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On Some Inequalities for Different Kinds of Convexity

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Abstract : In this paper, we examined the character of the function $f \circ \varphi$ according to character of f and φ functions and we obtained some inequalities for φ_s-convex function, $\varphi-$Godunova-Levin function, $\varphi - P-$function and $\log -\varphi-$convex function.

Keywords : $h-$convex function; $\log -\varphi-$convex function; $\varphi-$convex function; φ_h-convex function.
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1 Introduction

In [1], S. Varošanec defined the $h-$convex functions as below:

Definition 1.1. Let I, J are intervals in $\mathbb{R}$, $(0,1) \subseteq J$ and $h : J \to (0,\infty)$ be a non-negative function. We say that $f : I \to \mathbb{R}$ is an *$h-$convex function*, or that f belongs to $SX(h,I)$, if f is non-negative and for all $x, y \in I$, $\alpha \in (0,1)$ we have

$$f(\alpha x + (1-\alpha) y) \leq h(\alpha) f(x) + h(1-\alpha) f(y).$$

Obviously, if $h(\alpha) = \alpha$, then all non-negative convex functions belong to $SX(h,I)$; if $h(\alpha) = \frac{1}{\alpha}$, then $SX(h,I) = Q(I)$; if $h(\alpha) = 1$, then $SX(h,I) \supseteq P(I)$;

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and if $h(\alpha) = \alpha^s$ where $s \in (0,1)$, then $SX(h, I) \supseteq K_s^2$. Here Godunova-Levin class functions, $P-$functions and $s-$convex functions in the second sense are denoted by $Q(I)$, $P(I)$ and K_s^2 respectively.

For some results about Godunova-Levin class functions, $P-$functions and $s-$convex functions in the second sense, see [2–5].

E. A. Youness defined the generalized $\varphi-$convex sets and functions in [6]. G. Cristescu and L. Lupşa took into account the improved version of the definition of Youness in [7].

Let us consider a function $\varphi : [a, b] \to [a, b]$ where $[a, b] \subset \mathbb{R}$.

Definition 1.2. A function $f : [a, b] \to \mathbb{R}$ is said to be $\varphi-$*convex* on $[a, b]$ if for every two points $x \in [a, b]$, $y \in [a, b]$ and $t \in [0, 1]$ the following inequality holds:

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq tf(\varphi(x)) + (1-t) f(\varphi(y)).$$

In [8] and [9], M. Z. Sarikaya defined the following classes:

Definition 1.3. Let I be an interval in $\mathbb{R}$ and $h : (0, 1) \to (0, \infty)$ be a given function. We say that a function $f : I \to [0, \infty)$ is φ_h-*convex* if

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq h(t)f(\varphi(x)) + h(1-t)f(\varphi(y)) \tag{1.1}$$

for all $x, y \in I$ and $t \in (0, 1)$. If inequality (1.1) is reversed, then f is said to be φ_h-*concave*. In particular if f satisfies (1.1) with $h(t) = t$, $h(t) = t^s$ $(s \in (0, 1))$, $h(t) = \frac{1}{t}$ and $h(t) = 1$, then f is said to be $\varphi-$*convex*, φ_s-*convex*, $\varphi-$*Godunova-Levin function* and $\varphi - P-$*function*, respectively.

Definition 1.4. Let us consider a $\varphi : [a, b] \to [a, b]$ where $[a, b] \subset \mathbb{R}$ and I stands for a convex subset of $\mathbb{R}$. We say that a function $f : I \to \mathbb{R}^+$ is a $\log -\varphi-$*convex* if

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq [f(\varphi(x))]^t [f(\varphi(y))]^{1-t}$$

for all $x, y \in I$ and $t \in [0, 1]$.

In this paper, we examined the character of the function $f \circ \varphi$ according to character of f and φ functions and we obtained inequalities for $\log -\varphi-$convex function, φ_s-convex function, $\varphi-$Godunova-Levin function and $\varphi - P-$function.

2 Main Results

Theorem 2.1. *Let f be φ_s-convex function. Then*

i) If φ is linear, then $f \circ \varphi$ is $s-$convex in the second sense.

ii) If f is increasing and φ is convex, then $f \circ \varphi$ is $s-$convex in the second sense.

Proof. i) From φ_s-convexity of f and linearity of φ, we have

$$\begin{aligned} f\circ\varphi\left[\lambda x+(1-\lambda)y\right] &= f\left[\varphi\left(\lambda x+(1-\lambda)y\right)\right] \\ &= f\left[\lambda\varphi(x)+(1-\lambda)\varphi(y)\right] \\ &\leq \lambda^s f(\varphi(x))+(1-\lambda)^s f(\varphi(y)), \end{aligned}$$

which completes the proof for first case.

ii) From convexity of φ, we have

$$\varphi\left[\lambda x+(1-\lambda)y\right]\leq\lambda\varphi(x)+(1-\lambda)\varphi(y).$$

Since f is increasing, we can write

$$\begin{aligned} f\circ\varphi\left[\lambda x+(1-\lambda)y\right] &\leq f\left[\lambda\varphi(x)+(1-\lambda)\varphi(y)\right] \\ &\leq \lambda^s f(\varphi(x))+(1-\lambda)^s f(\varphi(y)). \end{aligned}$$

This completes the proof for this case. □

Theorem 2.2. *Let f be φ_s-convex and let $\sum_{i=1}^n t_i=T_n=1$, $t_i\in(0,1)$, $i=1,2,...,n$, $s\in(0,1)$, then*

$$f\left(\sum_{i=1}^n t_i\varphi(x_i)\right)\leq\sum_{i=1}^n t_i^s f(\varphi(x_i)).$$

Proof. From the above assumptions, we can write

$$\begin{aligned} f\left(\sum_{i=1}^n t_i\varphi(x_i)\right) &= f\left(T_{n-1}\sum_{i=1}^{n-1}\frac{t_i}{T_{n-1}}\varphi(x_i)+t_n\varphi(x_n)\right) \\ &\leq (T_{n-1})^s f\left(\sum_{i=1}^{n-1}\frac{t_i}{T_{n-1}}\varphi(x_i)\right)+t_n^s f(\varphi(x_n)) \\ &= (T_{n-1})^s f\left(\frac{T_{n-2}}{T_{n-1}}\sum_{i=1}^{n-2}\frac{t_i}{T_{n-2}}\varphi(x_i)+\frac{t_{n-1}}{T_{n-1}}\varphi(x_{n-1})\right)+t_n^s f(\varphi(x_n)) \\ &\leq (T_{n-2})^s f\left(\sum_{i=1}^{n-2}\frac{t_i}{T_{n-2}}\varphi(x_i)\right)+t_{n-1}^s f(\varphi(x_{n-1}))+t_n^s f(\varphi(x_n)) \\ &\vdots \\ &\leq \sum_{i=1}^n t_i^s f(\varphi(x_i)). \end{aligned}$$

This completes the proof. □

Theorem 2.3. *Let f be $\varphi-$Godunova-Levin function. Then*

i) If φ is linear, then $f\circ\varphi$ belongs to $Q(I)$.

ii) If f is increasing and φ is convex, then $f\circ\varphi\in Q(I)$.

Proof. i) Since f is $\varphi-$Godunova-Levin function and from linearity of φ, we have

$$\begin{aligned} f\circ\varphi\left[\lambda x+(1-\lambda)y\right] &= f\left[\varphi\left(\lambda x+(1-\lambda)y\right)\right] \\ &= f\left[\lambda\varphi(x)+(1-\lambda)\varphi(y)\right] \\ &\leq \frac{f\circ\varphi(x)}{\lambda}+\frac{f\circ\varphi(y)}{1-\lambda}, \end{aligned}$$

which completes the proof.

ii) From convexity of φ, we have

$$\varphi\left[\lambda x+(1-\lambda)y\right]\leq\lambda\varphi(x)+(1-\lambda)\varphi(y).$$

Since f is increasing we can write

$$\begin{aligned} f\circ\varphi\left[\lambda x+(1-\lambda)y\right] &\leq f\left[\lambda\varphi(x)+(1-\lambda)\varphi(y)\right] \\ &\leq \frac{f\circ\varphi(x)}{\lambda}+\frac{f\circ\varphi(y)}{1-\lambda}. \end{aligned}$$

This completes the proof. □

Theorem 2.4. *Let f be $\varphi-$Godunova-Levin function and let $\sum_{i=1}^{n}t_i=T_n=1$, $t_i\in(0,1)$, $i=1,2,...,n$, then*

$$f\left(\sum_{i=1}^{n}t_i\varphi(x_i)\right)\leq\sum_{i=1}^{n}\frac{f(\varphi(x_i))}{t_i}.$$

Proof. From the above assumptions, we can write

$$\begin{aligned} f\left(\sum_{i=1}^{n}t_i\varphi(x_i)\right) &= f\left(T_{n-1}\sum_{i=1}^{n-1}\frac{t_i}{T_{n-1}}\varphi(x_i)+t_n\varphi(x_n)\right) \\ &\leq \frac{f\left(\sum_{i=1}^{n-1}\frac{t_i}{T_{n-1}}\varphi(x_i)\right)}{T_{n-1}}+\frac{f(\varphi(x_n))}{t_n} \\ &= \frac{1}{T_{n-1}}f\left(\frac{T_{n-2}}{T_{n-1}}\sum_{i=1}^{n-2}\frac{t_i}{T_{n-2}}\varphi(x_i)+\frac{t_{n-1}}{T_{n-1}}\varphi(x_{n-1})\right)+\frac{f(\varphi(x_n))}{t_n} \\ &\leq \frac{f\left(\sum_{i=1}^{n-2}\frac{t_i}{T_{n-2}}\varphi(x_i)\right)}{T_{n-2}}+\frac{f(\varphi(x_{n-1}))}{t_{n-1}}+\frac{f(\varphi(x_n))}{t_n} \\ &\vdots \\ &\leq \sum_{i=1}^{n}\frac{f(\varphi(x_i))}{t_i}. \end{aligned}$$

This completes the proof. □

Theorem 2.5. *Let f be $\varphi - P-$convex function. Then*

i) If φ is linear, then $f \circ \varphi$ belongs to $P(I)$.

ii) If f is increasing and φ is convex, then $f \circ \varphi \in P(I)$.

Proof. i) From $\varphi - P-$convexity of f and linearity of φ, we have

$$\begin{aligned} f \circ \varphi\left[\lambda x + (1-\lambda)y\right] &= f\left[\varphi\left(\lambda x + (1-\lambda)y\right)\right] \\ &= f\left[\lambda\varphi(x) + (1-\lambda)\varphi(y)\right] \\ &\leq f(\varphi(x)) + f(\varphi(y)), \end{aligned}$$

which completes the proof.

ii) From convexity of φ, we have

$$\varphi\left[\lambda x + (1-\lambda)y\right] \leq \lambda\varphi(x) + (1-\lambda)\varphi(y).$$

Since f is increasing, we can write

$$\begin{aligned} f \circ \varphi\left[\lambda x + (1-\lambda)y\right] &\leq f\left[\lambda\varphi(x) + (1-\lambda)\varphi(y)\right] \\ &\leq f(\varphi(x)) + f(\varphi(y)). \end{aligned}$$

This completes the proof. □

Theorem 2.6. *Let f be $\varphi - P-$convex and let $\sum_{i=1}^{n} t_i = T_n = 1$, $t_i \in (0,1)$, $i = 1, 2, ..., n$, then*

$$f\left(\sum_{i=1}^{n} t_i\varphi(x_i)\right) \leq \sum_{i=1}^{n} f(\varphi(x_i)).$$

Proof. From the above assumptions, we can write

$$\begin{aligned} f\left(\sum_{i=1}^{n} t_i\varphi(x_i)\right) &= f\left(T_{n-1}\sum_{i=1}^{n-1}\frac{t_i}{T_{n-1}}\varphi(x_i) + t_n\varphi(x_n)\right) \\ &\leq f\left(\sum_{i=1}^{n-1}\frac{t_i}{T_{n-1}}\varphi(x_i)\right) + f(\varphi(x_n)) \\ &= f\left(\frac{T_{n-2}}{T_{n-1}}\sum_{i=1}^{n-2}\frac{t_i}{T_{n-2}}\varphi(x_i) + \frac{t_{n-1}}{T_{n-1}}\varphi(x_{n-1})\right) + f(\varphi(x_n)) \\ &\leq f\left(\sum_{i=1}^{n-2}\frac{t_i}{T_{n-2}}\varphi(x_i)\right) + f(\varphi(x_{n-1})) + f(\varphi(x_n)) \\ &\;\;\vdots \\ &\leq \sum_{i=1}^{n} f(\varphi(x_i)). \end{aligned}$$

This completes the proof. □

Theorem 2.7. *Let f be $\log-\varphi-$convex function. Then*

i) If φ is linear, then $f\circ\varphi$ is $\log-$convex.

ii) If f is increasing and φ is convex, then $f\circ\varphi$ is $\log-$convex function.

Proof. i) From $\log-\varphi-$convexity of f and linearity of φ, we have

$$\begin{aligned} f\circ\varphi\left[\lambda x+(1-\lambda)y\right] &= f\left[\varphi\left(\lambda x+(1-\lambda)y\right)\right] \\ &= f\left[\lambda\varphi(x)+(1-\lambda)\varphi(y)\right] \\ &\le \left[f(\varphi\left(x\right))\right]^{\lambda}\left[f(\varphi\left(y\right))\right]^{1-\lambda}, \end{aligned}$$

which completes the proof for first case.

ii) From convexity of φ, we have

$$\varphi\left[\lambda x+(1-\lambda)y\right]\le\lambda\varphi(x)+(1-\lambda)\varphi(y).$$

Since f is increasing, we can write

$$\begin{aligned} f\circ\varphi\left[\lambda x+(1-\lambda)y\right] &\le f\left[\lambda\varphi(x)+(1-\lambda)\varphi(y)\right] \\ &\le \left[f(\varphi\left(x\right))\right]^{\lambda}\left[f(\varphi\left(y\right))\right]^{1-\lambda}. \end{aligned}$$

This completes the proof for this case. □

Theorem 2.8. *Let $\varphi:[a,b]\to[a,b]$ be a function where $[a,b]\subset\mathbb{R}$ and I stands for a convex subset of $\mathbb{R}$. If $f:I\to\mathbb{R}^{+}$ is a $\log-\varphi-$convex function where $a,b\in I$ with $a<b$, for $\lambda\in[0,1]$ and $\varphi(b)\neq\varphi(a)$, then*

$$\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}G(f(x),f(\varphi(a)+\varphi(b)-x))dx\le G(f\left(\varphi(a)\right),f\left(\varphi(b)\right))$$

holds, where $G(,)$ is the geometric mean.

Proof. Since f is $\log-\varphi-$convex function, we have that

$$f\left(\lambda\varphi(a)+(1-\lambda)\varphi(b)\right)\le\left[f\left(\varphi(a)\right)\right]^{\lambda}\left[f\left(\varphi(b)\right)\right]^{1-\lambda},$$

$$f\left((1-\lambda)\varphi(a)+\lambda\varphi(b)\right)\le\left[f\left(\varphi(a)\right)\right]^{1-\lambda}\left[f\left(\varphi(b)\right)\right]^{\lambda}$$

for all $\lambda\in[0,1]$.

If we multiply the above inequalities and take square roots, we obtain

$$G\left(f\left(\lambda\varphi(a)+(1-\lambda)\varphi(b)\right),f\left((1-\lambda)\varphi(a)+\lambda\varphi(b)\right)\right)\le G(f\left(\varphi(a)\right),f\left(\varphi(b)\right)).$$

Integrating this inequality over λ on $[0,1]$, and changing the variable $x=\lambda\varphi(a)+(1-\lambda)\varphi(b)$, we have

$$\begin{aligned} &\int_0^1 G\left(f\left(\lambda\varphi(a)+(1-\lambda)\varphi(b)\right),f\left((1-\lambda)\varphi(a)+\lambda\varphi(b)\right)\right)d\lambda \\ \le\ & G(f\left(\varphi(a)\right),f\left(\varphi(b)\right)), \end{aligned}$$

$$\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}G(f(x),f(\varphi(a)+\varphi(b)-x))dx\le G(f\left(\varphi(a)\right),f\left(\varphi(b)\right))$$

which completes the proof. □

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