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Generalized $\tau_1\tau_2$-Closed Sets in Ideal Bitopological Spaces

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Abstract : This paper deals with the concept of $\mathscr{I}$-g-$\tau_1\tau_2$-closed sets in ideal bitopological spaces. $\mathscr{I}$-g-$\tau_1\tau_2$-normal and $\mathscr{I}$-g-$\tau_1\tau_2$-regular spaces are introduced and various characterizations are given. Several characterizations of $\mathscr{I}^{\star\star}$-R_0-spaces are discussed.

Keywords : $\mathscr{I}$-g-$\tau_1\tau_2$-closed set; $\mathscr{I}$-g-$\tau_1\tau_2$-normal space; $\mathscr{I}$-g-$\tau_1\tau_2$-regular space; $\mathscr{I}^{\star\star}$-R_0-space.
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1 Introduction

The notion of ideals in topological spaces has been studied by Kuratowski [1] and Vaidyanathaswamy [2] which is one of the important areas of research in the branch of mathematics. In 1990, Janković and Hamlett [3] further studied ideal topological spaces and their applications to various fields. Levine [4] introduced the notion of generalized closed sets in topological spaces. Donthev et al. [5] introduced and investigated the notion of $\mathscr{I}$-g-closed sets in ideal topological spaces as a modification of generalized closed sets due to Levine. Navaneethakrishnan and Josep [6] investigated generalized closed sets in ideal topological spaces. Mandal and Mukherjee [7] introduced the concept of $\star$-g-closed sets in ideal topological

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spaces and obtained several characterizations of $\star$-g-closed sets. Navaneethakrishnan et al. [8] introduced the concepts of $\mathscr{I}_g$-normal, $g\mathscr{I}$-normal and $\mathscr{I}_g$-regular spaces and investigated several characterizations of such spaces. Ozbakir and Yildirim [9] introduced ideal minimal spaces and investigated the relationships between minimal spaces and ideal minimal spaces. Recently, Noiri and Popa [10] introduced the notion of $\mathscr{I}$-mg-closed sets and obtained the unified characterizations for certain families of subsets between $\star$-closed sets and $\mathscr{I}$-g-closed sets in an ideal topological spaces. In this paper, we introduce the concept of $\mathscr{I}$-g-$\tau_1\tau_2$-closed sets in ideal bitopological spaces and investigate some properties of $\mathscr{I}$-g-$\tau_1\tau_2$-closed sets. Moreover, we define $\mathscr{I}$-g-$\tau_1\tau_2$-normal and $\mathscr{I}$-g-$\tau_1\tau_2$-regular spaces using $\mathscr{I}$-g-$\tau_1\tau_2$-open sets and give characterizations of such spaces. Finally, several characterizations of $\mathscr{I}^{\star\star}$-R_0-spaces are investigated.

2 Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) (or simply X) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopolgical space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*closed* [11] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed is said to be $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets containing A is called $\tau_1\tau_2$-*closure* of A and denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets contained in A is called $\tau_1\tau_2$-*interior* of A and denoted by $\tau_1\tau_2$-Int(A).

Lemma 2.1. [11] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ *and* $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) If $A \subseteq B$, *then* $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.

(3) $\tau_1\tau_2\text{-}Cl(A)$ *is* $\tau_1\tau_2$*-closed.*

(4) A *is* $\tau_1\tau_2$*-closed if and only if* $A = \tau_1\tau_2\text{-}Cl(A)$.

(5) $\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.

An ideal $\mathscr{I}$ on a topological space (X, τ) is a non-empty collection of subsets of X satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space (X, τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows: $A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I}$ for every open neighborhood U of $x\}$. In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [1], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ and $Cl^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ. A subset A is said to be $\star$-closed [3] if $A^\star \subseteq A$. The interior of a subset A in $(X, \tau^\star(\mathscr{I}))$ is denoted by $Int^\star(A)$. A bitopological space $(X, \tau_1, \tau_2,)$ with an ideal $\mathscr{I}$ on X is called an ideal bitopological space and is denoted by $(X, \tau_1, \tau_2, \mathscr{I})$.

3 Generalized $\tau_1\tau_2$-Closed Sets

We begin this section by introducing the notion of (τ_1, τ_2)-local functions.

Given a bitopological space (X, τ_1, τ_2) with an ideal $\mathscr{I}$ on X, the (τ_1, τ_2)*-local function* of A with respect to τ_1, τ_2 and $\mathscr{I}$ by

$A^{\star\star}(\tau_1, \tau_2, \mathscr{I}) = \{x \in X \mid A \cap U \notin \mathscr{I}$ for every $\tau_1\tau_2$-open set containing $x\}$.

In this case there is no confusion $A^{\star\star}(\tau_1, \tau_2, \mathscr{I})$ is briefly denoted by $A^{\star\star}$.

Proposition 3.1. *For subsets A, B of an ideal bitopological space* $(X, \tau_1, \tau_2, \mathscr{I})$*, the following properties hold:*

(1) If $A \subseteq B$, then $A^{\star\star} \subseteq B^{\star\star}$.

(2) $A^{\star\star} = \tau_1\tau_2\text{-}Cl(A^{\star\star}) \subseteq \tau_1\tau_2\text{-}Cl(A)$.

(3) $(A^{\star\star})^{\star\star} \subseteq A^{\star\star}$.

(4) $(A \cup B)^{\star\star} = A^{\star\star} \cup B^{\star\star}$.

Proof. (1). Suppose that $A \subseteq B$ and $x \notin B^{\star\star}$. Then, there exists a $\tau_1\tau_2$-open set U containing x such that $U \cap B \in \mathscr{I}$. Since $A \subseteq B$, we have $U \cap A \in \mathscr{I}$ and so $x \notin A^{\star\star}$. This shows that $A^{\star\star} \subseteq B^{\star\star}$.

(2). Let $x \in \tau_1\tau_2\text{-Cl}(A^{\star\star})$. Then $A^{\star\star} \cap U \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. Therefore, there exists $y \in A^{\star\star} \cap U$. Since U containing y and $y \in A^{\star\star}$, we have $U \cap A \notin \mathscr{I}$ and so $x \in A^{\star\star}$. Hence, $\tau_1\tau_2\text{-Cl}(A^{\star\star}) \subseteq A^{\star\star}$. Consequently, we obtain $\tau_1\tau_2\text{-Cl}(A^{\star\star}) = A^{\star\star}$. Again, let $x \in \tau_1\tau_2\text{-Cl}(A^{\star\star}) = A^{\star\star}$, then $U \cap A \notin \mathscr{I}$ for every $\tau_1\tau_2$-open set U containing x. This implies that $U \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. Therefore, we have $x \in \tau_1\tau_2\text{-Cl}(A)$. This proves $A^{\star\star} = \tau_1\tau_2\text{-Cl}(A^{\star\star}) \subseteq \tau_1\tau_2\text{-Cl}(A)$.

(3). Let $x \in (A^{\star\star})^{\star\star}$. Then for every $\tau_1\tau_2$-open set U containing x, $U \cap A^{\star\star} \notin \mathscr{I}$ and so $U \cap A^{\star\star} \neq \emptyset$. Therefore, there exists $y \in U \cap A^{\star\star}$. Since U containing y and $y \in A^{\star\star}$, we have $U \cap A \notin \mathscr{I}$ and so $x \in A^{\star\star}$. This shows that $(A^{\star\star})^{\star\star} \subseteq A^{\star\star}$.

(4). By (1), we have $A^{\star\star} \cup B^{\star\star} \subseteq (A \cup B)^{\star\star}$. For the reverse inclusion, suppose that $x \notin A^{\star\star} \cup B^{\star\star}$. Then, we have $x \notin A^{\star\star}$ and $x \notin B^{\star\star}$. There exist $\tau_1\tau_2$-open set U containing x and $\tau_1\tau_2$-open set V containing x such that $U \cap A \in \mathscr{I}$ and $V \cap B \in \mathscr{I}$. Therefore, $(U \cap V) \cap (A \cup B) = [(U \cap V) \cap A] \cup [(U \cap V) \cap B] \subseteq (U \cap A) \cup (V \cap B) \in \mathscr{I}$ and so $x \notin (A \cup B)^{\star\star}$. This implies that $(A \cup B)^{\star\star} \subseteq A^{\star\star} \cup B^{\star\star}$. Consequently, we obtain $(A \cup B)^{\star\star} = A^{\star\star} \cup B^{\star\star}$. □

Definition 3.2. Let $(X, \tau_1, \tau_2, \mathscr{I})$ be an ideal bitopological space. For any subset A of X, we put $\text{Cl}^{\star\star}(A) = A \cup A^{\star\star}$. The operator $\text{Cl}^{\star\star}$ is a Kuratowski closure operator. The topology generated by $\mathrm{Cl}^{\star\star}$ is denoted by $\tau^{\star\star}$, that is

$$\tau^{\star\star} = \{U \subseteq X \mid \mathrm{Cl}^{\star\star}(X - U) = X - U\}.$$

The elements of $\tau^{\star\star}$ are called $\star\star$-*open* sets and the complement of a $\star\star$-open set is called $\star\star$-*closed*. The closure and the interior of A with respect to $\tau^{\star\star}$ are denoted by $\mathrm{Cl}^{\star\star}(A)$ and $\mathrm{Int}^{\star\star}(A)$, respectively.

Proposition 3.3. *For subsets A and B of an ideal bitopological space $(X, \tau_1, \tau_2, \mathscr{I})$, the following properties hold:*

(1) $A \subseteq Cl^{\star\star}(A)$.

(2) $Cl^{\star\star}(\emptyset) = \emptyset$ and $Cl^{\star\star}(X) = X$.

(3) If $A \subseteq B$, then $Cl^{\star\star}(A) \subseteq Cl^{\star\star}(B)$.

(4) $Cl^{\star\star}(A) \cup Cl^{\star\star}(B) \subseteq Cl^{\star\star}(A \cup B)$.

Remark 3.1. *A subset A of an ideal bitopological space $(X, \tau_1, \tau_2, \mathscr{I})$ is $\star\star$-closed if and only if $A^{\star\star} \subseteq A$.*

Definition 3.4. A subset A of an ideal bitopological space $(X, \tau_1, \tau_2, \mathscr{I})$ is called $\star\star$-*dense in itself* (resp. $\star\star$-*perfect*) if $A \subseteq A^{\star\star}$ (resp. $A^{\star\star} = A$).

Lemma 3.5. *If A is $\star\star$-dense in itself in $(X, \tau_1, \tau_2, \mathscr{I})$, then $A^{\star\star} = \tau_1\tau_2\text{-}Cl(A^{\star\star}) = \tau_1\tau_2\text{-}Cl(A) = Cl^{\star\star}(A)$.*

Proof. Suppose that A is $\star\star$-dense in itself. Then, we have $A \subseteq A^{\star\star}$ and so $\tau_1\tau_2\text{-Cl}(A) \subseteq \tau_1\tau_2\text{-Cl}(A^{\star\star})$. By Proposition 3.1, $\tau_1\tau_2\text{-Cl}(A) \subseteq \tau_1\tau_2\text{-Cl}(A^{\star\star}) \subseteq \tau_1\tau_2\text{-Cl}(A)$ and hence $A^{\star\star} = \tau_1\tau_2\text{-Cl}(A^{\star\star}) = \tau_1\tau_2\text{-Cl}(A)$. Since $A^{\star\star} = \tau_1\tau_2\text{-Cl}(A)$, we have $\mathrm{Cl}^{\star\star}(A) = \tau_1\tau_2\text{-Cl}(A)$. Consequently, we obtain $A^{\star\star} = \tau_1\tau_2\text{-Cl}(A^{\star\star}) = \tau_1\tau_2\text{-Cl}(A) = \mathrm{Cl}^{\star\star}(A)$. □

Proposition 3.6. *For an ideal bitopologial space $(X, \tau_1, \tau_2, \mathscr{I})$, the following properties hold:*

(1) If G is $\tau_1\tau_2$-open, then $Int^{\star\star}(G) = G$.

(2) If F is $\tau_1\tau_2$-closed, then $Cl^{\star\star}(F)$ is $\tau_1\tau_2$-closed.

Proof. (1). Let G be a $\tau_1\tau_2$-open set. Then $X - G$ is $\tau_1\tau_2$-closed, by Proposition 3.1, we have $(X - G)^{\star\star} \subseteq \tau_1\tau_2\text{-Cl}(X - G) = X - G$ and so $\mathrm{Cl}^{\star\star}(X - G) = X - G$. This implies that G is a $\star\star$-open set and hence $\mathrm{Int}^{\star\star}(G) = G$.

(2). Let F be a $\tau_1\tau_2$-closed set. Then, we have

$$\begin{aligned}
\tau_1\tau_2\text{-Cl}(\mathrm{Cl}^{\star\star}(F)) &= \tau_1\tau_2\text{-Cl}(F^{\star\star} \cup F) \\
&= \tau_1\tau_2\text{-Cl}(F^{\star\star}) \cup \tau_1\tau_2\text{-Cl}(F) \\
&= \tau_1\tau_2\text{-Cl}(F) \\
&= F \subseteq \mathrm{Cl}^{\star\star}(F).
\end{aligned}$$

Consequently, $\tau_1\tau_2\text{-Cl}(\mathrm{Cl}^{\star\star}(F)) = \mathrm{Cl}^{\star\star}(F)$ and so $\mathrm{Cl}^{\star\star}(F)$ is $\tau_1\tau_2$-closed. □

Definition 3.7. A subset A of a bitopological space (X, τ_1, τ_2) is said to be *generalized $\tau_1\tau_2$-closed*(briefly *g-$\tau_1\tau_2$-closed*) if $\tau_1\tau_2$-Cl$(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_1\tau_2$-open. The complement of a generalized $\tau_1\tau_2$-closed set is said to be *generalized $\tau_1\tau_2$-open* (briefly *g-$\tau_1\tau_2$-open*).

Definition 3.8. A subset A of an ideal bitopological space $(X, \tau_1, \tau_2, \mathscr{I})$ is called *$\mathscr{I}$-g-$\tau_1\tau_2$-closed* if $A^{\star\star} \subseteq U$ whenever $A \subseteq U$ and U is $\tau_1\tau_2$-open. The complement of a $\mathscr{I}$-g-$\tau_1\tau_2$-closed set is called *$\mathscr{I}$-g-$\tau_1\tau_2$-open.*

Remark 3.2. *From the definitions one may deduce the following implications:*

$$\begin{array}{ccc} \tau_1\tau_2\text{-}closed & \Longrightarrow & g\text{-}\tau_1\tau_2\text{-}closed \\ \Downarrow & & \Downarrow \\ \star\star\text{-}closed & \Longrightarrow & \mathscr{I}\text{-}g\text{-}\tau_1\tau_2\text{-}closed \end{array}$$

However, none of these implications is reversible as shown by the following examples:

Example 3.3. Let $X = \{1, 2, 3\}$ with topologies $\tau_1 = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}$, $\tau_2 = \{\emptyset, \{1\}, X\}$ and an ideal $\mathscr{I} = \{\emptyset, \{1, 2\}\}$. Then $\{1, 2\}$ is $\star\star$-closed but $\{1, 2\}$ is not $\tau_1\tau_2$-closed. $\{2\}$ is $\mathscr{I}$-g-$\tau_1\tau_2$-closed but $\{2\}$ is not $\star\star$-closed. $\{1, 3\}$ is g-$\tau_1\tau_2$-closed but $\{1, 3\}$ is not $\tau_1\tau_2$-closed.

Example 3.4. Let $X = \{1, 2, 3\}$ with topologies $\tau_1 = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}$, $\tau_2 = \{\emptyset, \{1\}, X\}$ and an ideal $\mathscr{I} = \{\emptyset, \{1\}\}$. Then $\{1\}$ is $\mathscr{I}$-g-$\tau_1\tau_2$-closed but $\{1\}$ is not g-$\tau_1\tau_2$-closed.

Proposition 3.9. *If A is $\mathscr{I}$-g-$\tau_1\tau_2$-closed and $\star\star$-dense in itself in $(X, \tau_1\tau_2, \mathscr{I})$, then A is g-$\tau_1\tau_2$-closed.*

Proof. Suppose that A is $\mathscr{I}$-g-$\tau_1\tau_2$-closed and $\star\star$-dense in itself. Let $A \subseteq U$ and U be $\tau_1\tau_2$-open. Then, we have $A^{\star\star} \subseteq U$. Since A is $\star\star$-dense in itself, by Lemma 3.6, $\tau_1\tau_2$-Cl$(A) \subseteq U$ and so A is g-$\tau_1\tau_2$-closed. □

Proposition 3.10. *If A and B are $\mathscr{I}$-g-$\tau_1\tau_2$-closed in $(X, \tau_1, \tau_2, \mathscr{I})$, then $A \cup B$ is $\mathscr{I}$-g-$\tau_1\tau_2$-closed.*

Proof. Suppose that A and B are $\mathscr{I}$-g-$\tau_1\tau_2$-closed. Let $A \cup B \subseteq U$ and U be $\tau_1\tau_2$-open. Then $A \subseteq U$ and $B \subseteq B$. Since A and B are $\mathscr{I}$-g-$\tau_1\tau_2$-closed, $A^{\star\star} \subseteq U$ and $B^{\star\star} \subseteq U$. By Proposition 3.1, we have $(A \cup B)^{\star\star} = A^{\star\star} \cup B^{\star\star} \subseteq U$. Therefore, $A \cup B$ is $\mathscr{I}$-g-$\tau_1\tau_2$-closed. □

Remark 3.5. *The intersection of two $\mathscr{I}$-g-$\tau_1\tau_2$-closed sets need not be a $\mathscr{I}$-g-$\tau_1\tau_2$-closed set as shown by the following example.*

Example 3.6. Consider an ideal bitopological space $(X, \tau_1, \tau_2, \mathscr{I})$, where $X = \{1, 2, 3\}$, $\tau_1 = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}$, $\tau_2 = \{\emptyset, \{1\}, \{3\}, \{1, 3\}, X\}$ and $\mathscr{I} = \{\emptyset\}$. Then $\{1, 2\}$ and $\{1, 3\}$ are $\mathscr{I}$-g-$\tau_1\tau_2$-closed but $\{1, 2\} \cap \{1, 3\} = \{1\}$ is not $\mathscr{I}$-g-$\tau_1\tau_2$-closed.

Proposition 3.11. *If B is $\tau_1\tau_2$-closed and A is $\mathscr{I}$-g-$\tau_1\tau_2$-closed in $(X, \tau_1, \tau_2, \mathscr{I})$, then $A \cap B$ is $\mathscr{I}$-g-$\tau_1\tau_2$-closed.*

Proof. Suppose that B is $\tau_1\tau_2$-closed and A is $\mathscr{I}$-g-$\tau_1\tau_2$-closed. Let $A \cap B \subseteq V$ and V be $\tau_1\tau_2$-open. Then, we have $A \subseteq V \cup (X - B)$. Since $V \cup (X - B)$ is $\tau_1\tau_2$-open, $A^{\star\star} \subseteq V \cup (X - B)$ and so $A^{\star\star} \cap B \subseteq V \cap B \subseteq V$. By Proposition 3.1, $B^{\star\star} \subseteq B$ and $(A \cap B)^{\star\star} \subseteq A^{\star\star} \cap B^{\star\star} \subseteq A^{\star\star} \cap B \subseteq V$. This shows that $A \cap B$ is $\mathscr{I}$-g-$\tau_1\tau_2$-closed. □

The following theorem gives some characterizations of $\mathscr{I}$-g-$\tau_1\tau_2$-closed sets.

Theorem 3.12. *For a subset A of an ideal bitopological space $(X, \tau_1, \tau_2, \mathscr{I})$, the following properties are equivalent:*

(1) A is $\mathscr{I}$-g-$\tau_1\tau_2$-closed.

(2) $Cl^{\star\star}(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_1\tau_2$-open.

(3) $Cl^{\star\star}(A) \cap F = \emptyset$ whenever $A \cap F = \emptyset$ and F is $\tau_1\tau_2$-closed.

Proof. (1) ⇒ (2): Let $A \subseteq U$ and U be $\tau_1\tau_2$-open. Then, by (1) $A^{\star\star} \subseteq U$ and $\mathrm{Cl}^{\star\star}(A) = A^{\star\star} \cup A \subseteq U$.

(2) ⇒ (3): Let $A \cap F = \emptyset$ and F be $\tau_1\tau_2$-closed. Then $A \subseteq X - F$ and $X - F$ is $\tau_1\tau_2$-open. By (2), we have $\mathrm{Cl}^{\star\star}(A) \subseteq X - F$ and so $\mathrm{Cl}^{\star\star}(A) \cap F = \emptyset$.

(3) ⇒ (1): Let $A \subseteq U$ and U be $\tau_1\tau_2$-open. Then $A \cap (X - U) = \emptyset$ and $X - U$ is $\tau_1\tau_2$-closed. By (3), we have $\mathrm{Cl}^{\star\star}(A) \cap (X - U) = \emptyset$ and hence $A^{\star\star} \subseteq \mathrm{Cl}^{\star\star}(A) \subseteq U$. Therefore, A is $\mathscr{I}$-g-$\tau_1\tau_2$-closed. □

Proposition 3.13. *If A is $\mathscr{I}$-g-$\tau_1\tau_2$-closed in $(X, \tau_1, \tau_2, \mathscr{I})$ and $A \subseteq B \subseteq Cl^{\star\star}(A)$, then B is $\mathscr{I}$-g-$\tau_1\tau_2$-closed.*

Proof. Let $B \subseteq U$ and U be $\tau_1\tau_2$-open. Since $A \subseteq U$ and A is $\mathscr{I}$-g-$\tau_1\tau_2$-closed, we have $\mathrm{Cl}^{\star\star}(A) \subseteq U$ and so $\mathrm{Cl}^{\star\star}(B) \subseteq \mathrm{Cl}^{\star\star}(\mathrm{Cl}^{\star\star}(A)) = \mathrm{Cl}^{\star\star}(A) \subseteq U$. Therefore, B is $\mathscr{I}$-g-$\tau_1\tau_2$-closed. □

Corollary 3.14. *If A is $\mathscr{I}$-g-$\tau_1\tau_2$-open in $(X, \tau_1, \tau_2, \mathscr{I})$ and $Int^{\star\star}(A) \subseteq B \subseteq A$, then B is $\mathscr{I}$-g-$\tau_1\tau_2$-open.*

Proof. This follows from Proposition 3.19. □

Definition 3.15. [11] Let A be a subset of a bitopological space (X, τ_1, τ_2). The set $\cap\{G \mid A \subseteq G$ and G is $\tau_1\tau_2$-open$\}$ is called the $\tau_1\tau_2$-*kernel* of A and is denoted by $\tau_1\tau_2$-ker(A).

Theorem 3.16. *A subset A of an ideal bitopological space $(X, \tau_1, \tau_2, \mathscr{I})$ is $\mathscr{I}$-g-$\tau_1\tau_2$-closed if and only if $Cl^{\star\star}(A) \subseteq \tau_1\tau_2$-ker$(A)$.*

Proof. Suppose that A is $\mathscr{I}$-g-$\tau_1\tau_2$-closed. If $x \notin \tau_1\tau_2$-ker(A), then there exists a $\tau_1\tau_2$-open set U such that $A \subseteq U$ and $x \notin U$. Since A is $\mathscr{I}$-g-$\tau_1\tau_2$-closed, by Theorem 3.18, $\mathrm{Cl}^{\star\star}(A) \subseteq U$ and so $x \notin \mathrm{Cl}^{\star\star}(A)$. Consequently, we obtain $\mathrm{Cl}^{\star\star}(A) \subseteq \tau_1\tau_2$-ker$(A)$.

Conversely, suppose that $\mathrm{Cl}^{\star\star}(A) \subseteq \tau_1\tau_2$-ker$(A)$. Let $A \subseteq U$ and U be $\tau_1\tau_2$-open. Then $\mathrm{Cl}^{\star\star}(A) \subseteq \tau_1\tau_2$-ker$(A) \subseteq U$. Therefore, by Theorem 3.18, A is $\mathscr{I}$-g-$\tau_1\tau_2$-closed. □

Theorem 3.17. *A subset A of an ideal bitopological space $(X, \tau_1, \tau_2, \mathscr{I})$ is $\mathscr{I}$-g-$\tau_1\tau_2$-closed if and only if $\tau_1\tau_2$-Cl$(\{x\}) \cap A \neq \emptyset$ for each $x \in Cl^{\star\star}(A)$.*

Proof. Suppose that A is $\mathscr{I}$-g-$\tau_1\tau_2$-closed and $\tau_1\tau_2$-Cl$(\{x\}) \cap A = \emptyset$ for some $x \in \mathrm{Cl}^{\star\star}(A)$. Then, we have $A \subseteq X - \tau_1\tau_2$-Cl$(\{x\})$. Since A is $\mathscr{I}$-g-$\tau_1\tau_2$-closed and $X - \tau_1\tau_2$-Cl$(\{x\})$ is $\tau_1\tau_2$-open, by Theorem 3.18, $\mathrm{Cl}^{\star\star}(A) \subseteq X - \tau_1\tau_2$-Cl$(\{x\}) \subseteq X - \{x\}$. This contradicts that $x \in \mathrm{Cl}^{\star\star}(A)$. Therefore, $\tau_1\tau_2$-Cl$(\{x\}) \cap A \neq \emptyset$ for each $x \in \mathrm{Cl}^{\star\star}(A)$.

Conversely, suppose that A is not $\mathscr{I}$-g-$\tau_1\tau_2$-closed. Then, by Theorem 3.18, we have $\mathrm{Cl}^{\star\star}(A) - U \neq \emptyset$ for some $\tau_1\tau_2$-open set U containing A. There exists $x \in \mathrm{Cl}^{\star\star}(A) - U$. Since $x \notin U$, we have $\tau_1\tau_2$-Cl$(\{x\}) \cap U = \emptyset$ and hence

$$\tau_1\tau_2\text{-Cl}(\{x\}) \cap A \subseteq \tau_1\tau_2\text{-Cl}(\{x\}) \cap U = \emptyset.$$

This shows that $\tau_1\tau_2$-Cl$(\{x\}) \cap A = \emptyset$ for some $x \in \mathrm{Cl}^{\star\star}(A)$. □

Lemma 3.18. *For a subset A of an ideal bitopological space $(X, \tau_1, \tau_2, \mathscr{I})$, the following properties hold:*

(1) $x \in Cl^{\star\star}(A)$ if and only if $A \cap V \neq \emptyset$ for every $\star\star$-open set V containing x.

(2) $Cl^{\star\star}(X - A) = X - Int^{\star\star}(A)$ and $Int^{\star\star}(X - A) = X - Cl^{\star\star}(A)$.

Theorem 3.19. *A subset A of an ideal bitopological space $(X, \tau_1, \tau_2, \mathscr{I})$ is $\mathscr{I}$-g-$\tau_1\tau_2$-open if and only if $F \subseteq Int^{\star\star}(A)$ whenever $F \subseteq A$ and F is $\tau_1\tau_2$-closed.*

Proof. Suppose that A is $\mathscr{I}$-g-$\tau_1\tau_2$-open. Let $F \subseteq A$ and F be $\tau_1\tau_2$-closed. Then $X - A \subseteq X - F$. Since $X - F$ is $\tau_1\tau_2$-open and $X - A$ is $\mathscr{I}$-g-$\tau_1\tau_2$-closed, we have $\mathrm{Cl}^{\star\star}(X - A) \subseteq X - F$ and hence $X - \mathrm{Int}^{\star\star}(A) = \mathrm{Cl}^{\star\star}(X - A) \subseteq X - F$. Consequently, we obtain $F \subseteq \mathrm{Int}^{\star\star}(A)$.

Conversely, let $X - A \subseteq U$ and U be $\tau_1\tau_2$-open. Then $X - U \subseteq A$ and $X - U$ is $\tau_1\tau_2$-closed. By the hypothesis, we have $X - U \subseteq \mathrm{Int}^{\star\star}(A)$ and so

$$\mathrm{Cl}^{\star\star}(X - A) = X - \mathrm{Int}^{\star\star}(A) \subseteq U.$$

Therefore, $X - A$ is $\mathscr{I}$-g-$\tau_1\tau_2$-closed and so A is $\mathscr{I}$-g-$\tau_1\tau_2$-open. □

Theorem 3.20. *For a subset A of an ideal bitopological space $(X, \tau_1, \tau_2, \mathscr{I})$, the following properties are equivalent:*

(1) A is $\mathscr{I}$-g-$\tau_1\tau_2$-closed.

(2) $A^{\star\star} - A$ *contains no non-empty* $\tau_1\tau_2$*-closed set.*

(3) $A^{\star\star} - A$ *is* $\mathscr{I}$*-g-*$\tau_1\tau_2$*-open.*

(4) $A \cup (X - A^{\star\star})$ *is* $\mathscr{I}$*-g-*$\tau_1\tau_2$*-closed.*

(5) $Cl^{\star\star}(A) - A$ *contains no non-empty* $\tau_1\tau_2$*-closed set.*

Proof. (1) $\Rightarrow$ (2): Suppose that A is $\mathscr{I}$-g-$\tau_1\tau_2$-closed. Let $F \subseteq A^{\star\star} - A$ and F be $\tau_1\tau_2$-closed. Then $A \subseteq X - F$ and $X - F$ is $\tau_1\tau_2$-open. Since $X - F$ is $\tau_1\tau_2$-open and A is $\mathscr{I}$-g-$\tau_1\tau_2$-closed, we have $A^{\star\star} \subseteq X - F$ and hence $F \subseteq X - A^{\star\star}$. This implies that $F \subseteq A^{\star\star} \cap (X - A^{\star\star}) = \emptyset$.

(2) $\Rightarrow$ (3): Let $F \subseteq A^{\star\star} - A$ and F be $\tau_1\tau_2$-closed. By (2), we have $F = \emptyset$ and $F \subseteq \text{Int}^{\star\star}(A^{\star\star} - A)$. It follows from Theorem 3.25 that $A^{\star\star} - A$ is $\mathscr{I}$-g-$\tau_1\tau_2$-open.

(3) $\Rightarrow$ (1): Let $A \subseteq U$ and U be $\tau_1\tau_2$-open. Then $A^{\star\star} \cap (X - U) \subseteq A^{\star\star} - A$ and by (3) $A^{\star\star} - A$ is $\mathscr{I}$-g-$\tau_1\tau_2$-open. Since $A^{\star\star} \cap (X - U)$ is $\tau_1\tau_2$-closed and $A^{\star\star} - A$ is $\mathscr{I}$-g-$\tau_1\tau_2$-open, by Theorem 3.25, we obtain $A^{\star\star} \cap (X - U) \subseteq \text{Int}^{\star\star}(A^{\star\star} - A)$. Now, we have $\text{Int}^{\star\star}(A^{\star\star} - A) = \text{Int}^{\star\star}(A^{\star\star} \cap (X - A)) \subseteq A^{\star\star} \cap (X - \text{Cl}^{\star\star}(A)) = \emptyset$. Therefore, $A^{\star\star} \cap (X - U) = \emptyset$ and hence $A^{\star\star} \subseteq U$. This shows that A is $\mathscr{I}$-g-$\tau_1\tau_2$-closed.

(3) $\Leftrightarrow$ (4): This follows from the fact that $X - (A^{\star\star} - A) = (X - A^{\star\star}) \cup A$.

(2) $\Leftrightarrow$ (5): This is obvious by the fact that

$$\begin{aligned}\text{Cl}^{\star\star}(A) - A &= (A^{\star\star} \cup A) \cap (X - A)\\ &= (A^{\star\star} \cap (X - A)) \cup (A \cap (X - A))\\ &= A^{\star\star} - A.\end{aligned}$$ □

Corollary 3.21. *Let* A *be a* $\mathscr{I}$*-g-bilosed set of* $(X, \tau_1, \tau_2, \mathscr{I})$*. Then, the following are properties equivalent:*

(1) A *is a* $\star\star$*-closed set.*

(2) $Cl^{\star\star}(A) - A$ *is a* $\tau_1\tau_2$*-closed set.*

(3) $A^{\star\star} - A$ *is a* $\tau_1\tau_2$*-closed set.*

Proof. (1) $\Rightarrow$ (2): Suppose that A is a $\star\star$-closed set. Then $\text{Cl}^{\star\star}(A) - A = \emptyset$ and so $\text{Cl}^{\star\star}(A) - A$ is $\tau_1\tau_2$-closed.

(2) $\Rightarrow$ (3): This follows from the fact that $\text{Cl}^{\star\star}(A) - A = A^{\star\star} - A$.

(3) $\Rightarrow$ (1): Suppose that $A^{\star\star} - A$ is $\tau_1\tau_2$-closed. Since A is $\mathscr{I}$-g-$\tau_1\tau_2$-closed, by Theorem 3.26(2), we have $A^{\star\star} - A = \emptyset$ and so A is $\star\star$-closed. □

Theorem 3.22. *A subset* A *of an ideal bitopological space* $(X, \tau_1, \tau_2, \mathscr{I})$ *is* $\mathscr{I}$*-g-*$\tau_1\tau_2$*-closed if and only if* $A = F - N$*, where* F *is* $\star\star$*-closed and* N *contains no non-empty* $\tau_1\tau_2$*-closed set.*

Proof. Suppose that A is $\mathscr{I}$-g-$\tau_1\tau_2$-closed. Let $F = \mathrm{Cl}^{\star\star}(A)$ and $N = A^{\star\star} - A$. Then F is $\star\star$-closed and by, Theorem 3.26(2), N contains no non-empty $\tau_1\tau_2$-closed set. Moreover, we have

$$\begin{aligned} F - N &= (A^{\star\star} \cup A) - (A^{\star\star} - A) \\ &= (A^{\star\star} \cup A) \cap [X - (A^{\star\star} - A)] \\ &= (A^{\star\star} \cup A) \cap [(X - A^{\star\star}) \cup A] \\ &= [A^{\star\star} \cap (X - A^{\star\star})] \cup A = A. \end{aligned}$$

Conversely, suppose that $A = F - N$, where F is $\star\star$-closed and N contains no non-empty $\tau_1\tau_2$-closed set. Let $A \subseteq U$ and U be $\tau_1\tau_2$-open. Then, we have $F - N \subseteq U$ and $F \cap (X - U) \subseteq F \cap [X - (F - N)] = F \cap [(X - F) \cup N] = F \cap N \subseteq N$. By Proposition 3.1, $A^{\star\star}$ is $\tau_1\tau_2$-closed and $A^{\star\star} \cap (X - U)$ is $\tau_1\tau_2$-closed. Since $A \subseteq F$ and $F^{\star\star} \subseteq \mathrm{Cl}^{\star\star}(F) = F$, $A^{\star\star} \cap (X - U) \subseteq F^{\star\star} \cap (X - U) \subseteq F \cap (X - U) \subseteq N$. Therefore, $A^{\star\star} \cap (X - U) = \emptyset$ and so $A^{\star\star} \subseteq U$. This shows that A is $\mathscr{I}$-g-$\tau_1\tau_2$-closed. □

Corollary 3.23. *For a subset A of an ideal bitopological space $(X, \tau_1, \tau_2, \mathscr{I})$, the following properties are equivalent:*

(1) A is $\mathscr{I}$-g-$\tau_1\tau_2$-open.

(2) $A - Int^{\star\star}(A)$ contains no non-empty $\tau_1\tau_2$-closed set.

(3) $\tau_1\tau_2$-$Cl(\{x\}) \cap (X - A) \neq \emptyset$ for each $x \in X - Int^{\star\star}(A)$.

Proof. This follows from Theorem 3.23 and 3.26. □

4 Some Separation Axioms

In this section, we introduce the notions of $\mathscr{I}$-g-$\tau_1\tau_2$-normal, $\mathscr{I}$-g-$\tau_1\tau_2$-regular and $\mathscr{I}^{\star\star}$-R_0 spaces. Moreover, several interesting characterizations of these spaces are discussed.

Definition 4.1. An ideal bitopological space (X, τ_1, τ_2) is said to be $\mathscr{I}$-g-$\tau_1\tau_2$-*normal* if for every pair of disjoint $\tau_1\tau_2$-closed sets A and B, there exist disjoint $\mathscr{I}$-g-$\tau_1\tau_2$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Theorem 4.2. *For an ideal bitopological space $(X, \tau_1, \tau_2, \mathscr{I})$, the following properties are equivalent:*

(1) X is $\mathscr{I}$-g-$\tau_1\tau_2$-normal.

(2) For every $\tau_1\tau_2$-closed set F and every $\tau_1\tau_2$-open set V containing F, there exists a $\mathscr{I}$-g-$\tau_1\tau_2$-open set U such that $F \subseteq U \subseteq Cl^{\star\star}(U) \subseteq V$.

Proof. (1) $\Rightarrow$ (2): Let F be a $\tau_1\tau_2$-closed set and V be a $\tau_1\tau_2$-open set containing F. Since F and $X - V$ are disjoint $\tau_1\tau_2$-closed sets, there exist disjoint $\mathscr{I}$-g-$\tau_1\tau_2$-open sets U and W such that $F \subseteq U$ and $X - V \subseteq W$. Again, $U \cap W = \emptyset$ implies that $U \cap \text{Int}^{\star\star}(W) = \emptyset$ and so $\text{Cl}^{\star\star}(U) \subseteq X - \text{Int}^{\star\star}(W)$. Since $X - V$ is $\tau_1\tau_2$-closed and W is $\mathscr{I}$-g-$\tau_1\tau_2$-open, $X - V \subseteq W$ implies that $X - V \subseteq \text{Int}^{\star\star}(W)$ and so $X - \text{Int}^{\star\star}(W) \subseteq V$. Thus, we have $F \subseteq U \subseteq \text{Cl}^{\star\star}(U) \subseteq X - \text{Int}^{\star\star}(W) \subseteq V$ which proves (2).

(2) $\Rightarrow$ (1): Let A and B be two disjoint $\tau_1\tau_2$-closed sets. By hypothesis, there exists a $\mathscr{I}$-g-$\tau_1\tau_2$-open set U such that $A \subseteq U \subseteq \text{Cl}^{\star\star}(U) \subseteq X - B$. Put $W = X - \text{Cl}^{\star\star}(U)$, then U and W are the required disjoint $\mathscr{I}$-g-$\tau_1\tau_2$-open sets containing A and B, respectively. This shows that X is $\mathscr{I}$-g-$\tau_1\tau_2$-normal. □

Theorem 4.3. *For an ideal bitopological space $(X, \tau_1, \tau_2, \mathscr{I})$, the following properties are equivalent:*

(1) X is $\mathscr{I}$-g-$\tau_1\tau_2$-normal.

(2) For every $\tau_1\tau_2$-closed set A and every g-$\tau_1\tau_2$-closed set B such that $A \cap B = \emptyset$, there exists disjoint $\mathscr{I}$-g-$\tau_1\tau_2$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Proof. (1) $\Rightarrow$ (2): Let A be a $\tau_1\tau_2$-closed set and B be a $\mathscr{I}$-g-$\tau_1\tau_2$-closed set such that $A \cap B = \emptyset$. Then, we have $B \subseteq X - A$ and $X - A$ is $\tau_1\tau_2$-open. Therefore, by hypothesis, $\tau_1\tau_2\text{-Cl}(B) \subseteq X - A$ and so $\tau_1\tau_2\text{-Cl}(B) \cap A = \emptyset$. Since X is $\mathscr{I}$-g-$\tau_1\tau_2$-normal, there exist disjoint $\mathscr{I}$-g-$\tau_1\tau_2$-open sets U and V such that $B \subseteq \tau_1\tau_2\text{-Cl}(B) \subseteq U$ and $A \subseteq V$.

(2) $\Rightarrow$ (1): This is obvious. □

Proposition 4.4. *For an $\mathscr{I}$-g-$\tau_1\tau_2$-normal space $(X, \tau_1, \tau_2, \mathscr{I})$, the following properties hold:*

(1) For every $\tau_1\tau_2$-closed set F and every g-$\tau_1\tau_2$-open set U containing F, there exists a $\mathscr{I}$-g-$\tau_1\tau_2$-open set V such that $F \subseteq \text{Int}^{\star\star}(V) \subseteq V \subseteq U$.

(2) For every $\mathscr{I}$-g-$\tau_1\tau_2$-closed set F and every $\tau_1\tau_2$-open set U containing F, there exists a $\mathscr{I}$-g-$\tau_1\tau_2$-closed set H such that $F \subseteq H \subseteq \text{Cl}^{\star\star}(H) \subseteq U$.

Proof. (1). Let F be a $\tau_1\tau_2$-closed set and U be a g-$\tau_1\tau_2$-open set containing F. Then $F \cap (X - U) = \emptyset$, where F is $\tau_1\tau_2$-closed and $X - U$ is $\mathscr{I}$-g-$\tau_1\tau_2$-closed. By Theorem 4.3, there exist disjoint $\mathscr{I}$-g-$\tau_1\tau_2$-open sets V and W such that $F \subseteq V$ and $X - U \subseteq W$. Since $V \cap W = \emptyset$, we have $V \subseteq X - W$. By Theorem 3.25, we have $F \subseteq \text{Int}^{\star\star}(V)$. Therefore, $F \subseteq \text{Int}^{\star\star}(V) \subseteq V \subseteq X - W \subseteq U$. This proves (1).

(2). Let F be a $\mathscr{I}$-g-$\tau_1\tau_2$-closed set and U be a $\tau_1\tau_2$-open set containing F. Then $X - U$ is a $\tau_1\tau_2$-closed set contained in the $\mathscr{I}$-g-$\tau_1\tau_2$-open set $X - F$. By (1), there exists a $\mathscr{I}$-g-$\tau_1\tau_2$-open set V such that $X - U \subseteq \text{Int}^{\star\star}(V) \subseteq V \subseteq X - F$. Therefore, $F \subseteq X - V \subseteq \text{Cl}^{\star\star}(X - V) \subseteq U$. Put $H = X - V$, then $F \subseteq H \subseteq \text{Cl}^{\star\star}(H) \subseteq U$ and so H is the required $\mathscr{I}$-g-$\tau_1\tau_2$-closed set. □

Definition 4.5. An ideal bitopological space $(X, \tau_1, \tau_2, \mathscr{I})$ is said to be *$\mathscr{I}$-g-$\tau_1\tau_2$-regular* if for each $\tau_1\tau_2$-closed set F and each $x \notin F$, there exist disjoint $\mathscr{I}$-g-$\tau_1\tau_2$-open sets U and V such that $x \in U$ and $F \subseteq V$.

Theorem 4.6. *For an ideal bitopological space $(X, \tau_1, \tau_2, \mathscr{I})$, the following properties are equivalent:*

(1) X is $\mathscr{I}$-g-$\tau_1\tau_2$-regular.

(2) For each $x \in X$ and each $\tau_1\tau_2$-open set V containing x, there exists a $\mathscr{I}$-g-$\tau_1\tau_2$-open set U such that $x \in U \subseteq Cl^{\star\star}(U) \subseteq V$.

Proof. (1) ⇒ (2): For each $x \in X$ and any $\tau_1\tau_2$-open set V containing x, there exist disjoint $\mathscr{I}$-g-$\tau_1\tau_2$-open sets U and W such that $x \in U$ and $X - V \subseteq W$. Now, $X - V \subseteq W$ implies that $X - V \subseteq \mathrm{Int}^{\star\star}(W)$ and so $X - \mathrm{Int}^{\star\star}(W) \subseteq V$. Again, $U \cap W = \emptyset$ implies that $U \cap \mathrm{Int}^{\star\star}(W) = \emptyset$ and so $\mathrm{Cl}^{\star\star}(U) \subseteq X - \mathrm{Int}^{\star\star}(W)$. Therefore, $x \in U \subseteq \mathrm{Cl}^{\star\star}(U) \subseteq V$. This proves (2).

(2) ⇒ (1): Let $x \in X$ and F be a $\tau_1\tau_2$-closed set not containing x. Then by (2), there exists a $\mathscr{I}$-g-$\tau_1\tau_2$-open set U such that $x \in U \subseteq \mathrm{Cl}^{\star\star}(U) \subseteq X - F$. Put $W = X - \mathrm{Cl}^{\star\star}(U)$, then U and W are disjoint $\mathscr{I}$-g-$\tau_1\tau_2$-open sets such that $x \in U$ and $F \subseteq W$. This proves (1). □

Definition 4.7. An ideal bitopological space $(X, \tau_1, \tau_2, \mathscr{I})$ is said to be *$\mathscr{I}^{\star\star}$-R_0-space* if for each $\tau_1\tau_2$-open set U and each $x \in U$, $\{x\}^{\star\star} \subseteq U$.

The following theorem gives some characterizations of $\mathscr{I}^{\star\star}$-R_0-spaces.

Theorem 4.8. *For an ideal bitopological space $(X, \tau_1, \tau_2, \mathscr{I})$, the following properties are equivalent:*

(1) X is a $\mathscr{I}^{\star\star}$-R_0-space.

(2) For each $\tau_1\tau_2$-open set U and each $x \in U$, $Cl^{\star\star}(\{x\}) \subseteq U$.

(3) For each $\tau_1\tau_2$-closed set F and each $x \in X - F$, there exists a $\star\star$-open set U such that $F \subseteq U$ and $x \notin U$.

(4) For each $\tau_1\tau_2$-closed set F and each $x \in X - F$, $Cl^{\star\star}(\{x\}) \cap F = \emptyset$.

(5) For any two distinct points x and y of X, $x \notin \tau_1\tau_2$-$Cl(\{y\})$ implies

$$Cl^{\star\star}(\{x\}) \cap \tau_1\tau_2\text{-}Cl(\{y\}) = \emptyset.$$

Proof. (1) ⇒ (2): Let U be any $\tau_1\tau_2$-open set and $x \in U$. Then by (1), we have $\{x\}^{\star\star} \subseteq U$, Thus, $\mathrm{Cl}^{\star\star}(\{x\}) = \{x\}^{\star\star} \cup \{x\} \subseteq U$.

(2) ⇒ (3): Let F be any $\tau_1\tau_2$-closed set and $x \notin F$. Then $x \in X - F$. By (2), we have $\mathrm{Cl}^{\star\star}(\{x\}) \subseteq X - F$ and so $F \subseteq X - \mathrm{Cl}^{\star\star}(\{x\})$. Put $U = X - \mathrm{Cl}^{\star\star}(\{x\})$. Then U is a $\star\star$-open set such that $F \subseteq U$ and $x \notin U$.

(3) ⇒ (4): Let F be any $\tau_1\tau_2$-closed set and $x \notin F$. Then by (4), there exists a $\star\star$-open set U such that $F \subseteq U$ and $x \notin U$. Therefore, we obtain $\mathrm{Cl}^{\star\star}(\{x\}) \cap U = \emptyset$ and so $\mathrm{Cl}^{\star\star}(\{x\}) \cap F = \emptyset$.

(4) $\Rightarrow$ (5): The proof is obvious.

(5) $\Rightarrow$ (1): Let U be any $\tau_1\tau_2$-open set and $x \in U$. Then for each $y \in X - U$, we have $x \notin \tau_1\tau_2$-Cl$(\{y\})$. Therefore by (5), $\text{Cl}^{\star\star}(\{x\}) \cap \tau_1\tau_2\text{-Cl}(\{y\}) = \emptyset$ for each $y \in X - U$. Since $X - U$ is $\tau_1\tau_2$-closed, $y \in \tau_1\tau_2\text{-Cl}(\{y\}) \subseteq X - U$ and so $\cup_{y\in X-U}\tau_1\tau_2\text{-Cl}(\{y\}) = X - U$. Therefore,

$$\begin{aligned}\text{Cl}^{\star\star}(\{x\}) \cap (X - U) &= \text{Cl}^{\star\star}(\{x\}) \cap [\cup_{y\in X-U}\tau_1\tau_2\text{-Cl}(\{y\})]\\ &= \cup_{y\in X-U}[\text{Cl}^{\star\star}(\{x\}) \cap \tau_1\tau_2\text{-Cl}(\{y\})] = \emptyset.\end{aligned}$$

Consequently, we obtain $\text{Cl}^{\star\star}(\{x\}) \subseteq U$. This shows that X is a $\mathscr{I}^{\star\star}$-R_0-space. □

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