$. For in this case if $y \in (a)_{\mathcal{N}}$, then $y \in (Sb^3S]$. Hence $y \leq u_1b^3u_2 = (u_1b)b(bu_2)$ for some $u_1, u_2 \in S$. Therefore $u_1b \in (u_1b)_{\mathcal{N}} := (u_1)_{\mathcal{N}}(b)_{\mathcal{N}} = (u_1)_{\mathcal{N}}(y)_{\mathcal{N}} = (u_1)_{\mathcal{N}}(yu_1b^3u_2)_{\mathcal{N}} = (yu_1b^3u_2)_{\mathcal{N}} = (y)_{\mathcal{N}} = (a)_{\mathcal{N}}$. Similarly $bu_2 \in (bu_2)_{\mathcal{N}} := (b)_{\mathcal{N}}(u_2)_{\mathcal{N}} = (y)_{\mathcal{N}}(u_2)_{\mathcal{N}} = (yu_1b^3u_2)_{\mathcal{N}}(u_2)_{\mathcal{N}} = (yu_1b^3u_2)_{\mathcal{N}} = (y)_{\mathcal{N}} = (a)_{\mathcal{N}}$. Now since I is an ideal of $(a)_{\mathcal{N}}$, we get $u_1b,\, bu_2 \in (a)_{\mathcal{N}}$. This implies that $y \in (I] = I$, hence $(a)_{\mathcal{N}} \subseteq I$. Consequently $(a)_{\mathcal{N}} = I$.

Now to complete the proof we show that $(a)_{\mathcal{N}} \subseteq (Sb^3S]$. Since $b \in I \subseteq (a)_{\mathcal{N}}$, we have $(b, a) \in \mathcal{N}$. Then $N(b) = N(a)$. Thus $b \in N(a)$, so $bb \in N(a)$ because $N(a)$ is a subsemigroup. This implies $b^\star \in N(a)$ because $N(a)$ is a filter. Hence $N(b^\star) \subseteq N(a)$. On the other hand, since $(a^\star, b^\star) \in \mathcal{N}$, we have $a^\star \in N(b^\star)$. Similarly we can get that $N(a) \subseteq N(b^\star)$. Therefore $N(b^\star) = N(a)$, i.e. $(a)_{\mathcal{N}} = (b^\star)_{\mathcal{N}}$. This implies $(a)_{\mathcal{N}} \subseteq (Sb^3S]$ because $(b^\star)_{\mathcal{N}} \subseteq (Sb^3S]$ as claimed above.

(2) Let $y \in (a)_{\mathcal{N}}$. Then $yy \in (a)_{\mathcal{N}}$ because $(a)_{\mathcal{N}}$ is a subsemigroup. This implies $y^\star \in (a)_{\mathcal{N}}$ because $(yy, y^\star) \in \mathcal{N}$. Thus $y \in ((a)_{\mathcal{N}})^\star$, i.e. $(a)_{\mathcal{N}} \subseteq ((a)_{\mathcal{N}})^\star$. By symmetry, $((a)_{\mathcal{N}})^\star \subseteq (a)_{\mathcal{N}}$. So $((a)_{\mathcal{N}})^\star = (a)_{\mathcal{N}}$. □

Proposition 2.5. *Let S be an ordered $\star$-semigroup with order preserving involution $\star$. If S is intra-regular, then $\{(x^\star)_{\mathcal{N}} \mid x \in S\}$ is the set of all maximal simple subsemigroups of S.*

Proof. Denote T be the set of all maximal simple subsemigroups of S.

Let $a \in S$. By Proposition 2.4, $(a^\star)_{\mathcal{N}}$ is a simple subsemigroup of S. Let M be any simple subsemigroup of S such that $(a^\star)_{\mathcal{N}} \subseteq M$. Clearly $a^\star \in M$. Since $(Sa^\star S] \cap M$ is an ideal of M and $(a^\star)^3 \in (Sa^\star S] \cap M$, we have $(Sa^\star S] \cap M = M$ because M is simple. Thus $M \subseteq (Sa^\star S]$. Let $m \in M$. By Proposition 2.2, we have $a \in N(m)$ because $m \in (Sa^\star S]$. Hence $N(a) \subseteq N(m)$, so $N(a^\star) \subseteq N(m)$ because $N(a^\star) = N(a)$ (as shown in the proof of Proposition 2.1). Similarly $m^3 \in (SmS] \cap M$ and $(SmS] \cap M = M$ because $(SmS] \cap M$ is an ideal of M and M is simple. Hence $a^\star \in (SmS]$ because $a^\star \in M$. By Proposition 2.2, we get $m^\star \in N(a^\star)$. Hence $N(m^\star) \subseteq N(a^\star)$, so $N(m) \subseteq N(a^\star)$ because $N(m) = N(m^\star)$. Consequently $N(m) = N(a^\star)$, i.e. $m \in (a^\star)_{\mathcal{N}}$. Thus $M \subseteq (a^\star)_{\mathcal{N}}$. Therefore $M = (a^\star)_{\mathcal{N}}$. This means that $(a^\star)_{\mathcal{N}} \in T$. So $\{(a^\star)_{\mathcal{N}} \mid a \in S\} \subseteq T$.

Conversely, let $U \in T$ and $u \in U$. Clearly $(SuS] \cap U$ is an ideal of U. Since $u^3 \in (SuS] \cap U$ and U is simple, we get that $(SuS] \cap U = U$. Let $b \in U$. Then $b \in (SuS]$ because $U \subseteq (SuS]$. Proposition 2.2 implies $u^\star \in N(b)$. Hence $N(u^\star) \subseteq N(b)$. So $N(u) \subseteq N(b)$ because $N(u) = N(u^\star)$. Also since $b^3 \in (SbS] \cap U$ and U is simple, we get that $(SbS] \cap U = U$. Then $u \in (SbS]$ because $u \in U$. So $b^\star \in N(u)$ by Proposition 2.2. Thus $N(b^\star) \subseteq N(u)$. So $N(b) \subseteq N(u)$ because $N(b) = N(b^\star)$. Consequently we get $N(b) = N(u)$, i.e. $b \in (u)_{\mathcal{N}}$. Therefore $U \subseteq (u)_{\mathcal{N}}$. So $U = (u)_{\mathcal{N}}$ because U is a maximal simple subsemigroup of S. Since $u^\star \in S$, we have $(u)_{\mathcal{N}} = ((u^\star)^\star)_{\mathcal{N}} \in \{(a^\star)_{\mathcal{N}} \mid a \in S\}$. Thus $T \subseteq \{(a^\star)_{\mathcal{N}} \mid a \in S\}$. □

In ordered semigroups an equivalent definition for semilattice of simple semigroups was established by Kehayopulu (Definition 2.8). The analogous statement in ordered $\star$-semigroups will be induced by Proposition 2.9. Note that although in ordered $\star$-semigroups the semilattice of simple semigroups is defined exactly as in the case of ordered semigroups, the corresponding definitions of semilattice congruences are defined different (Definitions 1.5 and 2.7).

Definition 2.6. An ordered $\star$-semigroup S is called a *(complete) semilattice of simple semigroups* if there exists a (complete) semilattice congruence σ on S such that the class $(x)_\sigma$ of S containing x is a simple subsemigroup of S for any $x \in S$.

Definition 2.7. [4] A congruence σ on ordered semigroup S (without $\star$) is called *semilattice congruence* if $(aa, a) \in \sigma$ and $(ab, ba) \in \sigma$ for all $a, b \in S$. A semilattice congruence σ on S is called *complete* if $a \leq b$ implies $(a, ab) \in \sigma$.

Definition 2.8. [3] An ordered semigroup S (without $\star$) is called a *semilattice of simple semigroups* if there exists a semilattice congruence σ on S such that the class $(x)_\sigma$ of S containing x is a simple subsemigroup of S for any $x \in S$.

Equivalent definition: there exists a semilattice Y and a family $\{S_\alpha\}_{\alpha\in Y}$ of simple subsemigroups of S such that (1) $S_\alpha \cap S_\beta = \emptyset$ for each $\alpha, \beta \in Y$, $\alpha \neq \beta$, (2) $S = \cup_{\alpha\in Y} S_\alpha$, (3) $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for each α, $\beta \in Y$.

Proposition 2.9. *Let S be an ordered $\star$-semigroup. Then S is a semilattice of simple semigroups if and only if there exists a semilattice Y and a family $\{S_\alpha\}_{\alpha\in Y}$ of simple subsemigroups of S such that*

1) $S_\alpha \cap S_\beta = \emptyset$ for each $\alpha, \beta \in Y$, $\alpha \neq \beta$,
2) $S = \cup_{\alpha\in Y} S_\alpha$,
3) $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for each α, $\beta \in Y$,
4) $S_\alpha = (S_\alpha)^\star$ for $\alpha \in Y$.

Proof. ($\Longrightarrow$) Let S be a semilattice of simple semigroups. By Definition 2.6, there exists a semilattice congruence σ on S such that the σ-class $(x)_\sigma$ of S containing x is a simple subsemigroup of S for every $x \in S$. Let $y \in (x)_\sigma$. Since $yy \in (x)_\sigma$ and $(yy, y^\star) \in \sigma$, we have $y^\star \in (x)_\sigma$. It follows that $y \in (x)^\star_\sigma$, and $(x)_\sigma \subseteq (x)^\star_\sigma$. Therefore $(x)^\star_\sigma = (x)_\sigma$ by $(x)^\star_\sigma \subseteq ((x)^\star_\sigma)^\star = (x)_\sigma$, hence (4) follows. Note that $a^\star \in (a)^\star_\sigma$. So $a^\star \in (a)_\sigma$ because $(a)_\sigma = (a)^\star_\sigma$. Thus $(a^\star, a) \in \sigma$. Therefore $(a^\star a^\star, a^\star) \in \sigma$ since $(a^\star a^\star, a) \in \sigma$. Then $S = S^\star$ implies that $(aa, a) \in \sigma$. Consequently σ is a semilattice congruence on ordered semigroups S (without $\star$), and (1)-(3) follows by Definition 2.8.

($\Longleftarrow$) By hypothesis there exists a semilattice Y and a family $\{S_\alpha\}_{\alpha\in Y}$ of simple subsemigroups of S such that (1) $S_\alpha \cap S_\beta = \emptyset$ for each $\alpha, \beta \in Y$, $\alpha \neq \beta$, (2) $S = \cup_{\alpha\in Y} S_\alpha$, (3) $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for each α, $\beta \in Y$, and (4) $S_\alpha = (S_\alpha)^\star$ for $\alpha \in Y$. Since (1)-(3) are exactly the same as those in the equivalent definition of Definition 2.8, there exists a congruence σ on S such that the σ-class $(x)_\sigma$ of S containing x is a simple subsemigroup of S for every $x \in S$, and $(aa, a) \in \sigma$,

$(ab, ba) \in \sigma$ for any $a, b \in S$. In order to show that σ is a semilattice congruence on ordered $\star$-semigroup, we need to claim that $(a^\star a^\star, a) \in \sigma$ for any $a \in S$ according to Definition 1.5. Let $a \in S_\alpha$ for some $\alpha \in Y$. Since S_α is a subsemigroup, we have $aa \in S_\alpha$. Also $S_\alpha = (S_\alpha)^\star$ implies that $a^\star a^\star \in S_\alpha$. Observe that $S_\alpha = (x)_\sigma$ for some $x \in S$. Therefore $(aa, a^\star a^\star) \in \sigma$. Consequently $(a^\star a^\star, a) \in \sigma$ because $(a, aa) \in \sigma$. □

Proposition 2.10. *Let S be an ordered $\star$-semigroup. If S is a semilattice of simple subsemigroups, then S is intra-regular.*

Proof. By Proposition 2.9, there exists a semilattice Y and a family $\{S_\alpha\}_{\alpha \in Y}$ of simple subsemigroups of S such that $S = \cup_{\alpha \in Y} S_\alpha$ and $S_\alpha = (S_\alpha)^\star$. Let $x \in S$ and let $x \in S_\alpha = (S_\alpha)^\star$ for some $\alpha \in Y$. We will claim that $(Sx^\star x^\star S] \cap S_\alpha$ is an ideal of S_α. Then since S_α is simple, we have that $(Sx^\star x^\star S] \cap S_\alpha = S_\alpha$. Therefore $x \in S_\alpha = (Sx^\star x^\star S]$, that is S is intra-regular.

The proof of $(Sx^\star x^\star S] \cap S_\alpha$ is an ideal of S_α consists of combining three facts. (1) $((Sx^\star x^\star S] \cap S_\alpha)S_\alpha \subseteq (Sx^\star x^\star S] \cap S_\alpha$. (2) $S_\alpha((Sx^\star x^\star S] \cap S_\alpha) \subseteq (Sx^\star x^\star S] \cap S_\alpha$. (3) $b \in (Sx^\star x^\star S] \cap S_\alpha$ for any $b \in S_\alpha$ with $b \leq a \in (Sx^\star x^\star S] \cap S_\alpha$. These statements are justified as follows.

(1) Note that $(Sx^\star x^\star S] \cap S_\alpha \neq \emptyset$ because $(x^\star)^4 \in (Sx^\star x^\star S] \cap S_\alpha$. Let $y \in ((Sx^\star x^\star S] \cap S_\alpha)S_\alpha$. Then $y = y_1 y_2$ with $y_1 \in (Sx^\star x^\star S] \cap S_\alpha$ and $y_2 \in S_\alpha$. So $y_1 \leq u_1 x^\star x^\star u_2$ for some u_1, $u_2 \in S$. Clearly $y_1 y_2 \leq u_1 x^\star x^\star u_2 y_2 \in Sx^\star x^\star S$ and $y_1 y_2 \in S_\alpha$. This implies that $y_1 y_2 \in (Sx^\star x^\star S] \cap S_\alpha$. Therefore $((Sx^\star x^\star S] \cap S_\alpha)S_\alpha \subseteq (Sx^\star x^\star S] \cap S_\alpha$.

(2) Let $y \in S_\alpha((Sx^\star x^\star S] \cap S_\alpha)$. Then $y = y_1 y_2$ with $y_1 \in S_\alpha$, $y_2 \leq u_1 x^\star x^\star u_2$ and $y_2 \in S_\alpha$ for some u_1, $u_2 \in S$. Hence $y_1 y_2 \leq y_1 u_1 x^\star x^\star u_2 \in Sx^\star x^\star S$ and $y_1 y_2 \in S_\alpha$. This implies that $y \in (Sx^\star x^\star S] \cap S_\alpha$. Therefore $S_\alpha((Sx^\star x^\star S] \cap S_\alpha) \subseteq (Sx^\star x^\star S] \cap S_\alpha$.

(3) Since $a \leq u_1 x^\star x^\star u_2$ for some u_1, $u_2 \in S$ with $b \leq a$, we have $b \leq u_1 x^\star x^\star u_2 \in Sx^\star x^\star S$. So $b \in (Sx^\star x^\star S]$. Thus $b \in (Sx^\star x^\star S] \cap S_\alpha$ because $b \in S_\alpha$. □

Theorem 2.11. *Let S be an ordered $\star$-semigroup with order preserving involution $\star$. Then S is intra-regular if and only if S is a semilattice of simple subsemigroups.*

Proof. By Propositions 2.4 and 2.10. □

Proposition 2.12. ([2]; Proposition 2.10) *Let S be an ordered $\star$-semigroup. Then S is intra-regular if and only if the ideals of S are semiprime.*

Corollary 2.13. *Let S be an ordered $\star$-semigroup with order preserving involution $\star$. Then the following are equivalent:*

1. *S is a union of simple subsemigroups of S,*
2. *S is intra-regular,*
3. *any ideal of S is semiprime,*

4. *The relation $\mathcal{N}$ is a complete semilattice congruence on S and the class $(a)_{\mathcal{N}}$ is a simple subsemigroup of S for any $a \in S$,*
5. *S is a complete semilattice of simple semigroups,*
6. *S is a semilattice of simple semigroups,*
7. *There exists a congruence σ on S such that the class $(a)_\sigma$ of S is a simple subsemigroup of S for any $a \in S$.*

Proof. (1) $\Longrightarrow$ (2). By Theorem 2.11.

(2) $\Longrightarrow$ (3). Use Proposition 2.12.

(3) $\Longrightarrow$ (4). Proposition 2.12 implies that S is intra-regular. Then Proposition 1.6 shows that $\mathcal{N}$ is a complete semilattice congruence on S. Furthermore $(a)_{\mathcal{N}}$ is a simple subsemigroup of S by Proposition 2.4.

(4) $\Longrightarrow$ (5). By Definition 2.6.

(5) $\Longrightarrow$ (6). The result is obtained immediately since a complete semilattice congruence is a semilattice congruence.

(6) $\Longrightarrow$ (7). Clearly, because a semilattice congruence is a congruence.

(7) $\Longrightarrow$ (1). By hypothesis, $(x)_\sigma$ is a congruence class containing x, hence there exists $A \subseteq S$ such that $S = \cup_{x \in A}(x)_\sigma$. Therefore S is a union of simple subsemigroups of S because $(x)_\sigma$ is a simple subsemigroup of S for any $x \in S$. □

Theorem 2.14. ([2]; Theorem 2.13) *Let S be an ordered $\star$-semigroup with order preserving involution $\star$. (S is intra-regular and any two ideals are comparable under the inclusion relation $\subseteq$) if and only if the ideals of S are prime.*

Theorem 2.15. *Let S be an ordered $\star$-semigroup with order preserving involution $\star$. Then (S is intra-regular and any two ideals are comparable under the inclusion relation $\subseteq$) if and only if for any $x, y \in S$ we have $x \in (Sx^\star y^\star S]$ or $y \in (Sx^\star y^\star S]$.*

Proof. ($\Longrightarrow$) By Theorem 2.14, we must show that all ideals of S are prime. Let I be an ideal of S and $ab \in I$, $a, b \in S$. By hypothesis we have $b \in (Sb^\star a^\star S]$ or $a \in (Sb^\star a^\star S]$. If $b \in (Sb^\star a^\star S]$, then $b \in (S(ab)^\star S] \subseteq (I^\star] = I^\star$, hence $b^\star \in I$. If $a \in (Sb^\star a^\star S]$, then $a \in (S(ab)^\star S] \subseteq (I^\star] = I^\star$, hence $a^\star \in I$. This means that I is prime.

($\Longleftarrow$) Let x, $y \in S$. Since $(Sx^\star y^\star S]$ is an ideal of S, we have $(Sx^\star y^\star S]$ is prime by Theorem 2.14. This implies that $x^2 \in (Sx^\star y^\star S]$ or $y^2 \in (Sx^\star y^\star S]$ because $(x^\star)^2(y^\star)^2 \in (Sx^\star y^\star S]$. If $x^2 \in (Sx^\star y^\star S]$, then $x^\star \in (Sx^\star y^\star S]$. So $x^\star \leq u_1 x^\star y^\star u_2$ for some $u_1, u_2 \in S$. Thus $(x^\star)^2 \leq u_1 x^\star y^\star u_2 x^\star \in Sx^\star y^\star S$. Hence $(x^\star)^2 \in (Sx^\star y^\star S]$. This implies $x \in (Sx^\star y^\star S]$. Similarly, if $y^2 \in (Sx^\star y^\star S]$, then $y \in (Sx^\star y^\star S]$. □

Theorem 2.16. *Let S be an ordered $\star$-semigroup with order preserving involution $\star$. Then (S is intra-regular and any two ideals are comparable under the inclusion relation $\subseteq$) if and only if there exists a semilattice congruence σ such that the following hold:*

1. *For any $a \in S$, $(a)_\sigma$ is a simple subsemigroup of S,*
2. *For any $x, y \in S$, either $(x, x^\star y^\star) \in \sigma$ or $(y, x^\star y^\star) \in \sigma$.*

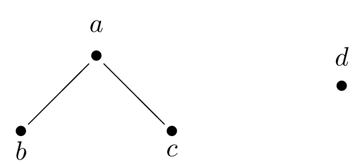
Proof. ($\Longrightarrow$) Let σ be $\mathcal{N}$. Since S is intra regular, by Proposition 2.4, we have $(a)_{\mathcal{N}}$ is a simple subsemigroup of S for any $a \in S$. Let $x, y \in S$. Since S is intra-regular and any two ideals are comparable under the inclusion relation $\subseteq$, we have $x \in (Sx^\star y^\star S]$ or $y \in (Sx^\star y^\star S]$ by Theorem 2.15. If $x \in (Sx^\star y^\star S]$, then $x \leq u_1 x^\star y^\star u_2$ for some $u_1, u_2 \in S$. So $u_1 x^\star y^\star u_2 \in N(x)$ and $(u_1 x^\star y^\star)^\star \in N(x)$ by Definition 1.4. Hence $yxu_1^\star \in N(x)$ and $(yx)^\star \in N(x)$ by Definition 1.4 again. Thus $N(x^\star y^\star) \subseteq N(x)$ because $x^\star y^\star \in N(x)$. Also since $x^\star y^\star \in N(x^\star y^\star)$, we have $x \in N(x^\star y^\star)$, so $N(x) \subseteq N(x^\star y^\star)$. Therefore $N(x) = N(x^\star y^\star)$. This means that $(x, x^\star y^\star) \in \mathcal{N}$. Similarly if $y \in (Sx^\star y^\star S]$, we can get $N(y) = N(x^\star y^\star)$. Thus $(y, x^\star y^\star) \in \mathcal{N}$.

($\Longleftarrow$) By Theorem 2.14, we must show that the ideals of S are prime. Let I be an ideal of S with $ab \in I$. Then since $(ab)_\sigma$ is a subsemigroup of S by hypothesis, we have $((ab)_\sigma \cap I)(ab)_\sigma = ((ab)_\sigma)^2 \cap I(ab)_\sigma \subseteq (ab)_\sigma \cap I$ and $(ab)_\sigma((ab)_\sigma \cap I) = ((ab)_\sigma)^2 \cap (ab)_\sigma I \subseteq (ab)_\sigma \cap I$. Thus $(ab)_\sigma \cap I$ is an ideal of $(ab)_\sigma$. So $(ab)_\sigma \cap I = (ab)_\sigma$ because $(ab)_\sigma$ is simple. Also by hypothesis we have that $(a^\star, ab) \in \sigma$ or $(b^\star, ab) \in \sigma$. If $(a^\star, ab) \in \sigma$, i.e. $(a^\star)_\sigma = (ab)_\sigma$, then $a^\star \in (a^\star)_\sigma = (ab)_\sigma = (ab)_\sigma \cap I \subseteq I$. If $(b^\star, ab) \in \sigma$, i.e. $(b^\star)_\sigma = (ab)_\sigma$, then $b^\star \in (b^\star)_\sigma = (ab)_\sigma = (ab)_\sigma \cap I \subseteq I$. Therefore I is a prime ideal. $\square$

Example 2.17. Let $S = \{a, b, c, d\}$ be an ordered semigroup. The multiplication "$\cdot$", the order "$\leq$" and the corresponding Hasse diagram are given below. Define the involution $\star$ by $a^\star = a$ and $b^\star = c$ (hence $c^\star = b$), $d^\star = d$. It is easy to check that S is an ordered $\star$-semigroup with order preserving involution $\star$.

$$\leq := \{(a, a), (b, a), (b, b), (c, a), (c, c), (d, d)\}$$

$\cdot$	a	b	c	d
a	a	a	a	d
b	a	b	a	d
c	a	a	c	d
d	d	d	d	d



S is intra-regular because $(Sa^\star a^\star S] = (Sb^\star b^\star S] = (Sc^\star c^\star S] = S$ and $(Sd^\star d^\star S] = \{d\}$ by Definition 1.4. $N(a) = N(b) = N(c) = \{a, b, c\}$ and $N(d) = S$ by Definition 3.3, thus $(a)_{\mathcal{N}} = (b)_{\mathcal{N}} = (c)_{\mathcal{N}} = \{a, b, c\}$, $(d)_{\mathcal{N}} = \{d\}$ because $\mathcal{N} := \{(x, y) \in S \times S \mid N(x) = N(y)\}$. Clearly $(x)_{\mathcal{N}}$ is a simple subsemigroup of S for any $x \in S$ and $S = \cup\{(x)_{\mathcal{N}} \mid x \in S\}$. Also $\mathcal{N} = \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c), (d, d)\}$. Furthermore $I(a) = (a \cup Sa \cup aS \cup SaS]$ implies $I(a) = S$. Similarly, $I(b) = I(c) = S$ and $I(d) = \{d\}$. Therefore $\mathcal{I} = \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c), (d, d)\}$, whence $\mathcal{N} = \mathcal{I}$.

Clearly, any two ideals are comparable under the inclusion relation $\subseteq$ because all the ideals are $\{d\}$ and S. On the other hand $\mathcal{N}$ is semilattice congruences on S. Also if x or $y \in \{a, b, c\}$, then $x^\star y^\star \in \{a, b, c\}$. If $x = y = d$, then $x^\star y^\star = d$.

Therefore either $(x, x^\star y^\star) \in \mathcal{N}$ or $(y, x^\star y^\star) \in \mathcal{N}$ for any $x, y \in S$. Thus Theorem 2.16 coincides on this example.

3 Decomposition of Left (resp. Right) Regular Ordered $\star$-Semigroup

In this paragraph, we first get some equivalent relations referring to left (resp. right) ideals. Then we find a characterization of left (resp. right) regular ordered $\star$-semigroup in terms of left (resp. right) ideals. Now since left (resp. right) regular ordered $\star$-semigroup is intra-regular, a decomposition can be made by Theorem 2.15.

Let S be an ordered $\star$-semigroup. We denote by $L(a)$, $R(a)$ and $I(a)$ the left ideal, right ideal and the ideal of S, respectively, generated by a. Clearly $L(a) = (a \cup Sa]$, $R(a) = (a \cup aS]$, $I(a) = (a \cup Sa \cup aS \cup SaS]$ [2].

Let S be an ordered $\star$-semigroup with order preserving involution $\star$. Then $L^\star$ is a right ideal for any left ideal L of S, $R^\star$ is a left ideal for any right ideal R of S and $I^\star$ is an ideal for any ideal I of S ([2] Proposition 2.2). As usual we define $a\mathcal{L}b$ if and only if $L(a) = L(b)$. $a\mathcal{R}b$ if and only if $R(a) = R(b)$. $a\mathcal{I}b$ if and only if $I(a) = I(b)$.

Proposition 3.1. *Let S be an ordered $\star$-semigroup. The following are equivalent:*

1. *$S \subseteq (Sa^\star a]$ for any $a \in S$,*
2. *$L(a) \subseteq L(a^\star a)$ for any $a \in S$,*
3. *$a\mathcal{L}(a^\star a)$ for any $a \in S$.*

Proof. (1) $\Longrightarrow$ (2). Let $a \in S$. Since $a \in (Sa^\star a]$, we have $a \leq u_1 a^\star a$ for some $u_1 \in S$. This implies that $u_2 a \leq u_2 u_1 a^\star a \in Sa^\star a$ for any $u_2 \in S$, so $Sa \subseteq (Sa^\star a]$. Thus $a \cup Sa \subseteq (Sa^\star a]$. Therefore $L(a) = (a \cup Sa] \subseteq (Sa^\star a] \subseteq (a^\star a \cup Sa^\star a] = L(a^\star a)$.

(2) $\Longrightarrow$ (3). Let $a \in S$. Clearly $a^\star a \in Sa$, so $Sa^\star a \subseteq Sa$. Hence $L(a^\star a) = (a^\star a \cup Sa^\star a] \subseteq (Sa] \subseteq (a \cup Sa] = L(a)$. Then by hypothesis $L(a) = L(a^\star a)$, i.e. $a\mathcal{L}(a^\star a)$.

(3) $\Longrightarrow$ (1). Let $a \in S$. Since $a \in (a \cup Sa] = L(a) = L(a^\star a) = (a^\star a \cup Sa^\star a]$, we have that $a \leq a^\star a$ or $a \leq ua^\star a$ for some $u \in S$. If $a \leq a^\star a$, then $a \leq a^\star(a) \leq a(a^\star a) \in Sa^\star a$, so $a \in (Sa^\star a]$. If $a \leq ua^\star a$ for some $u \in S$, then $a \leq ua^\star(a) \leq ua^\star(ua^\star a) \in Sa^\star a$, so $a \in (Sa^\star a]$. Therefore $S \subseteq (Sa^\star a]$. □

Proposition 3.2. *Let S be an ordered $\star$-semigroup. The following are equivalent:*

1. *$S \subseteq (aa^\star S]$ for any $a \in S$,*
2. *$R(a) \subseteq R(aa^\star)$ for any $a \in S$,*
3. *$a\mathcal{R}(aa^\star)$ for any $a \in S$.*

Proof. (1) $\Longrightarrow$ (2). Let $a \in S$. Since $a \in (aa^\star S]$, we have $a \leq aa^\star u_1$ for some $u_1 \in S$. This implies that $au_2 \leq aa^\star u_1 u_2 \in aa^\star S$ for any $u_2 \in S$, so $aS \subseteq (aa^\star S]$. Thus $a \cup aS \subseteq (aa^\star S]$. Therefore $R(a) = (a \cup aS] \subseteq (aa^\star S] \subseteq (aa^\star \cup aa^\star S] = R(aa^\star)$.

(2) $\Longrightarrow$ (3). Let $a \in S$. Clearly $aa^\star \in aS$, so $aa^\star S \subseteq aS$. Hence $R(aa^\star) = (aa^\star \cup aa^\star S] \subseteq (aS] \subseteq (a \cup aS] = R(a)$. Then by hypothesis $R(a) = R(aa^\star)$, i.e. $a\mathcal{R}(aa^\star)$.

(3) $\Longrightarrow$ (1). Let $a \in S$. Since $a \in (a \cup aS] = R(a) = R(aa^\star) = (aa^\star \cup aa^\star S]$, we have that $a \leq aa^\star$ or $a \leq aa^\star u$ for some $u \in S$. If $a \leq aa^\star$, then $a \leq (a)a^\star \leq (aa^\star)a^\star \in aa^\star S$, so $a \in (aa^\star S]$. If $a \leq aa^\star u$ for some $u \in S$, then $a \leq (a)a^\star u \leq (aa^\star u)a^\star u \in aa^\star S$, so $a \in (aa^\star S]$. Therefore $S \subseteq (aa^\star S]$. $\square$

Definition 3.3. An ordered $\star$-semigroup S is called *left (resp. right) regular* if $a \in (Sa^\star a^\star]$ (resp. $a \in (a^\star a^\star S]$) for any $a \in S$.

Definition 3.4. An ordered $\star$-semigroup S is called a *(complete) semilattice of left (resp. right) regular and simple semigroups* if there exists a (complete) semilattice congruence σ on S such that the class $(x)_\sigma$ of S, which is a subsemigroup of S, is left (resp. right) regular and simple for any $x \in S$.

Proposition 3.5. *Let S be an ordered $\star$-semigroup. Then S is left (resp. right) regular if and only if the left (resp. right) ideals of S are semiprime.*

Proof. ($\Longrightarrow$) Let L be a left ideal and $a^\star a^\star \in L$ for some $a \in S$. Clearly $Sa^\star a^\star \subseteq SL \subseteq L$. Since S is left regular, we have $a \in (Sa^\star a^\star] \subseteq (SL] \subseteq (L] = L$. Thus L is semiprime.

($\Longleftarrow$) Let $a \in S$. It is easy to see that $(Sa^\star a^\star]$ is a left ideal of S. Since $(a^\star a^\star)(a^\star a^\star) \in Sa^\star a^\star \subseteq (Sa^\star a^\star]$, we have $aa \in (Sa^\star a^\star]$ because $(Sa^\star a^\star]$ is semiprime by hypothesis. Hence $a^\star \in (Sa^\star a^\star]$ similarly. This implies $a^\star a^\star \in (Sa^\star a^\star]$ because $(Sa^\star a^\star]$ is a left ideal. Thus $a \in (Sa^\star a^\star]$. So S is left regular. The rest of the proof (when replacing the word "left" by "right") is similar. $\square$

Proposition 3.6. *Let S be an ordered $\star$-semigroup with order preserving involution $\star$. If S is left (resp. right) regular, then S is intra-regular.*

Proof. Let $a \in S$. Then $a \in (Sa^\star a^\star]$ because S is left regular. Hence $a \leq ua^\star a^\star$ for some $u \in S$, so $a^\star \leq aau^\star$. This implies that $a \leq ua^\star a^\star \leq u(aau^\star)a^\star \leq u((ua^\star a^\star)au^\star)a^\star \in Sa^\star a^\star S$. Therefore $a \in (Sa^\star a^\star S]$, i.e. S is intra-regular.

The rest of the proof (when replacing the word "left" by "right") is similar. $\square$

Proposition 3.7. *Let S be an ordered $\star$-semigroup. If S is a union of left (resp. right) regular subsemigroups of S, then S is left (resp. right) regular.*

Proof. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ where $S'_\alpha s$ are left regular subsemigroups of S. Let $a \in S$. Then $a \in S_\beta$ for some $\beta \in Y$. Since S_β is left regular, we get that $a \in (S_\beta a^\star a^\star] \subseteq (Sa^\star a^\star]$, so S is left regular.

The rest of the proof (when replacing the word "left" by "right") is similar. $\square$

Proposition 3.8. *Let S be an ordered $\star$-semigroup with order preserving involution $\star$. If S is left (resp. right) regular, then $(a)_\mathcal{N}$ is a left (resp. right) regular subsemigroup of S for any $a \in S$.*

Proof. In view of Theorem 2.3 and Proposition 3.6, we have that $\mathcal{N} = \mathcal{I}$. Therefore it suffices to prove that $(a)_{\mathcal{I}}$ is left regular subsemigroup. Proposition 2.1 implies that $(a)_{\mathcal{I}}$ is a subsemigroup. We need only show that it is left regular as well.

We first prove the fact that $I(xb^\star x) = I(a)$ for any $b \in (a)_{\mathcal{I}}$ and $x \in S$, which is an immediate consequence of combining three statements (1) $I(b) = I(b^\star)$. (2) $I(xb^\star x) = I(b)$. (3) $I(xb^\star x) = I(a)$. They are justified as follows.

(1) Since S is intra-regular by Proposition 3.6, we have $b \in (Sb^\star b^\star S]$. Thus $b \leq y$ for some $y \in Sb^\star b^\star S$. Since $Sb^\star b^\star S \subseteq Sb^\star S \subseteq b^\star \cup b^\star S \cup Sb^\star \cup Sb^\star S$, we have $b \leq y \in b^\star \cup b^\star S \cup Sb^\star \cup Sb^\star S$. Thus $b \in (b^\star \cup b^\star S \cup Sb^\star \cup Sb^\star S] = I(b^\star)$. Hence $I(b) \subseteq I(b^\star)$. By symmetry we have $I(b^\star) \subseteq I(b)$. Therefore $I(b) = I(b^\star)$.

(2) Clearly $xb^\star x \in I(b^\star)$. Thus $I(xb^\star x) \subseteq I(b)$ because $I(b^\star) = I(b)$. Also since $b \leq xb^\star b^\star \leq (xb^\star x)b^\star b^\star \in I(xb^\star x)$, we have $b \in (I(xb^\star x)] = I(xb^\star x)$. Thus $I(b) \subseteq I(xb^\star x)$. So $I(xb^\star x) = I(b)$. Clearly $I(b) = I(a)$ because $b \in (a)_{\mathcal{I}}$. Therefore $I(xb^\star x) = I(a)$.

(3) Since $b \in (a)_{\mathcal{I}}$, we have $I(b) = I(a)$. Thus $I(xb^\star x) = I(a)$ because $I(xb^\star x) = I(b)$.

Finally we use the fact that $I(xb^\star x) = I(a)$ to complete the proof. Let $b \in (a)_{\mathcal{I}}$. Since $(a)_{\mathcal{I}} \subseteq S$ and S is left regular, we have $b \in (Sb^\star b^\star]$. Thus $b \leq xb^\star b^\star$ for some $x \in S$. Therefore $b \leq xb^\star xb^\star b^\star = (xb^\star x)b^\star b^\star$. Now since $I(xb^\star x) = I(a)$, we have $xb^\star x \in (a)_{\mathcal{I}}$, hence $b \leq xb^\star xb^\star b^\star = (xb^\star x)b^\star b^\star \in (a)_{\mathcal{I}} b^\star b^\star$. Consequently $b \in ((a)_{\mathcal{I}} b^\star b^\star]$ and $(a)_{\mathcal{I}}$ is left regular.

The rest of the proof (when replacing the word "left" by "right") is similar. □

Corollary 3.9. *Let S be an ordered $\star$-semigroup with order preserving involution $\star$. Then the following are equivalent:*

1. *S is a union of left (resp. right) regular subsemigroups of S,*
2. *S is left (resp. right) regular,*
3. *Any left (resp. right) ideal of S is semiprime,*
4. *S is a complete semilattice of left (resp. right) regular and simple semigroups,*
5. *S is a semilattice of left (resp. right) regular and simple semigroups,*
6. *There exists a congruence σ on S such that the class $(x)_\sigma$ of S is a left (resp. right) regular and simple subsemigroup of S for any $x \in S$.*

Proof. (1) $\Longrightarrow$ (2). By Proposition 3.7.

(2) $\Longrightarrow$ (3) By Proposition 3.5.

(3) $\Longrightarrow$ (4). Since any left (resp. right) ideal of S is semiprime, Proposition 3.5 implies that S is left regular. Let $a \in S$. In view of Proposition 3.8, $(a)_{\mathcal{N}}$ is a left (resp. right) regular subsemigroup of S. Also Proposition 3.6 shows that S is intra-regular because S is left (resp. right) regular. Then by Proposition 2.4, $(a)_{\mathcal{N}}$ is a simple subsemigroup of S. So there exists a complete semilattice congruence $\mathcal{N}$ on S such that the class $(x)_{\mathcal{N}}$ of S, which is a subsemigroup of S, is left (resp. right) regular and simple for any $x \in S$. Consequently we get that S is a complete semilattice of left (resp. right) regular and simple semigroups by Definition 3.4.

(4) $\Longrightarrow$ (5). Clearly, because a complete semilattice congruence is a semilattice congruence.

(5) $\Longrightarrow$ (6). By Definition 3.4.

(6) $\Longrightarrow$ (1). Since $(x)_\sigma$ is a congruence class containing x, there exists $A \subseteq S$ such that $S = \cup_{x\in A}(x)_\sigma$. So S is a union of left regular subsemigroups of S because $(x)_\sigma$ is a left regular and subsemigroup of S for any $x \in S$.

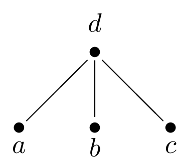
The rest of the proof (when replacing the word "left" by "right") is similar. $\square$

The following are two examples of left regular ordered $\star$-semigroups with order preserving involution $\star$.

Example 3.10. Let $S = \{a, b, c, d\}$ be an ordered semigroup. The multiplication "$\cdot$", the order "$\leq$" and the corresponding Hasse diagram are given below. Define the involution $\star$ by $a^\star = a$ and $b^\star = c$ (hence $c^\star = b$), $d^\star = d$. It is easy to check that S is an ordered $\star$-semigroup with order preserving involution $\star$.

$$\leq := \{(a, a), (a, d), (b, b), (b, d), (c, c), (c, d), (d, d)\}$$

$\cdot$	a	b	c	d
a	a	a	a	d
b	a	b	a	d
c	a	a	c	d
d	d	d	d	d

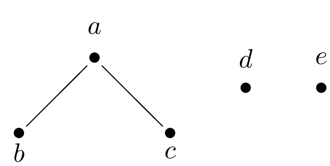


S is left regular because $(Sa^\star a^\star] = (Sb^\star b^\star] = (Sc^\star c^\star] = (Sd^\star d^\star] = S$ by Definition 1.4. $N(a) = N(b) = N(c) = N(d) = S$ by Definition 3.3, thus $(a)_\mathcal{N} = (b)_\mathcal{N} = (c)_\mathcal{N} = (d)_\mathcal{N} = S$ because $\mathcal{N} := \{(x, y) \in S \times S \mid N(x) = N(y)\}$, hence $S = \cup\{(x)_\mathcal{N} \mid x \in S\}$. Since the only non trivial left ideal of S is itself, S is semiprime. Clearly $(x)_\mathcal{N}$ is a simple semigroup of for any $x \in S$. Furthermore $\mathcal{N}$ is a (complete) semilattice congruence because $\mathcal{N} = S \times S$.

Example 3.11. Let $S = \{a, b, c, d\}$ be an ordered semigroup. The multiplication "$\cdot$", the order "$\leq$"and the corresponding Hasse diagram are given below. Define the involution $\star$ by $a^\star = a$ and $b^\star = c$ (hence $c^\star = b$), $d^\star = d$, $e^\star = e$. It is easy to check that S is an ordered $\star$-semigroup with order preserving involution $\star$.

$$\leq := \{(a, a), (b, a), (b, b), (c, a), (c, c), (d, d), (e, e)\}$$

$\cdot$	a	b	c	d	e
a	a	a	a	d	e
b	a	b	a	d	e
c	a	a	c	d	e
d	d	d	d	d	e
e	e	e	e	e	e



S is left regular because $(Sa^\star a^\star] = (Sb^\star b^\star] = (Sc^\star c^\star] = S$, $(Sd^\star d^\star] = \{d, e\}$ and $(Se^\star e^\star] = \{e\}$ by Definition 1.4. $N(a) = N(b) = N(c) = \{a, b, c\}$, $N(d) = \{a, b, c, d\}$ and $N(e) = S$ by Definition 3.3, thus $(a)_{\mathcal{N}} = (b)_{\mathcal{N}} = (c)_{\mathcal{N}} = \{a, b, c\}$, $(d)_{\mathcal{N}} = \{d\}$ and $(e)_{\mathcal{N}} = \{e\}$, hence $S = \cup\{(x)_{\mathcal{N}} \mid x \in S\}$. Clearly $\{a, b, c\}$, $\{d\}$, $\{e\}$ are simple subsemigroups and left regular. All left ideals are S, $\{d, e\}$ and $\{e\}$, obviously they are semiprime. Furthermore $(c^\star c^\star, c) = (bb, c) = (b, c) \in \mathcal{N}$, $(b^\star b^\star, b) = (cc, b) = (c, b) \in \mathcal{N}$, $(b, ab) = (b, a) \in \mathcal{N}$ and $(c, ac) = (c, a) \in \mathcal{N}$. Then by the facts that (1) "$\cdot$" is a commutative multiplication, (2) $c^\star = b$ and $z^\star = z$ for any $S \setminus \{b, c\}$, (3) $b \le a$ and $c \le a$, we have that $\mathcal{N}$ is a (complete) semilattice congruence.

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