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Study on Smallest (Fuzzy) Ideals of $\mathcal{LA}$-Semigroups

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Abstract : In this paper, we characterize a weakly regular $\mathcal{LA}$-semigroup by using the smallest ideals and fuzzy ideals of an $\mathcal{LA}$-semigroup.

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1 Introduction

A left almost semigroup ($\mathcal{LA}$-semigroup) was studied by Kazim and Naseeruddin in 1972 [1]. An $\mathcal{LA}$-semigroup S is a groupoid satisfying the identity $(ab)c = (cb)a$ for all $a, b, c \in S$. This identity is sometimes called the left invertive law. Such a groupoid is a non-associative and non-commutative algebraic structure "mid-way" between a groupoid and a commutative semigroup. This structure is closely related to a commutative semigroup because a commutative $\mathcal{LA}$-semigroup

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is a semigroup [2, 3]. In an $\mathcal{LA}$-semigroup, the medial law [1] $(ab)(cd) = (ac)(bd)$ holds for all $a, b, c, d \in S$. An $\mathcal{LA}$-semigroup may or may not contain a left identity. The left identity of an $\mathcal{LA}$-semigroup allows us to introduce the inverses of elements in an $\mathcal{LA}$-semigroup. If an $\mathcal{LA}$-semigroup contains a left identity, then it is unique [2]. In an $\mathcal{LA}$-semigroup S with left identity (unitary $\mathcal{LA}$-semigroup), the paramedial law $(ab)(cd) = (dc)(ba)$ holds for all $a, b, c, d \in S$. By using medial law with left identity, we get $a(bc) = b(ac)$ for all $a, b, c \in S$. The connection of a commutative inverse semigroup with an $\mathcal{LA}$-semigroup has been given by Yousafzai et al. in [4] as, a commutative inverse semigroup $(S, \cdot)$ becomes an $\mathcal{LA}$-semigroup $(S, *)$ under $a * b = ba^{-1}r^{-1}$ for all $a, b, r \in S$. An $\mathcal{LA}$-semigroup S with left identity e becomes a semigroup under the binary operation "$\circ_e$" defined as, $x \circ_e y = (xe)y$ for all $x, y \in S$ [5]. An $\mathcal{LA}$-semigroup is the generalization of a semigroup theory [2] and has vast applications in collaboration with semigroup like other branches of mathematics.

The concept of fuzzy set was first proposed by Zadeh [6] in 1965, which has a wide range of applications in various fields such as computer engineering, artificial intelligence, control engineering, operation research, management science, robotics and many more. It gives us a tool to model the uncertainty present in phenomena that do not have sharp boundaries. Many papers on fuzzy sets have been published which shows the importance and its applications to set theory, algebra, real analysis, measure theory and topology etc.

Several algebraists extended the concepts and results of algebra to the boarder frame work of fuzzy set theory. Rosenfeld [7] was the first to consider the case when S is a groupoid. Kuroki and Mordeson have widely explored fuzzy semigroups in [8] and [9]. Fuzzy algebra is going popular day by day due to wide applications of fuzzification in almost every field.

In the current paper, we give some characterizations of weakly regular $\mathcal{LA}$-semigroups by using the smallest ideals and fuzzy ideals of an $\mathcal{LA}$-semigroup.

2 Preliminaries, Definitions and Basic Results

A fuzzy subset or a fuzzy set f of a non-empty set S is an arbitrary mapping $f : S \to [0, 1]$, where $[0, 1]$ is the unit segment of real line. A fuzzy subset f is a class of objects with a grades of membership having the form $f = \{(s, f(s)) \mid s \in S\}$.

Let f and g be any two fuzzy subsets of an $\mathcal{LA}$-semigroup S. Then the product $f \circ g$ is defined by,

$$(f \circ g)(a) = \begin{cases} \bigvee\limits_{a=bc} \{f(b) \wedge g(c)\} & \text{if there exist } b, c \in S \text{ such that } a = bc, \\ 0 & \text{otherwise.} \end{cases}$$

The order relation $\subseteq$ between any two fuzzy subsets f and g of an $\mathcal{LA}$-semigroup S is defined as:

$$f \subseteq g \text{ if and only if } f(x) \leq g(x) \text{ for all } x \in S.$$

The symbols $f \cap g$ and $f \cup g$ will means the following fuzzy subsets of an $\mathcal{LA}$-semigroup S:

$$(f \cap g)(x) = \min\{f(x), g(x)\} = f(x) \wedge g(x) \text{ for all } x \in S$$

and

$$(f \cup g)(x) = \max\{f(x), g(x)\} = f(x) \vee g(x) \text{ for all } x \in S.$$

If S is an $\mathcal{LA}$-semigroup with product $\cdot : S \times S \to S$, then $ab \cdot c = (ab)c$ and $a \cdot bc = a(bc)$ both will denote the product $(a \cdot b) \cdot c$ and $a \cdot (b \cdot c)$. Similarly, $ab \cdot cd = (ab)(cd)$ will denote the product $(a \cdot b) \cdot (c \cdot d)$.

If A is a non-empty set of an $\mathcal{LA}$-semigroup S, then we define

$$A^n = (...((AA)A)...)A,$$

where AA is a usual product and $n \in \mathbb{N}$. Similarly, we can define x^n for any $x \in S$ and f^n for any $f \in f(S)$ where $f(S)$ denotes the set of all fuzzy subsets of an $\mathcal{LA}$-semigroup S and $n \in \mathbb{N}$.

Definition 2.1. A fuzzy subset f of an $\mathcal{LA}$-semigroup S is called a *fuzzy left (right) ideal* of S if $f(xy) \geq f(y)$ $(f(xy) \geq f(x))$ for all $x, y \in S$. A fuzzy subset f of an $\mathcal{LA}$-semigroup S is called a *fuzzy two-sided ideal* or simply a *fuzzy ideal* of S if it is both a fuzzy left and a fuzzy right ideal of S.

The proof of the following four lemmas are same as in [9].

Lemma 2.2. *Let f, g be two fuzzy subsets of an $\mathcal{LA}$-semigroup S. Then the following assertions hold:*
(i) f is a fuzzy right ideal of S if and only if $f \circ S \subseteq f$;
(ii) g is a fuzzy left ideal of S if and only if $S \circ g \subseteq g$.

For $\emptyset \neq A \subseteq S$, the characteristic function, C_A is defined as follows:

$$C_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Lemma 2.3. *A non-empty subset A of an $\mathcal{LA}$-semigroup S is a left (right) ideal of S if and only if C_A is a fuzzy left (right) ideal of S.*

Lemma 2.4. *Let S be an $\mathcal{LA}$-semigroup. For $\emptyset \neq A \subseteq S$ and $\emptyset \neq B \subseteq S$, the following assertions hold:*
(i) $C_A \cap C_B = C_{A \cap B}$;
(ii) $C_A \circ C_B = C_{AB}$.

Definition 2.5. A fuzzy subset g of an $\mathcal{LA}$-semigroup S is called *fuzzy idempotent* if $g^2 = g \circ g = g$.

Definition 2.6. A fuzzy subset β of an $\mathcal{LA}$-semigroup S is called *fuzzy semiprime* if $\beta(a) \geq \beta(a^2)$ for all $a \in S$.

Lemma 2.7. *Let L be any left ideal of an $\mathcal{LA}$-semigroup S. Then L is semiprime if and only if C_L is fuzzy semiprime.*

Definition 2.8. An element a of an $\mathcal{LA}$-semigroup S is called a *weakly regular element* of S if there exist some $x, y \in S$ such that $a = ax \cdot ay$ and S is called *weakly regular* if all elements of S are weakly regular.

Lemma 2.9. *Let f be any fuzzy right ideal and g be any fuzzy left ideal of a weakly regular unitary $\mathcal{LA}$-semigroup S, then the following assertions hold:*
(i) *$f = f \circ S$ and $g = S \circ g$;*
(ii) *f and g are fuzzy idempotent.*

Proof. The proof is same as in [10]. □

If S is a unitary $\mathcal{LA}$-semigroup, then $S \circ S = S = SS$. Also if S is a weakly $\mathcal{LA}$-semigroup, then $S \circ S = S = SS$. Note that every fuzzy right ideal of a unitary $\mathcal{LA}$-semigroup S is a fuzzy left ideal of S but the converse is not true in general.

3 Characterizations of Weakly Regular $\mathcal{LA}$-Semigroups in Terms of Smallest (Fuzzy) Ideals

Example 3.1. Let $S = \{a, b, c, d, e\}$ be a unitary $\mathcal{LA}$-semigroup with the following multiplication table.

.	a	b	c	d	e
a	a	a	a	a	a
b	a	b	b	b	b
c	a	b	d	e	c
d	a	b	c	d	e
e	a	b	e	c	d

Note that S is non-commutative as $ed \neq de$ and also S is non-associative because $(cc)d \neq c(cd)$.

Define a fuzzy subset f of S as follows: $f(a) = 1$ and $f(b) = f(c) = f(d) = f(e) = 0$, then clearly f is a fuzzy two-sided ideal of S.

It is easy to see that every fuzzy left (right, two-sided) ideal of an $\mathcal{LA}$-semigroup S is a fuzzy $\mathcal{LA}$-subsemigroup of S but the converse is not true in general. Let us define a fuzzy subset f of S as follows: $f(a) = 1, f(b) = 0$ and $f(c) = f(d) = f(e) = 0.5$, then by routine calculation one can easily check that f is a fuzzy $\mathcal{LA}$-subsemigroup of S but it is not a fuzzy left (right, two-sided) ideal of S because $f(bd) \ngeq f(d)$ or $f(db) \ngeq f(d)$.

Lemma 3.2. *Let S be a unitary $\mathcal{LA}$-semigroup such that S is idempotent. Then $R_a = Sa \cup Sa^2$ is the smallest right ideal of S containing a, providing $a = a^2$ for all $a \in S$.*

Proof. Assume that $a = a^2$ for all $a \in S$, then

$$\begin{aligned}(Sa \cup Sa^2)S &= Sa \cdot S \cup Sa^2 \cdot S = Sa \cdot SS \cup Sa^2 \cdot SS = S \cdot aS \cup S \cdot a^2 S \\ &= a \cdot SS \cup a^2 \cdot SS = a^2 \cdot SS \cup a^2 \cdot SS \\ &= SS \cdot a^2 \cup SS \cdot a^2 = Sa \cup Sa^2,\end{aligned}$$

which shows that $Sa \cup Sa^2$ is a right ideal of S. Since $a \in Sa$, therefore $a \in Sa \cup Sa^2$. Let R be another right ideal of S containing a. Since

$$Sa = SS \cdot a = aS \cdot S \subseteq RS \cdot S \subseteq R,$$

and

$$Sa^2 = a \cdot Sa \subseteq RS \subseteq R,$$

we have $Sa \cup Sa^2 \subseteq R$. Hence $Sa \cup Sa^2$ is the smallest right ideal of S containing a. □

It is well known that if all the elements of a unitary $\mathcal{LA}$-semigroup S are idempotent (also called a *unitary $\mathcal{LA}$-band*), then S becomes a commutative monoid.

Lemma 3.3. *Let S be a unitary $\mathcal{LA}$-semigroup and $a \in S$. Then $L_a = Sa$ is the smallest left ideal of S containing a.*

Proof. It is simple. □

Theorem 3.4. *Let S be a unitary $\mathcal{LA}$-semigroup. Then the following conditions are equivalent:*
(i) *S is weakly regular;*
(ii) *$L_a = L_a L_a$, where L_a is the smallest left ideal of S containing a;*
(iii) *$L \cap M = ML$, where both L and M are any left ideals of S;*
(iv) *$f \cap g = g \circ f$, where both f and g are any fuzzy left ideals of S.*

Proof. $(i) \Rightarrow (iv)$: Let f and g be both fuzzy left ideals of a weakly regular S. Now for $a \in S$, there exist some $x, y \in S$ such that

$$a = ax \cdot ay = ya \cdot xa,$$

therefore

$$(g \circ f)(a) = \bigvee_{a = ya \cdot xa} \{g(ya) \wedge f(xa)\} \geq g(ya) \wedge f(xa) \geq g(a) \wedge f(a),$$

which implies that $g \circ f \supseteq f \cap g$. Now by using Lemmas 2.2 and 2.9, it is easy to see that $g \circ f \subseteq f \cap g$. Thus $f \cap g = g \circ f$.

$(iv) \Rightarrow (iii)$: Let L and M be any left ideals of S. Then by Lemma 2.3, C_L and C_M are fuzzy left ideals of S. Let $x \in L \cap M$. Then by using Lemma 2.4, we have

$$1 = C_{L\cap M}(x) = (C_L \cap C_M)(x) \leq (C_M \circ C_L)(x) = C_{ML}(x),$$

which implies that $a \in ML$ and therefore $L \cap M \subseteq ML$. It is easy to see that $ML \subseteq L \cap M$ and therefore $L \cap M = ML$.

$(iii) \Rightarrow (ii)$: It is obvious.

$(ii) \Rightarrow (i)$: Since Sa is the smallest left ideal of S containing a, we have

$$a \in Sa = Sa \cdot Sa = aS \cdot aS.$$

Hence S is weakly regular. □

Theorem 3.5. *Let S be a unitary $\mathcal{LA}$-semigroup. Then the following conditions are equivalent:*

(i) *S is a weakly regular $\mathcal{LA}$-semigroup;*

(ii) *S is a weakly regular commutative monoid;*

(iii) *$R_a \cap L_a = R_aL_a \cdot R_a$, for the smallest right ideal R_a and smallest left ideal L_a of S provided that S is an $\mathcal{LA}$-band;*

(iv) *$R \cap L = RL \cdot R$, for every right ideal R and left ideal L of S;*

(v) *$f \cap g = (f \circ g) \circ f$, for every fuzzy right ideal f and every fuzzy left ideal g of S;*

(vi) *$f \cap g = (f \circ g) \circ f$, for every fuzzy ideal f and g of S.*

Proof. $(i) \Rightarrow (vi)$: Let f and g be any fuzzy ideal of a weakly regular S. Now for $a \in S$, there exist some $x, y \in S$ such that

$$\begin{aligned} a &= ax \cdot ay = ax \cdot (ax \cdot ay)y = ((ax \cdot ay)y \cdot x)a \\ &= (xy \cdot (ax \cdot ay))a = (ax \cdot (xy \cdot ay))a \\ &= (ax \cdot (a \cdot (xy)y))a, \end{aligned}$$

therefore

$$\begin{aligned} ((f \circ g) \circ f)(a) &= \bigvee_{a=(ax\cdot(a\cdot(xy)y))a} \{(f \circ g)(ax \cdot (a \cdot (xy)y)) \wedge g(a)\} \\ &\geq \bigvee_{ax\cdot(a\cdot(xy)y)=ax\cdot(a\cdot(xy)y)} \{f(ax) \wedge g(a \cdot (xy)y)\} \wedge g(a) \\ &\geq f(ax) \wedge g(a \cdot (xy)y) \wedge g(a) \geq f(a) \wedge g(a), \end{aligned}$$

which shows that $(f \circ g) \circ f \supseteq f \cap g$. Now by using Lemma 2.2, it is easy to see that $(f \circ g) \circ f \subseteq f \cap g$.

$(vi) \Rightarrow (v)$: It is simple.

$(v) \Rightarrow (iv)$: Let R and L be any left and right ideals of S. Then by Lemma 2.3, C_R and C_L are the fuzzy right and fuzzy left ideals of S. Now by using Lemma 2.4, we have

$$C_{R\cap L} = C_R \cap C_L = (C_R \circ C_L) \circ C_L = C_{RL\cdot R},$$

which shows that $R \cap L = RL \cdot R$.

$(iv) \Rightarrow (iii)$: Straightforward.

$(iii) \Rightarrow (ii)$: Since $Sa \cup Sa^2$ is the smallest right ideal of S containing a and Sa is the smallest left ideal of S containing a, therefore

$$\begin{aligned} a &\in (Sa \cup Sa^2) \cap (Sa) = (Sa \cup Sa^2)(Sa) \cdot (Sa \cup Sa^2) \\ &= (Sa \cdot Sa \cup Sa^2 \cdot Sa) \cdot (Sa \cup Sa^2) \\ &= (Sa \cdot Sa)(Sa) \cup (Sa \cdot Sa)(Sa^2) \cup (Sa^2 \cdot Sa)(Sa) \cup (Sa^2 \cdot Sa)(Sa^2) \\ &= (aS \cdot aS)(Sa) \cup (aS \cdot aS)(a^2S) \cup (aS \cdot a^2S)(Sa) \cup (aS \cdot a^2S)(a^2S) \\ &= ((aS \cdot S)a)(Sa) \cup ((aS \cdot S)a)(a^2S) \cup ((a^2S \cdot S)a)(Sa) \cup ((a^2S \cdot S)a)(a^2S) \\ &\subseteq Sa \cdot Sa \cup Sa \cdot Sa \cup Sa \cdot Sa \cup Sa \cdot Sa = Sa \cdot Sa = aS \cdot aS. \end{aligned}$$

Hence S is a weakly regular commutative monoid.

$(ii) \Rightarrow (i)$: It is obvious. □

Lemma 3.6. *Let R be a right ideal and L be a left ideal of a unitary $\mathcal{LA}$-semigroup S. Then RL is a left ideal of S.*

Proof. Let R and L be any left and right ideals of a unitary $\mathcal{LA}$-semigroup S. Then

$$S \cdot RL = SS \cdot RL = SR \cdot SL \subseteq SR \cdot L = (SS \cdot R)L = (RS \cdot S)L \subseteq RL,$$

which shows that RL is a left ideal of S. □

Theorem 3.7. *Let S be a unitary $\mathcal{LA}$-semigroup. Then the following conditions are equivalent:*

(i) S is a weakly regular $\mathcal{LA}$-semigroup;

(ii) S is a weakly regular commutative monoid;

(iii) $R_aL_a \cap L_a = R_a \cdot R_aL_a^3$, for the smallest right ideal R_a and smallest left ideal L_a of S provided that S is an $\mathcal{LA}$-band;

(iv) $RL \cap L = R \cdot RL^3$, for every right ideal R and left ideal L of S such that L is semiprime;

(v) $f \cap g = (f \circ g) \circ f$, for every fuzzy left ideal f and g of S such that f is fuzzy semiprime.

Proof. $(i) \Rightarrow (v)$: Let f and g be any fuzzy left ideals of a weakly regular S. By using Lemmas 2.2 and 2.9, it is easy to show that $(f \circ g) \circ f \subseteq f \cap g$. Now for $a \in S$, there exist some $x, y \in S$ such that

$$\begin{aligned} a &= ax \cdot ay = ya \cdot xa = y(ax \cdot ay) \cdot xa = (ax)(y \cdot ay) \cdot xa \\ &= (ay \cdot y)(xa) \cdot xa = (y^2a \cdot xa)(xa). \end{aligned}$$

Therefore

$$\begin{aligned}((f\circ g)\circ f)(a) &= \bigvee_{a=(y^2a\cdot xa)(xa)} \left\{(f\circ g)((y^2a)(xa))\wedge f(xa)\right\}\\ &\geq \bigvee_{(y^2a)(ax\cdot e)=(y^2a)(ax\cdot e)} \{f(y^2a)\wedge g(xa)\}\wedge f(xa)\\ &\geq f(y^2a)\wedge g(xa)\wedge f(xa)\geq f(a)\wedge g(a),\end{aligned}$$

which shows that $f\cap g\subseteq (f\circ g)\circ f$. Thus $f\cap g=(f\circ g)\circ f$ and by Lemma 2.9, f is fuzzy semiprime.

$(v)\Rightarrow(iv)$: Let R and L be any right and left ideals of S. Then by using Lemmas 3.6 and 2.3, C_{RL} and C_L are fuzzy left ideals of S. Now by using Lemma 2.4, we get

$$C_{RL\cap L}=C_{RL}\cap C_L=(C_{RL}\circ C_L)\circ C_{RL}=C_{(RL\cdot L)(RL)},$$

which give us $RL\cap L=(RL\cdot L)(RL)$, and

$$\begin{aligned}(RL\cdot L)(RL) &= L^2R\cdot RL=LR\cdot RL^2=R(LR\cdot L^2)\\ &= R(L^2\cdot RL)=R(R\cdot L^2L)=R\cdot RL^3,\end{aligned}$$

which implies that $RL\cap L=R\cdot RL^3$ and by using Lemma 2.7, L is semiprime.

$(iv)\Rightarrow(iii)$: Straightforward.

$(iii)\Rightarrow(ii)$: Now $Sa\cup Sa^2$ is the smallest right ideal of S containing a and Sa is the smallest left ideal of S containing a. Setting $R=Sa\cup Sa^2$ and $L=Sa$, then by using given assumption and Lemma 3.6, we have

$$a^2\in(Sa\cup Sa^2)(Sa)=RL\Rightarrow a\in RL.$$

Therefore

$$\begin{aligned}a &\in (Sa\cup Sa^2)(Sa)\cap(Sa)=(Sa\cup Sa^2)\cdot(Sa\cup Sa^2)(Sa)^3\\ &= (Sa\cup Sa^2)\cdot(Sa\cup Sa^2)(Sa^3)=(Sa\cup Sa^2)\cdot(Sa\cdot Sa^3\cup Sa^2\cdot Sa^3)\\ &= Sa\cdot(Sa\cdot Sa^3)\cup Sa\cdot(Sa^2\cdot Sa^3)\cup Sa^2\cdot(Sa\cdot Sa^3)\cup Sa^2\cdot(Sa^2\cdot Sa^3)\\ &= Sa\cdot(Sa\cdot S(a^2a))\cup Sa\cdot(a^2S\cdot S(a^2a))\cup a^2S\cdot(Sa\cdot S(a^2a))\\ &\quad \cup a^2S\cdot(Sa^2\cdot S(a^2a))\\ &= Sa\cdot(Sa\cdot a^2(Sa))\cup Sa\cdot(a^2S\cdot a^2(Sa))\cup(Sa)a\cdot(Sa\cdot a^2(Sa))\\ &\quad \cup(Sa)a\cdot(Sa^2\cdot a^2(Sa))\\ &\subseteq Sa\cdot(a^2\cdot(Sa)(Sa))\cup Sa\cdot(a^2\cdot(a^2S)(Sa))\cup Sa\cdot(a^2\cdot(Sa)(Sa))\\ &\quad \cup Sa\cdot(a^2\cdot(Sa^2)(Sa))\\ &\subseteq Sa\cdot a^2S\cup Sa\cdot a^2S\cup Sa\cdot a^2S\cup Sa\cdot a^2S\\ &= Sa\cdot(Sa)a\cup Sa\cdot(Sa)a\cup Sa\cdot(Sa)a\cup Sa\cdot(Sa)a\\ &\subseteq Sa\cdot Sa\cup Sa\cdot Sa\cup Sa\cdot Sa\cup Sa\cdot Sa\\ &= aS\cdot aS\cup aS\cdot aS\cup aS\cdot aS\cup aS\cdot aS=aS\cdot aS.\end{aligned}$$

Hence S is a weakly regular commutative monoid.

$(ii) \Rightarrow (i)$: It is obvious. □

Corollary 3.8. *Let S be a unitary $\mathcal{LA}$-semigroup. Then the following conditions are equivalent:*

(i) *S is a weakly regular $\mathcal{LA}$-semigroup;*

(ii) *S is a weakly regular commutative monoid;*

(iii) *$R_aL_a \cap L_a = L_a^2R_a \cdot R_aL_a$, for the smallest right ideal R_a and smallest left ideal L_a of S provided that S is an $\mathcal{LA}$-band;*

(iv) *$RL \cap L = L^2R \cdot RL$, for every right ideal R and left ideal L of S such that L is semiprime;*

(v) *$f \cap g = (f \circ g) \circ f$, for every fuzzy left ideal f and g of S such that f is fuzzy semiprime.*

Theorem 3.9. *Let S be a unitary $\mathcal{LA}$-semigroup. Then the following conditions are equivalent:*

(i) *S is a weakly regular $\mathcal{LA}$-semigroup;*

(ii) *S is a weakly regular commutative monoid;*

(iii) *$R_a \cap L_a = R_a^2L_a^2$, for the smallest right ideal R_a and smallest left ideal L_a of S provided that S is an $\mathcal{LA}$-band;*

(iv) *$R \cap L = R^2L^2$, for every right ideal R and left ideal L of S.*

Proof. $(i) \Rightarrow (iv)$: Let R and L be any right and left ideals of a weakly regular S. It is easy to see that $R^2L^2 \subseteq R \cap L$. Let $a \in R \cap L$. Then there exist some $x, y \in S$ such that

$$\begin{aligned} a &= ax \cdot ay = (ax \cdot ay)x \cdot (ax \cdot ay)y = (ax \cdot ay) \cdot ((ax \cdot ay)x)y \\ &= (ax \cdot ay) \cdot (yx)(ax \cdot ay) = (ax \cdot ay) \cdot (ax)(yx \cdot ay) \\ &= (ax \cdot ay) \cdot (ay \cdot yx)(xa) = (ax \cdot ay) \cdot ((yx \cdot y)a)(xa) \\ &\in (aS \cdot aS)(Sa \cdot Sa) \in (RS \cdot RS)(SL \cdot SL) \subseteq R^2L^2, \end{aligned}$$

which shows that $R \cap L = R^2L^2$.

$(iv) \Rightarrow (iii)$: Straightforward.

$(iii) \Rightarrow (ii)$: Since $Sa \cup Sa^2$ is the smallest right ideal of S containing a and Sa is the smallest left ideal of S containing a, we have

$$\begin{aligned} a &\in (Sa \cup Sa^2) \cap Sa = (Sa \cup Sa^2)(Sa \cup Sa^2) \cdot (Sa)(Sa) \\ &= (Sa \cdot Sa \cup Sa \cdot Sa^2 \cup Sa^2 \cdot Sa \cup Sa^2 \cdot Sa^2)(Sa^2) \\ &= (Sa^2 \cup a^2S \cdot aS \cup Sa^3 \cup Sa^4)(Sa^2) \\ &= Sa^2 \cdot Sa^2 \cup a^3S \cdot Sa^2 \cup a^2(Sa) \cdot Sa^2 \cup a^2(Sa^2) \cdot Sa^2 \\ &= a^2S \cdot a^2S \cup (Sa^2)a \cdot a^2S \cup (Sa \cdot a)a \cdot a^2S \cup (Sa^2 \cdot a)a \cdot a^2S \\ &\subseteq (Sa \cdot a) \cdot (Sa \cdot a) \cup Sa \cdot (Sa \cdot a) \cup Sa \cdot (Sa \cdot a) \cup Sa \cdot (Sa \cdot a) \\ &\subseteq Sa \cdot Sa \cup Sa \cdot Sa \cup Sa \cdot Sa \cup Sa \cdot Sa \\ &= aS \cdot aS \cup aS \cdot aS \cup aS \cdot aS \cup aS \cdot aS = aS \cdot aS. \end{aligned}$$

This implies that S is a weakly regular commutative monoid.

$(ii) \Rightarrow (i)$: It is obvious. □

Corollary 3.10. *Let S be a unitary $\mathcal{LA}$-semigroup. Then the following conditions are equivalent:*

(i) *S is a weakly regular $\mathcal{LA}$-semigroup;*

(ii) *S is a weakly regular commutative monoid;*

(iii) *$R_a \cap L_a = L_a^2 R_a^2$, for the smallest right ideal R_a and smallest left ideal L_a of S provided that S is an $\mathcal{LA}$-band;*

(iv) *$R \cap L = L^2 R^2$, for every right ideal R and left ideal L of S.*

Theorem 3.11. *Let S be a unitary $\mathcal{LA}$-semigroup. Then the following conditions are equivalent:*

(i) *S is a weakly regular $\mathcal{LA}$-semigroup;*

(ii) *S is a weakly regular commutative monoid;*

(iii) *$R_a \cap L_a = L_a^2 R_a^2$, for the smallest right ideal R_a and smallest left ideal L_a of S provided that S is an $\mathcal{LA}$-band;*

(iv) *$R \cap L = L^2 R^2$, for every right ideal R and left ideal L of S;*

(v) *$f \cap g = (f \circ g) \circ (f \circ g)$, for every fuzzy right ideal f and fuzzy left ideal g of S.*

Proof. $(i) \Rightarrow (v)$: Let f and g be any fuzzy right and left ideals of a weakly regular S. From Lemma 2.2, it is easy to show that $(f \circ g) \circ (f \circ g) \subseteq f \cap g$. Now from Theorem 3.9, for $a \in S$, there exist some $x, y \in S$ such that

$$\begin{aligned} a &= (ax \cdot ay) \cdot ((yx \cdot y)a)(xa) = (ax)((yx \cdot y)a) \cdot (ay)(xa) \\ &= (ax)(ba) \cdot (ay)(xa), \text{ where } yx \cdot y = b. \end{aligned}$$

Therefore

$$\begin{aligned} ((f \circ g) \circ (f \circ g))(a) &= \bigvee_{a=(ax)(ba)\cdot(ay)(xa)} \{(f \circ g)(ax \cdot ba) \wedge (f \circ g)(ay \cdot xa)\} \\ &\geq \bigvee_{ax\cdot ba = ax \cdot ba} \{f(ax) \wedge g(ba)\} \wedge \bigvee_{ay\cdot xa=ay\cdot xa} \{f(ay) \wedge g(xa)\} \\ &\geq f(ax) \wedge g(ba) \wedge f(ay) \wedge g(xa) \geq f(a) \wedge g(a), \end{aligned}$$

which shows that $f \cap g \subseteq (f \circ g) \circ (f \circ g)$. Thus $f \cap g = (f \circ g) \circ (f \circ g)$.

$(v) \Rightarrow (iv)$: Let R and L be any right and left ideals of S. Then by using Lemma 2.3, C_R and C_L are the fuzzy right and left ideals of S. Now by using Lemma 2.4, we get

$$\begin{aligned} C_{R\cap L} &= C_R \cap C_L = (C_R \circ C_L) \circ (C_R \circ C_L) = (C_R \circ C_R) \circ (C_L \circ C_L) \\ &= C_{R^2} \circ C_{L^2} = C_{R^2 L^2} = C_{L^2 R^2}, \end{aligned}$$

which implies that $R \cap L = L^2 R^2$.

$(iv) \Rightarrow (iii)$: Straightforward.

$(iii) \Rightarrow (ii)$: It can be followed from Corollary 3.10.

$(ii) \Rightarrow (i)$: It is obvious. □

Theorem 3.12. *Let S be a unitary $\mathcal{LA}$-semigroup. Then the following conditions are equivalent:*

(i) *S is a weakly regular $\mathcal{LA}$-semigroup;*

(ii) *S is a weakly regular commutative monoid;*

(iii) *$R_a \cap L_a = R_a^3 L_a$, for the smallest right ideal R_a and smallest left ideal L_a of S provided that S is an $\mathcal{LA}$-band;*

(iv) *$R \cap L = R^3 L$, for every right ideal R and left ideal L of S;*

(v) *$f \cap g = f^3 \circ g$, for every fuzzy right ideal f and fuzzy left ideal g of S.*

Proof. $(i) \Rightarrow (v)$: Let f and g be any fuzzy right and left ideals of a weakly regular S. From Lemma 2.2, it is easy to show that $f^3 \circ g \subseteq f \cap g$. Now for $a \in S$, there exist some $x, y \in S$ such that

$$\begin{aligned} a &= ax \cdot ay = (ax \cdot ay)x \cdot (ax \cdot ay)y = y(ax \cdot ay) \cdot x(ax \cdot ay) \\ &= (ax)(y \cdot ay) \cdot (ax)(x \cdot ay) = (ax)(ay^2) \cdot (ax)(a \cdot xy) \\ &= (y^2 a)(xa) \cdot (ax)(a \cdot xy) = ((ax)(a \cdot xy))(xa) \cdot y^2 a \\ &= ((ax)(a \cdot xy))(ex \cdot a) \cdot y^2 a = ((ax)(a \cdot xy))(ax \cdot e) \cdot y^2 a \\ &= bc \cdot y^2 a = d \cdot y^2 a, \text{ where } d = bc = ((ax)(a \cdot xy))(ax \cdot e). \end{aligned}$$

Now

$$\begin{aligned} ((f \circ f) \circ f)(d) &= \bigvee_{d=bc} \{(f \circ f)(b) \wedge f(c)\} \geq (f \circ f)(b) \wedge f(c) \\ &= \bigvee_{b=(ax)(a \cdot xy)} \{f(ax) \wedge f(a \cdot xy)\} \wedge f(ax \cdot e) \\ &\geq f(ax) \wedge f(a \cdot xy) \wedge f(ax \cdot e) \geq f(a). \end{aligned}$$

Therefore

$$(f^3 \circ g)(a) = \bigvee_{a=d \cdot y^2 a} \{((f \circ f) \circ f)(d) \wedge g(y^2 a)\} \geq f(a) \wedge g(a),$$

which shows that $f \cap g \subseteq f^3 \circ g$. Thus $f \cap g = f^3 \circ g$.

$(v) \Rightarrow (iv)$: Let R and L be any right and left ideals of S. Then by using Lemma 2.3, C_R and C_L are the fuzzy right and left ideals of S. Now by using Lemma 2.4, we get

$$C_{R \cap L} = C_R \cap C_L = ((C_R \circ C_R) \circ C_R) \circ C_L = C_{R^3} \circ C_L = C_{R^3 L},$$

which implies that $R \cap L = R^3 L$.

$(iv) \Rightarrow (iii)$: Straightforward.

$(iii) \Rightarrow (ii)$: Since $Sa \cup Sa^2$ is the smallest right ideal of S containing a and

Sa is the smallest left ideal of S containing a, we have

$$
\begin{aligned}
a &\in (Sa \cup Sa^2) \cap Sa = ((Sa \cup Sa^2)(Sa \cup Sa^2) \cdot (Sa \cup Sa^2))(Sa) \\
&= (Sa^2 \cup Sa \cdot Sa^2 \cup Sa^3 \cup Sa^4)(Sa) = (Sa^2 \cup Sa \cdot Sa^2 \cup Sa^3 \cup Sa^4)(Sa) \\
&= Sa^2 \cdot Sa \cup (Sa \cdot Sa^2)(Sa) \cup Sa^3 \cdot Sa \cup Sa^4 \cdot Sa \\
&= a^2S \cdot Sa \cup (Sa \cdot Sa) \cdot (Sa^2 \cdot Sa) \cup aS \cdot a^3S \cup aS \cdot a^4S \\
&= (Sa \cdot a)(Sa) \cup Sa^2 \cdot (aS \cdot a^2S) \cup Sa^3 \cdot Sa \cup Sa^4 \cdot Sa \\
&\subseteq Sa \cdot Sa \cup a^2S \cdot (a^2S \cdot S)a \cup a^2(Sa) \cdot Sa \cup (a^2a)(Sa) \cdot Sa \\
&= Sa \cdot Sa \cup (Sa)a \cdot Sa \cup (Sa \cdot a)a \cdot Sa \cup (aS \cdot aa^2) \cdot Sa \\
&\subseteq Sa \cdot Sa \cup Sa \cdot Sa \cup Sa \cdot Sa \cup (aa^2 \cdot S)a \cdot Sa \\
&\subseteq Sa \cdot Sa \cup Sa \cdot Sa \cup Sa \cdot Sa \cup Sa \cdot Sa \\
&= aS \cdot aS \cup aS \cdot aS \cup aS \cdot aS \cup aS \cdot aS = aS \cdot aS.
\end{aligned}
$$

Thus S is a weakly regular commutative monoid.

$(ii) \Rightarrow (i)$: It is obvious. □

From the duality of Theorem 3.12, we have the following theorem:

Theorem 3.13. *Let S be a unitary $\mathcal{LA}$-semigroup. Then the following conditions are equivalent:*

(i) *S is weakly regular $\mathcal{LA}$-semigroup;*

(ii) *S is weakly regular commutative monoid;*

(iii) *$R_a \cap L_a = L_a^3 R_a$, for the smallest right ideal R_a and smallest left ideal L_a of S provided that S is an $\mathcal{LA}$-band;*

(iv) *$R \cap L = L^3 R$, for every right ideal R and left ideal L of S;*

(v) *$f \cap g = g^3 \circ f$, for every fuzzy right ideal f and fuzzy left ideal g of S.*

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