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Fuzzy Congruences on Strongly π-Inverse Semigroups

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Abstract : In this paper, we use the notion of a fuzzy congruence relation on semigroups to study group congruence on strongly π-inverse semigroups and give several forms of the group congruence. Finally, sufficient and necessary condition for a fuzzy congruence on strongly π-inverse semigroups to be a fuzzy group congruence are proved.

Keywords : strongly π-inverse semigroup; fuzzy congruence; fuzzy group congruence.

1 Introduction and Preliminaries

Congruence and fuzzy congruence play an important role in studying semigroups and fuzzy semigroups [1–6]. In recent decades, many semigroup scholars pay attention to congruence theories on various of generalized regular semigroups [7–9]. Li Chun hua used the nation of a fuzzy congruence relation on semigroups to study some properties of fuzzy congruence on strictly π-regular semigroups and obtained the group congruence on such semigroups. In this paper, we use the notion of a fuzzy congruence relation on semigroups to study group congruence on strongly π-inverse semigroups and give several forms of the group congruence. Finally, sufficient and necessary condition for a fuzzy congruence on strongly π-inverse semigroup to be a fuzzy group congruence are proved. An ele-

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ment a of a semigroup S is called *regular* if there exists x in S such that $axa = a$. A semigroup S is called *π-regular* if a power of each element is regular. A semigroup S is called *strongly π-inverse semigroup* if its idempotents commute. Throughout this paper, let $E(S)$ denote the set of idempotents and $Reg(S)$ denote the set of regular elements of S. If a is a regular element of S, $V(a)$ denotes the set of inverses of a.

Let X be a non-empty, a map $f : X \to [0, 1]$ is called a *fuzzy subset of X*. Let S be a semigroup, a function $\mu : S \times S \to [0, 1]$ is called a *fuzzy relation on S*.

2 Definitions and Basic Results

Definition 2.1. [2] Let S be a semigroup. Let μ and ν be two fuzzy relations on S, then the product $\mu \circ \nu$ and $\mu \subseteq \nu$ is defined by
(1) $\mu \circ \nu = \vee_{x \in S}\{\mu(a, x) \wedge \nu(x, b)\}$;
(2) $\mu \subseteq \nu \Leftrightarrow \forall x, y \in S, \mu(x, y) \leq \nu(x, y)$
for all a, b in S.

Definition 2.2. [2] A fuzzy relation μ on S is called a *fuzzy equivalence relation on S* if
(1) (fuzzy reflexive) $\mu(a, a) = 1$ for all a in S;
(2) (fuzzy symmetric) $\mu(a, b) = \mu(b, a)$ for all a, b in S;
(3) (fuzzy transitive) $\mu \circ \mu \subseteq \mu$.

Definition 2.3. [2] A fuzzy relation μ on S is called *compatible* if

$$\mu(ax, bx) \geq \mu(a, b) \text{ and } \mu(xa, xb) \geq \mu(a, b)$$

for all a, b in S.

A fuzzy equivalence relation on a semigroup S which is compatible is called a *fuzzy congruence relation on S*, let μ_a denote the fuzzy subset of semigroup S that have fuzzy equivalence relation μ with a, let μ be a fuzzy congruence relation on S, then we can define a multiplication "$*$" on the set $S/\mu = \{\mu_a | a \in S\}$ as follows:

$$\mu_a * \mu_b = \mu_{ab}$$

for all a, b in S.

It is easy to verify $(S/\mu, \text{“}*\text{”})$ is a semigroup with $(\mu_e)^2 = \mu_e$ for every e in $E(S)$.

Lemma 2.4. [2] *Let μ be a fuzzy congruence relation on S. The following properties hold for all a, b in S.*
(1) $\mu_a = \mu_b \Leftrightarrow \mu(a, b) = 1$*;*
(2) $\mu^{-1} = \{(a, b) \in S \times S | \mu(a, b) = 1\}$ *is a congruence relation on S.*

A fuzzy congruence relation μ on a semigroup S is called a *fuzzy group congruence relation*, if $(S/\mu, \text{“}*\text{”})$ is a group.

Lemma 2.5. [10] *Let S be a π-regular semigroup, if μ is a fuzzy congruence relation on S, then $(S/\mu, \text{“}*\text{”})$ is a π-regular semigroup.*

Lemma 2.6. [10] *Let S be a π-regular semigroup, if μ is a fuzzy congruence relation on S, then the following conditions are equivalent:*
(1) $(\forall a \in S)\ \mu_a \in E(S/\mu)$;
(2) $(\exists e \in E(S))\ \mu_a = \mu_e$.

3 Fuzzy Congruences

Theorem 3.1. *Let S be a strongly π-inverses semigroup, let μ be a fuzzy congruence relation on S, then $(S/\mu, \text{“}*\text{”})$ is a strongly π-inverse semigroup. We define a map:*

$$\mu^\sharp : \ S \to S/\mu, \ a\mu^\sharp = \mu_a (\forall a \in S).$$

Then $\mu^\sharp$ is morphism from S onto S/μ.

Conversely, suppose that $\mu^\sharp : S \to T$ is a morphism, then $S\mu^\sharp$ is a strongly π-inverse semigroup, and for each $g \in E(S\mu^{\sharp)}$ there exists $e \in E(S)$ such that $\mu_e = g$.

Proof. From Lemma 2.5 know that S/μ is a π-regular semigroup. We shall show that the idempotents of S/μ are commuted. For all e, f in $E(S)$, we have $\mu_e, \mu_f \in E(S/\mu)$ and

$$\mu_e * \mu_f = \mu_{ef} = \mu_{fe} = \mu_f * \mu_e.$$

Namely, S/μ is a strongly π-inverse semigroup.

Since

$$(a\mu^\sharp)(b\mu^\sharp) = \mu_a * \mu_b = \mu_{ab} = (ab)\mu^\sharp$$

for all a, b in S. Namely, $\mu^\sharp$ is morphism from S onto S/μ.

If a is an element of a strongly π-inverse semigroup S, there exists $n \in N^+$ such that $a^n \in Reg(S)$. Since $\mu^\sharp : S \to T$ is a morphism, then

$$(a^n\mu^\sharp) = (a\mu^\sharp)^n.$$

Let $e, f \in E(S)$, then $e\mu^\sharp, f\mu^\sharp \in E(S\mu^\sharp)$, and

$$(e\mu^\sharp)(f\mu^\sharp) = (ef)\mu^\sharp = (fe)\mu^\sharp = (f\mu^\sharp)(e\mu^\sharp).$$

Hence, $S\mu^\sharp$ is a strongly π-inverse semigroup. Let $a\mu^\sharp$ be an idempotent of $S\mu^\sharp$, then there exist a^2, x in S, we still have

$$a\mu^\sharp = a^m\mu^\sharp \ (m \in N^+) \text{ and } \ (a^2)^n = (a^2)^n x (a^2)^n, \ x = x(a^2)^n x.$$

Since

$$(a^n x a^n)^2 = (a^n x a^n)(a^n x a^n) = a^n x a^{2n} x a^n = a^n x a^n \in E(S),$$

then

$$\begin{aligned} g = a\mu^\sharp = a^2\mu^\sharp &= (a^2)^n\mu^\sharp = ((a^2)^n x (a^2)^n)\mu^\sharp \\ &= ((a^2)^n\mu^\sharp)(x\mu^\sharp)((a^2)^n\mu^\sharp) \\ &= (a^n\mu^\sharp)(x\mu^\sharp)(a^n\mu^\sharp) = (a^n x a^n)\mu^\sharp. \end{aligned}$$

Hence, there exists $e = a^n x a^n \in E(S)$ such that $\mu_e = g$ as required. □

Theorem 3.2. *Let S be a strongly π-inverses semigroup, μ be a fuzzy congruence relation on S and $E(S)$ be the set of idempotents, then the relation defined by:*

$$\tilde{\mu} = \{(a,b) \in S \times S | \exists e \in E(S), \mu(ae, be) = 1\},$$

and $\tilde{\mu}$ is a group congruence on S.

Proof. We show first that $\tilde{\mu}$ is an equivalence. It is clear that $\tilde{\mu}$ is reflexive and symmetric (see Lemma 2.5). To show that it is transitive, let $(a,b) \in \tilde{\mu}, (b,c) \in \tilde{\mu}$, then there exist $e, f \in E(S)$ such that

$$\mu(ae, be) = 1,\ \mu(bf, cf) = 1, \text{ and } \mu_{ae} = \mu_{be},\ \mu_{bf} = \mu_{cf},$$

hence

$$\begin{aligned}\mu_{aef} &= \mu_{ae} * \mu_f = \mu_{be} * \mu_f = \mu_{bef} = \mu_{bfe}\\ &= \mu_{bf} * \mu_e = \mu_{cf} * \mu_e = \mu_{cfe}\\ &= \mu_{cef},\end{aligned}$$

then $\mu(aef, bef) = 1$, and so $(a,c) \in \tilde{\mu}$ as required.

To show $\tilde{\mu}$ is a congruence, suppose that $(a,b) \in \tilde{\mu}$ and that $c \in S$. Thus there exists $e \in E(S)$, such that $\mu(ae, be) = 1$, then $\mu_{ae} = \mu_{be}$. It follows that $\mu_c * \mu_{ae} = \mu_c * \mu_{be}$, then $\mu_{cae} = \mu_{cbe}$, thus $(ca, cb) \in \tilde{\mu}$.

To show that $(ac, bc) \in \tilde{\mu}$, notice that $e \in E(S)$, then $c^{n-1}(c^n)^{'}ec \in E(S)$, for $(c^n)^{'}$ in $V(c^n)$. Hence

$$\begin{aligned}\mu_{acc^{n-1}(c^n)'ec} &= \mu_{ac^n(c^n)'ec} = \mu_{aec^n(c^n)'c} = \mu_{ae} * \mu_{c^n(c^n)'c}\\ &= \mu_{be} * \mu_{c^n(c^n)'c} = \mu_{bec^n(c^n)'c} = \mu_{bc^n(c^n)'ec}\\ &= \mu_{bcc^{n-1}(c^n)'ec}.\end{aligned}$$

Thus there exists $c^{n-1}(c^n)^{'}ec \in E(S)$, such that $\mu(acc^{n-1}(c^n)^{'}ec, bcc^{n-1}(c^n)^{'}ec) = 1$, thus $(ac, bc) \in \tilde{\mu}$.

We now verify $\tilde{\mu}$ is a group congruence on S. Let $a \in S, e \in E(S)$, there exists $n \in N^+$ such that $a^n \in Reg(S)$, then $\mu_{aee} = \mu_{ae}$. Thus $\mu(aee, ae) = 1$, and so $(ae, a) \in \tilde{\mu}$. Notice that $c^{n-1}(c^n)^{'}ec \in E(S)$, for $(c^n)^{'}$ in $V(c^n)$, then

$$\mu_{ea(a^{n-1}(a^n)'ea)} = \mu_{a(a^{n-1}(a^n)'ea)},$$

and so $(a, ea) \in \tilde{\mu}$. Then

$$a\tilde{\mu}e\tilde{\mu} = a\tilde{\mu} = e\tilde{\mu}a\tilde{\mu}.$$

Thus $e\tilde{\mu}$ is identical of $S/\tilde{\mu}$ for every e in $E(S)$. It is clear that for all e, f in $E(S)$, we have $e\tilde{\mu} = f\tilde{\mu}$. Since $a^n(a^n)^{'}, a^{n-1}(a^n)^{'}a \in E(S)$, Then

$$a\tilde{\mu}(a^{n-1}(a^n)^{'})\tilde{\mu} = (a^{n-1}(a^n)^{'})\tilde{\mu}a\tilde{\mu} = e\tilde{\mu}.$$

Hence $(a^{n-1}(a^n)^{'})\tilde{\mu}$ is an inverse of $a\tilde{\mu}$. Thus $\tilde{\mu}$ is a group congruence on S. □

Theorem 3.3. *Let S be a strongly π-inverses semigroup, μ be a fuzzy congruence relation on S and $E(S)$ be the set of idempotents, $Reg(S)$ be the set of regular elements, then the following relations defined by:*

$\tilde{\mu}_1 = \{(a,b) \in S \times S | \exists x \in Reg(S), \mu(ax,bx) = 1\}$,
$\tilde{\mu}_2 = \{(a,b) \in S \times S | \forall e,f \in E(S), \exists g \in E(S), \mu(aeg,bfg) = 1\}$,
$\tilde{\mu}_3 = \{(a,b) \in S \times S | \forall e,f \in E(S), \exists x \in Reg(S), \mu(aex,bfx) = 1\}$,
$\tilde{\mu}_4 = \{(a,b) \in S \times S | \exists e \in E(S), \mu(ea,eb) = 1\}$,
$\tilde{\mu}_5 = \{(a,b) \in S \times S | \exists x \in Reg(S), \mu(xa,xb) = 1\}$,
$\tilde{\mu}_6 = \{(a,b) \in S \times S | \forall e,f \in E(S), \exists g \in E(S), \mu(gea,gfb) = 1\}$,
$\tilde{\mu}_7 = \{(a,b) \in S \times S | \forall e,f \in E(S), \exists x \in Reg(S), \mu(xea,xfb) = 1\}$,
$\tilde{\mu}_8 = \{(a,b) \in S \times S | \exists e \in E(S), \mu(eae,ebe) = 1\}$.

Then $\tilde{\mu}_1 = \tilde{\mu}_2 = \cdots = \tilde{\mu}_8 = \tilde{\mu}$. Namely, $\tilde{\mu}_1, \tilde{\mu}_2, \cdots, \tilde{\mu}_8$ are group congruences on S.

Proof. From Theorem 3.2 know that $\tilde{\mu} = \{(a,b) \in S \times S | \exists e \in E(S), \mu(ae,be) = 1\}$ is a group congruence on strongly π -inverses semigroup S. It is clear that $\tilde{\mu} \subseteq \tilde{\mu}_1$. Let $(a,b) \in \tilde{\mu}_1$, then there exists $x \in Reg(S)$ such that $\mu(ax,bx) = 1$, then $\mu_{ax} = \mu_{bx}$. Since $x^{'} \in V(x)$, we in fact have

$$\mu_{ax} * \mu_{x'} = \mu_{bx} * \mu_{x'},$$

then

$$\mu_{axx'} = \mu_{bxx'}.$$

Denoted the idempotents $xx^{'}$ by e, then $\mu(ae,be) = 1$. Hence $\tilde{\mu}_1 \subseteq \tilde{\mu}$. And so $\tilde{\mu}_1 = \tilde{\mu}$. Similarly we can easily verified $\tilde{\mu} = \tilde{\mu}_2 = \tilde{\mu}_3$ and $\tilde{\mu}_4 = \tilde{\mu}_5 = \tilde{\mu}_6 = \tilde{\mu}_7$.

We now verify $\tilde{\mu} = \tilde{\mu}_4 = \tilde{\mu}_8$. Let $(a,b) \in \tilde{\mu}$, then there exists $e \in E(S)$ such that $\mu(ae,be) = 1$, then $\mu_{ae} = \mu_{be}$. Let $a^n, b^m \in Reg(S), (a^n)^{'} \in V(a^n), (b^m)^{'} \in V(b^m)$, then

$$\mu_{a^n(a^n)'} * \mu_{b^m(b^m)'} * \mu_{ae} = \mu_{a^n(a^n)'} * \mu_{b^m(b^m)'} * \mu_{be}$$
$$\mu_{a^n(a^n)'b^m(b^m)'} * \mu_{ae} = \mu_{a^n(a^n)'b^m(b^m)'} * \mu_{be}$$
$$\mu_{b^m(b^m)'aa^{n-1}(a^n)'} * \mu_{ae} = \mu_{a^n(a^n)'b^m(b^m)'} * \mu_{be}$$
$$\mu_{b^m(b^m)'aa^{n-1}(a^n)'ae} = \mu_{a^n(a^n)'b^m(b^m)'be},$$

since $a^n(a^n)^{'}, b^m(b^m)^{'}, aea^{n-1}(a^n)^{'}, beb^{m-1}(b^m)^{'}$ are idempotents, left multiplied $\mu_{a^n(a^n)'} * \mu_{b^m(b^m)'} * \mu_{aea^{n-1}(a^n)'} * \mu_{beb^{m-1}(b^m)'}$ on each sides of the above equation. Notice that the idempotents commute, it is easy to deduce

$$\mu_{a^n(a^n)'b^m(b^m)'aea^{n-1}(a^n)'beb^{m-1}(b^m)'a} = \mu_{a^n(a^n)'b^m(b^m)'aea^{n-1}(a^n)'beb^{m-1}(b^m)'b}.$$

Hence $\tilde{\mu} \subseteq \tilde{\mu}_4$. Similarly $\tilde{\mu}_4 \subseteq \tilde{\mu}$, and so $\tilde{\mu} = \tilde{\mu}_4$.

Finally, we show that $\tilde{\mu}_4 = \tilde{\mu}_8$. It is clear that $\tilde{\mu}_4 \subseteq \tilde{\mu}_8$, the proof of $\tilde{\mu}_8 \subseteq \tilde{\mu}_4$ is similar to the front. And so $\tilde{\mu}_4 = \tilde{\mu}_8$.

Thus we verified $\tilde{\mu}_1 = \tilde{\mu}_2 = \cdots = \tilde{\mu}_8 = \tilde{\mu}$, namely, they are group congruence on S. □

Inverse semigroups are special strongly π-inverse semigroups. Hence, as a Corollary of Theorem 3.2 and Theorem 3.3, we have nine equivalent forms of the group congruence on inverse semigroups.

Corollary 3.4. *Let S be a inverses semigroup, μ be a fuzzy congruence relation on S and $E(S)$ be the set of idempotents, then the following relations defined by:*

$\tilde{\mu}_1 = \{(a,b) \in S \times S | \exists e \in E(S), \mu(ae, be) = 1\}$,
$\tilde{\mu}_2 = \{(a,b) \in S \times S | \exists x \in S, \mu(ax, bx) = 1\}$,
$\tilde{\mu}_3 = \{(a,b) \in S \times S | \forall e, f \in E(S), \exists g \in E(S), \mu(aeg, bfg) = 1\}$,
$\tilde{\mu}_4 = \{(a,b) \in S \times S | \forall e, f \in E(S), \exists x \in S, \mu(aex, bfx) = 1\}$,
$\tilde{\mu}_5 = \{(a,b) \in S \times S | \exists e \in E(S), \mu(ea, eb) = 1\}$,
$\tilde{\mu}_6 = \{(a,b) \in S \times S | \exists x \in S, \mu(xa, xb) = 1\}$,
$\tilde{\mu}_7 = \{(a,b) \in S \times S | \forall e, f \in E(S), \exists g \in E(S), \mu(gea, gfb) = 1\}$,
$\tilde{\mu}_8 = \{(a,b) \in S \times S | \forall e, f \in E(S), \exists x \in S, \mu(xea, xfb) = 1\}$,
$\tilde{\mu}_9 = \{(a,b) \in S \times S | \exists e \in E(S), \mu(eae, ebe) = 1\}$.

Then $\tilde{\mu}_1 = \tilde{\mu}_2 = \cdots = \tilde{\mu}_9$. Namely, $\tilde{\mu}_1, \tilde{\mu}_2, \cdots, \tilde{\mu}_9$ are group congruences on S.

Theorem 3.5. *Let S be a strongly π-inverses semigroup, μ be a fuzzy congruence relation on S, then μ is a fuzzy group congruence on S if and only if $\tilde{\mu} \subseteq \mu^{-1}$.*

Proof. On the one hand, $\tilde{\mu}$ is a group congruence on S, then $e\tilde{\mu} = f\tilde{\mu}$ for all e, f in $E(S)$, and so $(e,f) \subseteq \tilde{\mu}$. Since $\tilde{\mu} \subseteq \mu^{-1}$, we deduce that $(e,f) \subseteq \mu^{-1}$, and so $\mu(e,f) = 1$. From Lemma 2.4 know that $\mu_e = \mu_f$. On the other hand, since $\tilde{\mu}$ is a group congruence on S, we have $(ae, e) \subseteq \tilde{\mu}, (e, ea) \subseteq \tilde{\mu}$, for all e in $E(S)$ and a in S. Then $(ae, e) \subseteq \mu^{-1}, (e, ea) \subseteq \mu^{-1}$. By Lemma2.4, we deduce that

$$\mu_{ae} = \mu_e = \mu_{ea},$$

and so

$$\mu_a * \mu_e = \mu_a = \mu_e * \mu_a.$$

Hence, μ_e is identical of S/μ. Thus μ is a fuzzy group congruence on S.

Conversely, if $(a,b) \subseteq \tilde{\mu}$ for all a, b in S, then there exists e in $E(S)$ such that $\mu(ae, be) = 1$, and so $\mu_{ae} = \mu_{be}$. Since μ be a fuzzy group congruence on S, we deduce that

$$\mu_a = \mu_{ae} = \mu_{be} = \mu_b.$$

Hence, $\mu(a,b) = 1$, and so $(a,b) \subseteq \mu^{-1}$ as required. □

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