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On Strongly Semiprime Modules and Submodules

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Abstract : We provide the notion of strongly semiprime submodules of a given right R-module M and describe properties of them as a generalization of completely semiprime ideals in associative rings. We show that a proper fully invariant submodule of M is strongly prime if and only if it is prime and strongly semiprime.

Keywords : strongly prime submodules; strongly semiprime submodules; multiplicative systems and strongly prime radical; dr-submodules; Kaplansky-Cohen theorem.

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1 Introduction and Preliminaries

Throughout this paper, all rings are associative rings with identity and all modules are unitary right R-modules. Let R be a ring and M, a right R-module. Denote $S = \mathrm{End}_R(M)$, the endomorphism ring of the module M. A submodule X of M is called a fully invariant submodule if $f(X) \subset X$ for any $f \in S$. Especially, a right ideal of R is a fully invariant submodule of R_R if it is a two-sided ideal of R. The class of all fully invariant submodules of M is non-empty and closed under intersections and sums. A right R-module M is called a self-generator if it generates all its submodules. Following [1], a fully invariant proper submodule X of M is called a prime submodule of M if for any ideal I of $S = \mathrm{End}_R(M)$, and any fully invariant submodule U of M, if $I(U) \subset X$, then either $I(M) \subset X$ or $U \subset X$. A fully invariant submodule X of M is called a strongly prime submodule of M if for any $\phi \in S = \mathrm{End}_R(M)$ and $m \in M$, if $\phi(m) \in X$, then either $\phi(M) \subset X$ or $m \in X$. The basic Theorem 2.1 in [1] shows that the class of prime submodules of a given module has some properties similar to that of prime ideals in an associative ring. Following that theorem, a fully invariant proper submodule X of M is prime if and only if for any $\phi \in S$ and $m \in M$, $\phi Sm \subset X$ implies that $\phi(M) \subset X$ or $m \in X$. Using this property one can see that every strongly prime submodule is prime.

Definition 1.1. [2, Definition 2.1] A submodule of a right R-module M is said to have *insertion factor property* (briefly, an *IFP-submodule*) if for any endomorphism ϕ of M and any element $m \in M$, if $\phi(m) \in X$, then $\phi Sm \in X$. A right ideal I is an *IFP-right ideal* if it is an IFP-submodule of R_R, that is for any $a, b \in R$, if $ab \in I$, then $aRb \subseteq I$. A right R-module M is called an *IFP-module* if 0 is an IFP-submodule of M. A ring R is *IFP* if 0 is an IFP-ideal.

Definition 1.2. [1, Definition 2.1] A fully invariant submodule X of a right R-module M is called a *semiprime submodule* if it is an intersection of prime submodules of M. A right R-module M is called a *semiprime module* if 0 is a semiprime submodule of M. Consequently, the ring R is a *semiprime ring* if R_R is semiprime. By symmetry, the ring R is a semiprime ring if ${}_RR$ is a semiprime left R-module.

Proposition 1.3. [3, Proposition 2.3] *Let M be a right R-module which is a self-generator and X, a fully invariant submodule of M. Then X is a semiprime submodule if and only if whenever $f \in S$ with $fSf(M) \subset X$, then $f(M) \subset X$.*

In what follows, by $\mathbb{Z}, \mathbb{Q}, \mathbb{Z}_n$ and $\mathbb{Z}/n\mathbb{Z}$ we denote, respectively, integers, rational numbers, the ring of integers modulo n and the $\mathbb{Z}$-module of integers modulo n.

2 Completely Semiprime Modules and Submodules

In this section, we investigate the properties of completely semiprime submodules and modules by our definition. We now give the notion of a completely semiprime submodule.

Definition 2.1. A fully invariant proper submodule X of M is called *completely semiprime* if for any $\psi \in S$ and $m \in M$, $\psi^2(m) \in X$ implies $\psi S m \subseteq X$.

We provide a relationship between a completely semiprime submodule and semiprime submodule as follows.

Remark 2.2. *Every completely semiprime submodule is semiprime.*

Proof. Suppose that X is a completely semiprime submodule of M. It follows from Proposition 1.3 that X will be semiprime if we can prove that for every $f \in S = \text{End}_R(M)$, $fSf(M) \subseteq X$ implies $f(M) \subseteq X$. So, let $f \in S = \text{End}_R(M)$ such that $fSf(M) \subseteq X$. Therefore, $f^2(m) \in X$ for every $m \in M$. Since X is a completely semiprime submodule of M, we have $fSm \subseteq X$ for every $m \in M$. Hence, $f(M) \subseteq fS(M) \subseteq X$ and we are done. □

An R-module M is completely semiprime if the zero submodule of M is a completely semiprime submodule of M. In general, an R-module M/P is a completely semiprime module if and only if P is a completely semiprime prime submodule of M. To illustrate, we give an example of completely semiprime module.

Example 2.3. Let p be any prime integer and $M = (\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Q}$ a $\mathbb{Z}$-module. Then the endomorphism ring S of the module M is isomorphic to the matrix ring $\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a \in \mathbb{Z}_p, b \in \mathbb{Q} \right\}$. It is evident that M is a completely semiprime module.

We provide a characterization of completely semiprime submodules as follows.

Proposition 2.4. *Let M be a right R-module and $S = End_R(M)$ be a reduced ring, i.e., S has no nonzero nilpotent elements. Assume that for each element $m \in M$, there exists $g \in S$ such that $mR = gM$. Then M is completely semiprime.*

Proof. Let $f \in S$ and $m \in M$ such that $f^2(m) = 0$. From our assumption there exists $g \in S$ such that $mR = gM$. Hence $0 = f^2(mR) = f^2g(M)$. Hence $f^2g = 0$. Since S is reduced, we have $fg = 0$ and consequently $fSg = 0$. Thus $0 = fSg(M) = fS(mR)$ and we can see that $fS(m) = 0$. □

The following corollary is a direct consequence of Proposition 2.4.

Corollary 2.5. *A free R-module is completely semiprime if $S = End_R(M)$ is a reduced ring.*

Proof. Let F be a free R-module. Clearly for every $m \in F$ there exists $f \in S = \text{End}_R(M)$ such that $fF = Rm$. By Proposition 2.4, F is completely semiprime. □

Some characterizations of completely semiprime submodules are given in the following result.

Proposition 2.6. *Let M be a right R-module and $S = End_R(M)$. For a fully invariant proper submodule X of M, the followings are equivalent:*

1. *X is completely semiprime.*
2. *For all $a \in S$ and $m \in M$, if $am \in X$, then $Sm \cap aM \subseteq X$.*
3. *(a) For all $a \in S$ and $m \in M$ such that $am \in X$, we have $aSm \subseteq X$ and*

   (b) $a^2m \in X$ implies $am \in X$.

Proof. 1. ⇒ 2. Assume that $a^2m \in X$. Then $aSm \subseteq X$. Let $a \in S$ and $m \in M$ such that $am \in X$ and $x \in Sm \cap aM$. Now, $x = bm = am_1$, for $b \in S$ and $m_1 \in M$. Since $am \in X$ and X is a fully invariant submodule, $a^2m \in X$. From our assumption, we have $aSm \subseteq X$. From $am_1 \in Sm$, we can see that $a^2m_1 \in aSm \subseteq X$. Again, by our assumption, $aSm_1 \subseteq X$. It implies that $x = am_1 \in aSm_1 \subseteq X$. Hence, $Sm \cap aM \subseteq X$.

2. ⇒ 3. Let $a \in S$ and $m \in M$ such that $am \in X$. From (2), we have $aSm \subseteq Sm \cap aM \subseteq X$ and $(3a)$ is satisfied. Now, let $a \in S$ and $m \in M$ such that $a^2m = a(am) \in X$. From (2), we have $am \in S(am) \cap aM \subseteq Sm \cap aM \subseteq X$ and $(3b)$ is proved.

3. ⇒ 2. Let $a \in S$ and $m \in M$ such that $am \in X$. If $x \in Sm \cap aM$, then $x = bm = an$ for some $b \in S$ and $n \in M$. From $(3a)$, we have $abm \in aSm \subseteq X$. Since $a^2n = ax = abm \in X$, applying $(3b)$, we have $x = an \in X$. Hence, $Sm \cap aM \subseteq X$ and (2) is proved.

3. ⇒ 1. Let $a \in S$ and $m \in M$ such that $a^2m \in X$. From $(3b)$, we have $am \in X$ and from $(3a)$, we have $aSm \subseteq X$. □

We consider the relationship between completely semiprime and IFP submodules in the following lemma.

Lemma 2.7. *If a fully invariant submodule X of a right R-module M is completely semiprime, then*

1. *X is an IFP-submodule of M.*
2. *If $\alpha, \beta \in S$ and $m \in M$ such that $\alpha\beta(m) \in X$, then $\beta\alpha(m) \in X$.*

Proof. 1. Let $\alpha(m) \in X$. Since X is a fully invariant submodule, we have $\alpha^2(m) \in X$. Now, since X is a completely semiprime submodule of M, we have $\alpha S(m) \subseteq X$.

2. Let $\alpha, \beta \in S$ and $m \in M$ such that $\alpha\beta(m) \in X$. It implies that $(\beta\alpha\beta)^2(m) \in X$. Because X is completely semiprime, we have $\beta\alpha\beta S(m) \subseteq X$. Hence $(\beta\alpha)^2(m) \in X$ and again, since X is completely semiprime, we have $\beta\alpha S(m) \subseteq X$ and consequently $\beta\alpha(m) \in X$.

□

It is well known from [4] that a strongly prime submodule is prime. The following result shows that a strongly prime submodule is also completely semiprime.

Proposition 2.8. *If a fully invariant submodule X of M is a strongly prime submodule of M, then it is completely semiprime.*

Proof. Let $\psi \in S$ and $m \in M$ such that $\psi^2(m) \in X$. Since X is a strongly prime submodule of M, we have $\psi(M) \subseteq X$ or $\psi(m) \in X$. If $\psi(M) \subseteq X$, then $\psi S(M) = \psi(M) \subseteq X$ and we have $\psi S(m) \subseteq X$. If $\psi(m) \in X$, then since X is a strongly prime submodule of M, we have $\psi(M) \subseteq X$ or $m \in X$. We only have to show that if $m \in X$, then $\psi S(m) \subseteq X$. Now $g(m) \in X$ for every $g \in S$. Hence $S(m) \subseteq X$ and consequently $\psi S(m) \subseteq X$. □

As for a semiprime submodule in [1] we now define strongly semiprime submodule as follows:

Definition 2.9. A fully invariant submodule X of a right R-module M is called a *strongly semiprime submodule* if it is an intersection of strongly prime submodules of M. A right R-module M is called a strongly semiprime module if 0 is a strongly semiprime submodule of M.

Similar to a prime submodule, the following shows that if X is a strongly semiprime submodule, then I_X is a completely semiprime ideal of S. The converse is aslo considered.

Proposition 2.10. *Let M be a right R-module.*

1. *If X is a strongly semiprime submodule of M, then I_X is a completely semiprime ideal of S.*
2. *If M is a self-generator and P is a completely semiprime ideal of S, then $X := P(M)$ is a strongly semiprime submodule of M and $I_X = P$.*

Proof.
1. Since X is a strongly semiprime submodule of M, we can write $X = \bigcap_{P \in \mathcal{F}} P$, where each P is a strongly prime submodule of M.
So $I_X = I_{\bigcap_{P \subseteq M, P \in \mathcal{F}} P} = \bigcap_{P \subseteq M, P \in \mathcal{F}} I_P$. By [4, Theorem 2.13] it is easy to see that I_X is a completely semiprime ideal of S.
2. Since M is a self-generator, we can write $P = I_{P(M)} = I_X$, which is a completely semiprime ideal of S. Hence
$I_X = \bigcap_{K \subset S, K \text{ completely prime}} K = \text{Hom}(M, (\bigcap_{K \subset S, K \text{ completely prime}} K)(M))$. Let

$X = P(M)$, where P is a completely semiprime ideal of S. Since M is a self-generator, we have $P = I_{P(M)} = I_X$ and by our assumption $P = \bigcap_{K \in \Lambda} K$, for some set Λ of completely prime ideals of S. Thus $I_X = \mathrm{Hom}(M, I_X(M))$ $= \mathrm{Hom}(M, \bigcap_{K\in\Lambda} K(M))$. Thus $(\bigcap_{K\in\Lambda} K)(M) = \bigcap_{K\in\Lambda} K(M)$, and therefore $X = \bigcap_{K\in\Lambda} K(M)$. Since K is a completely prime ideal of S, $K(M)$ is a strongly prime submodule of M, proving that X is a strongly semiprime submodule of M.

□

The following result provides an important property of completely semiprime submodules.

Proposition 2.11. *Let M be a right R-module which is a self-generator and X a fully invariant submodule of M. Then X is a completely semiprime submodule of M if and only if it is strongly semiprime submodule of M.*

Proof. Let X be a completely semiprime submodule of M. We prove that I_X is a completely semiprime ideal of S. Let $a \in S$ such that $a^2 \in I_X$. Hence $a^2(M) \subseteq X$. Now for all $m \in M$ we have $a^2m \in X$. Since X is a completely semiprime submodule of M, we have $aSm \subseteq M$. Hence $aM = aSM \subseteq X$ and $a \in I_X$. Thus I_X is a completely semiprime ideal of S. From Proposition 2.10, X is a strongly semiprime submodule of M.

For the converse, suppose $X = \bigcap_{P\subseteq M, P\in\Lambda} P$, where Λ is a family of strongly prime submodules of M. Let $f \in S$ and $m \in M$ such that

$f^2m \in X = \bigcap_{P\subseteq M, P\in\Lambda} P$. Hence $f^2m \in P$ for every $P \in \Lambda$. Since P is strongly prime and $f(m) \in M$, we have $f(M) \subseteq P$ or $f(m) \in P$ for every $P \in \Lambda$. If $f(M) \subseteq P$ for every $P \in \Lambda$, then $fS(m) \subseteq P$ for every $P \in \Lambda$. Hence $fS(m) \subseteq X$. If $f(m) \in P$ for every $P \in \Lambda$, then since every P is strongly prime, we have $f(M) \subseteq P$ or $m \in P$. Now $m \in P$ implies that $g(m) \in P$ for all $g \in S$. Hence $S(m) \subseteq P$ and consequently $fS(m) \subseteq P$ for every $P \in \Lambda$. Thus $fS(m) \subseteq X$ and we are done. □

We consider the quotient module of a completely semiprime submodule as follows.

Lemma 2.12. *Let X be a submodule of a right R-module M. If X is a completely semiprime submodule and M is quasi-projective, then M/X is a completely semiprime module. Conversely, if M/X is completely semiprime and X is fully invariant, then X is a completely semiprime submodule of M.*

Proof. Suppose X is a completely semiprime submodule of M and $\overline{\phi}^2(\overline{m}) = \overline{0}$ where $\overline{\phi} \in \overline{S} = \mathrm{End}_R(M/X)$ and $\overline{m} \in M/X$. By the quasi-projectivity of M, there is a $\phi \in S$ such that $v\phi = \overline{\phi}v$, where $v : M \to M/X$ is the natural epimorphism. It follows that $\phi^2(m) \in X$. Let $\overline{\eta}$ be any element of $\overline{S}$. As above there is a $\eta \in S$ such that $v\eta = \overline{\eta}v$. Since X is completely semiprime $\phi S(m) \subseteq X$. Hence $v\phi\eta(m) = 0$ and thus $\overline{\phi}\overline{\eta}(\overline{m}) = \overline{\phi}\overline{\eta}v(m) = \overline{\phi}v\eta(m) = v\phi\eta(m) = 0$. Thus $\overline{\phi S}(\overline{m}) = \overline{0}$ and consequently M/X is a completely semiprime module. For the converse, suppose X is a fully invariant submodule of M with M/X completely semiprime. Let $\phi^2(m) \in X$ with $\phi \in S$ and $m \in M$. Since M is quasi-projective, there is a $\overline{\phi} \in \overline{S}$ such that $v\phi = \overline{\phi}v$. Now $\phi^2(m) \in X$ implies $v\phi^2(m) = \overline{0}$. Hence $\overline{\phi}^2(\overline{m}) = \overline{\phi}^2 v(m) = v\phi^2(m) = \overline{0}$. Let $\eta \in S$. Using the fact that M is quasi-projective, there is a $\overline{\mu} \in \overline{S}$ such that $v\eta = \overline{\mu}v$. Since M/X is completely semiprime, we have that $\overline{\phi S}(\overline{m}) = \overline{0}$. Now $\overline{\phi}\overline{\eta}(\overline{m}) = \overline{\phi}\overline{\eta}v(m) = \overline{\phi}v\eta(m) = v\phi\eta(m) = \overline{0}$. Hence $\phi\eta(m) \in X$ for every $\eta \in S$. Thus $\phi S(m) \subseteq X$ and therefore X is a completely semiprime submodule of M. □

The following theorem can be considered as a generalization of [2, Proposition 2.4].

Theorem 2.13. *Let X be a fully invariant submodule of a right R-module M. X is strongly prime if and only if it is prime and completely semiprime.*

Proof. From [4] and Proposition 2.8, if X is strongly prime, then X is prime and completely semiprime.

For the converse, let X be prime and completely semiprime. From Lemma 2.7, X has IFP and from [2, Proposition 2.4] X is strongly prime. □

We give the relationship between a completely semiprime submodule and its endomorphism ring.

Proposition 2.14. *Let M be a right R-module and $S =End_R(M)$. If M is completely semiprime, then S is reduced. The converse is true if M is a self generator.*

Proof. Let $\phi^2 = 0 \in S$. Then $\phi^2(m) = 0$ for all $m \in M$. If M is completely semiprime, then $\phi S(m) = 0$ for all $m \in M$. Hence $\phi S(M) = \phi(M) = 0$. Thus $\phi = 0$ and S is reduced.

Conversely, since $I_0 = \{f \in S | f(M) = 0 \subset M\} = 0$ is a completely semiprime ideal, it follows from Proposition 2.10 that 0 is a completely semiprime submodule. Hence, M is completely semiprime. □

We provide a method to examine when an R-module is completely semiprime.

Proposition 2.15. *Let M be a right R-module and $S = End_R(M)$. If M is quasi-projective, then the followings are equivalent:*

1. *M is completely semiprime.*
2. *For any $\phi \in S$, $ker(\phi)$ is a completely semiprime submodule of M.*
3. *$M/ker(I)$ is a completely semiprime module for any subset I of S.*

Proof. 1. $\Rightarrow$ 2. Let $\psi^2(m) \in \ker(\phi)$. If ϕ is any element from S, then $\phi\psi^2(m) = 0$. From Lemma 2.7, we have $\psi\phi\psi(m) = 0$. Hence $\phi\psi\phi\psi(m) = 0$ and since M is completely semiprime, we have $\phi\psi S(m) = 0$. Thus $\psi S(m) \subseteq \ker(\phi)$ and therefore $\ker(\phi)$ is a completely semiprime submodule of M.

2. $\Rightarrow$ 3. We note that $\ker(I) = \bigcap_{f\in I} \ker(f)$ and each $\ker(f)$ is a completely semiprime submodule of M. Hence, $\ker(I)$ is a completely semiprime submodule of M. Since M is quasi-projective, by applying Lemma 2.12, we have that $M/\ker(I)$ is a completely semiprime module.

3. $\Rightarrow$ 1. This part is clear by taking $I = \{1_M\}$, 1_M is the identity map of M. □

By Proposition 2.15, we can check that M is completely semiprime if and only if for any $\phi \in S$, $\ker(\phi)$ is a completely semiprime submodule of M. The following result provides another method to check when M is completely semiprime.

Theorem 2.16. *Let M be a right R-module which is a self-generator and $S = End_R(M)$. The followings are equivalent:*

1. *M is completely semiprime.*
2. *For any $m \in M$, $l_S(m)$ is a completely semiprime ideal of S.*
3. *$S/l_S(A)$ is a reduced ring for any subset $A \subset M$.*

Proof. 1. $\Rightarrow$ 2. Let $\alpha^2 \in l_S(m)$. Hence $\alpha^2(m) = 0$. Since M is completely semiprime, we have $\alpha S(m) = 0$. Thus $\alpha(m) = 0$ and $\alpha \in l_S(m)$.

2. $\Rightarrow$ 3. $l_S(A) = \bigcap_{\alpha\in A} l_S(a)$. Since each $l_S(a)$ is a completely semiprime ideal of S, $l_S(A)$ is a completely semiprime ideal of S. Hence, $S/l_S(A)$ is a reduced ring.

3. $\Rightarrow$ 1. Taking $A = M$, then it is clear that S is a reduced ring. Since M is a self-generator, by applying Theorem 2.13, we can see that M is a strongly semiprime module. □

3 Completely Prime Radical

We start this section by the following lemma.

Lemma 3.1. *Let M be a quasi-projective module and A a fully invariant submodule of M. If $\overline{P} \subset M/A$ is a strongly prime submodule of M/A, then $v^{-1}(\overline{P})$ is a strongly prime submodule of M*

Proof. Put $\overline{M} = M/P(M)$. Let $P = v^{-1}(\overline{P})$. Suppose $f \in S$ and $m \in M$ such that $f(m) \in P$. Since M is quasi-projective, there is $\overline{f} \in \overline{S}$ such that $\overline{f}v = vf$, where $v : M \to M/A$ is the canonical projection. From $f(m) \in P$, we have $vf(m) \in v(P) = \overline{P}$ or $\overline{f}v(m) \in \overline{P}$. From our assumption $\overline{f}(\overline{M}) \subseteq \overline{P}$ or $v(m) \in \overline{P}$. If $\overline{f}(\overline{M}) \subseteq \overline{P}$, we have $\overline{f}v(M) \subseteq \overline{P}$ or $vf(M) \subset \overline{P}$, that is $f(M) \subset P$. If $v(m) \in \overline{P}$, then $m \in P$. Hence, P is strongly prime. □

Lemma 3.2. *Let M be a quasi-projective module and P a strongly prime submodule of M. If $A \subset P$ is a fully invariant submodule of M, then P/A is a strongly prime submodule of M/A.*

Proof. Let $\overline{S}$ =End$_R(M/A)$ and let $\overline{f} \in \overline{S}$ and $\overline{f}(m + A) \in M/A$ such that $\overline{f}(m + A) \in P/A$. Since M is quasi-projective, we can find $f \in S$ such that $\overline{f}v$ $= vf$, where $v : M \to M/A$ is the canonical projection. Then $vf(m) = \overline{f}v(m) = \overline{f}(m + A) \in P/A$. Hence $f(m) \in P$. Now, since P is a strongly prime submodule of M, we have $f(M) \subseteq P$ or $m \in P$. It implies that $(f(M) + A)/A \subset P/A$ or $(m+A) \in P/A$. Thus $\overline{f}(M/A) \subset P/A$ or $(m+A) \in P/A$. Hence P/A is a strongly prime submodule of M/A. □

For a right R-module M, let $\mathcal{C}(M)$ be the intersection of all strongly prime prime submodules of M. By our definition, M is a strongly semiprime module if $\mathcal{C}(M) = 0$. We want to get some properties similar to that of completely prime radicals of rings and as first step, the following theorem is true for quasi-projective modules.

Theorem 3.3. *Let M be a quasi projective module. Then $M/\mathcal{C}(M)$ is a strongly semiprime module, that is $\mathcal{C}(M/\mathcal{C}(M)) = 0$.*

Proof. Put $\overline{M} = M/\mathcal{C}(M)$. By Lemma 3.1 and Lemma 3.2, we have
$\mathcal{C}(\overline{M}) = \bigcap_{\overline{X} \subset \overline{M}\ ,\ \overline{X} \text{ is strongly prime}} \overline{X} = \bigcap_{X \subset M, X \text{ is strongly prime}} X/\mathcal{C}(M)$
$= (\bigcap_{X \subset M, X \text{ is strongly prime}} X)/\mathcal{C}(M) = \mathcal{C}(M)/\mathcal{C}(M) = 0$. This shows that $M/\mathcal{C}(M)$ is a strongly semiprime module. □

4 Multiplicative Systems

The following proposition offers several other characterizations of strongly prime submodules.

Proposition 4.1. *Let M be a right R module and S =End$_R(M)$. For a proper fully invariant submodule P of M, the following are equivalent:*

1. *P is a strongly prime submodule of M.*
2. *For all $a \in S$ and every $m \in M$, if $\langle am\rangle \subseteq P$ then either $\langle m\rangle \subseteq P$ or $aM \subseteq P$.*

Proof. 1. $\Rightarrow$ 2. Let $a \in S$ and $m \in M$ such that $\langle am\rangle \subseteq P$. Since $am \in P$, it follows from 1. that $m \in P$ or $aM \subseteq P$, i.e. $\langle m\rangle \subseteq P$ or $aM \subseteq P$.

2. $\Rightarrow$ 1. Let $a \in S$ and $m \in M$ such that $am \in P$. Now $\langle am\rangle \subseteq P$ and it follows from 2. that $\langle m\rangle \subseteq P$ or $aM \subseteq P$. Hence $m \in \langle m\rangle \subseteq P$ or $aM \subseteq P$ and we are done. □

The notion of multiplicative systems of rings is generalized to modules as follows.

Definition 4.2. Let M_R be a module and $S = \text{End}_R(M)$. A nonempty set $X \subseteq M\backslash\{0\}$ is called a *multiplicative system* of M_R if for each $a \in S$, $m \in M$ and for all submodules K of M such that $(K + \langle m\rangle) \cap X \neq \phi$ and $(K + \langle am\rangle) \cap X \neq \phi$, then $(K + \langle aM\rangle) \cap X \neq \phi$.

Using multiplicative systems, we can check when a proper fully invariant submodule is strongly prime.

Lemma 4.3. *Let M be a right R module and $S = End_R(M)$. A proper fully invariant submodule P of M is strongly prime if and only if $X = M\backslash P$ is a multiplicative system of M.*

Proof. $\Rightarrow$ Put $X = M\backslash P$. Let $a \in R$ and $m \in M$. If K is a submodule of M, then $(K + \langle m\rangle) \cap X \neq \phi$ and $(K + \langle aM\rangle) \cap X \neq \phi$. If $(K + \langle am\rangle) \cap X = \phi$, then $\langle am\rangle \subseteq P$. since P is strongly prime, we have either $\langle m\rangle \subseteq P$ or $aM \subseteq P$. Thus, $(K + \langle m\rangle) \cap X = \phi$ or $(K + \langle aM\rangle) \cap X = \phi$, a contradiction.

$\Leftarrow$ Let $a \in S$ and $m \in M$ such that $\langle am\rangle \subseteq P$ but $\langle m\rangle \nsubseteq P$ and $aM \nsubseteq P$. Then, $\langle m\rangle) \cap X \neq \phi$ and $aM \cap X \neq \phi$. By the definition of a multiplicative system, $\langle am\rangle \cap X \neq \phi$ such that $\langle am\rangle \nsubseteq P$, a contradiction. □

The following is a property of strongly prime submodules.

Lemma 4.4. *Let M be an R-module, $X \subseteq M$ a multiplicative system of M and P a fully invariant submodule of M maximal with respect to the property that $P \cap X = \phi$. Then P is a strongly prime submodule of M.*

Proof. Suppose $a \in S$ and $m \in M$ such that $\langle am\rangle \subseteq P$. If $\langle m\rangle \nsubseteq P$ and $aM \nsubseteq P$ then $(P + \langle m\rangle) \cap X \neq \phi$ and $(P + aM) \cap X \neq \phi$. Since X is a multiplicative system of M, $(\langle am\rangle + P) \cap X \neq \phi$. Since $\langle am\rangle \subseteq P$, we have $P \cap X \neq \phi$, a contradiction. Hence, P must be a strongly prime submodule. □

We give the definition of $st(N)$, where $st(N)$ is the intersection of all strongly prime submodules containing N.

Definition 4.5. Let R be a ring and M an R-module. For a fully invariant submodule N of M, if there is a strongly prime submodule containing N, we define $st(N) := \{m \in M :$ every multiplicative system containing m meets $N\}$. We write $st(N) = M$ when there are no strongly prime submodules of M containing N.

Using definition above, we have the following result.

Theorem 4.6. *Let M be a right R-module and N a fully invariant submodule of M. Then, either $st(N) = M$ or $st(N)$ equals the intersection of all strongly prime submodules containing N, which is denoted by $\beta_{st}(N)$.*

Proof. Suppose $st(N) \neq M$. Then, $\beta_{st}(N) \neq \phi$. Both $st(N)$ and N are contained in the same strongly prime submodules. By definition of $st(N)$, it is clear that $N \subseteq st(N)$. Hence, any strongly prime submodule of M which contains $st(N)$ must necessarily contain N. Suppose P is a strongly prime submodule of M such that $N \subseteq P$, and let $t \in st(N)$. If $t \notin P$, then the complement of P, $C(P)$ in M is a multiplicative system containing t and therefore we would have $C(P) \cap N \neq \phi$. However, since $N \subseteq P$, $C(P) \cap P = \phi$ and this contradiction shows that $t \in P$. Hence $st(N) \subseteq P$ as we wished to show. Thus, $st(N) \subseteq \beta_{st}(N)$. Conversely, assume $s \notin st(N)$, then there exists a multiplicative system X such that $s \in X$ and $X \cap N = \phi$. From Zorn's Lemma, there exists a fully invariant submodule $P \supseteq N$ which is maximal with respect to $P \cap X = \phi$. From Lemma 4.4, P is a strongly prime submodule of M and $s \notin P$. Therefore, we have $st(N) = \beta_{st}(N)$. □

Let I be an ideal of a ring R. Recall from [5, Theorem 3] that, if there exists a completely prime ideal of R containing I, then we define $\mathcal{N}(I)$ is the intersection of all completely prime ideals of R containing I. If there is no completely prime ideal containing I, we put $\mathcal{N}(I) = R$.

Proposition 4.7. *Let M be a right R-module and N a fully invariant submodule of M. Then $\mathcal{N}(I_N)(M) \subseteq st(N)$.*

Proof. If $st(N) = M$, then the result is immediate. Otherwise, if T is any strongly prime submodule of M that contains N, then I_T is a completely prime ideal of S and $I_T \supset I_N$. Thus $\mathcal{N}(I_N) \subseteq I_T$ and hence $\mathcal{N}(I_N)(M) \subseteq I_T(M) \subseteq T$. Since T is an arbitrary strongly prime submodule of M containing N, we have $\mathcal{N}(I_N)(M) \subseteq st(N)$. □

Applying Proposition 4.7, we have the following proposition.

Proposition 4.8. *Let M be a quasi-projective finitely generated right R-module which is a self-generator. Let N be a fully invariant submodule of M. Then $\mathcal{N}(I_N)(M) = st(N)$.*

Proof. By Proposition 4.7, we have $\mathcal{N}(I_N)(M) \subseteq st(N)$. Now, we write $st(N) = I_{st(N)}(M)$ and we will show that $I_{st(N)} \subseteq \mathcal{N}(I_N)$. Let P be a completely prime ideal of S such that $I_N \subseteq P$. Then PM is a strongly prime submodule of M and $PM \supset I_N(M) = N$. Hence $PM \supset st(N)$. Since $I_{st(N)} = \mathrm{Hom}(M, st(N)) \subseteq \mathrm{Hom}(M, PM) = P$, we have $I_{st(N)} \subseteq \mathcal{N}(I_N)$. It follows that $st(N) \subseteq \mathcal{N}(I_N)(M)$. □

5 Dr Prime Submodules

In a more recent paper, S. I. Bilavska and B. V. Zabavsky studied dr-prime right ideals of rings and they suggested a version of Kaplansky-Cohen's theorem for noncommutative rings (see [6] for more details). Many authors studied Cohen's theorem and Kaplansky-Cohen's theorem for noncommutative rings. Some of them also extended these results for modules. For example, in [6], S. I. Bilavska and B. V. Zabavsky gave a noncommutative version of the Kaplansky-Cohen theorem. Following them, a right ideal P of R is called a dr-prime right ideal if $P \subseteq cR$, where c is a duo element and for any $p \in P$ the condition $p = cx$ implies $x \in P$. It is easy to verify that any maximal right ideal J of a ring R is a dr-prime right ideal. This result is introduced in [6].

Theorem 5.1. [6, Theorem 2] *If every dr-prime right ideal of a ring R is principal, then every right ideal of R is principal.*

We now give the defnition of dr-prime submodules as an extension of dr-prime right ideals for rings.

Definition 5.2. A submodule X of M is called a *dr-prime submodule* if $X \subseteq \varphi S(M)$ where φ is a duo-element of S and if $\eta \in S$ such that $\varphi\eta(M) \subseteq X$ then $\eta(M) \subseteq X$.

It is easy to see that if $M = R$, then the defnition of dr-prime right ideals and dr-prime submodules coincide.

Proposition 5.3. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Then X is a dr-prime submodule if and only if I_X is a dr-prime right ideal of S.*

Proof. Suppose that X is a dr-prime submodule of M and $I_X \subseteq \eta S$ where $\eta \in S$ is a duo-element of S. We have $X \subseteq \eta S(M)$. If $\rho \in S$ such that $\eta\rho \in I_X$, then $\eta\rho(M) \subseteq X$. Since is X is a dr-prime submodule of M, we have $\rho(M) \subseteq X$ i.e. $\rho \in I_X$.

For the converse, assume that I_X is a dr-prime right ideal of S and $X \subseteq \varphi S(M)$, where φ is a duo-element of S. Hence $I_X \subseteq \varphi S$. If $\eta \in S$ such that $\varphi\eta(M) \subseteq X$, then $\varphi\eta \in I_X$. Since I_X is a dr-prime right ideal of S, we have $\eta \in I_X$ i.e. $\eta(M) \subseteq X$. □

Recall that a module N is said to be M-generated if there is an epimorphism $M^{(I)} \longrightarrow N$ for some index set I. If I is finite, then N is called a finitely M-generated module. In particular, a module N is called M-cyclic if there is an epimorphism from $M \longrightarrow N$.

Proposition 5.4. *Let M be a quasi-projective module and X an M-cyclic submodule of M. Then I_X is a principal right ideal of S.*

Proof. Since M is M-cyclic, there exists an epimorphism $\varphi : M \longrightarrow X$ such that $X = \varphi(M)$. It follows that $\varphi S \subset I_X$. By the quasi-projectivity of M, for any $f \in I_X$, we can find a $\alpha \in S$ such that $f = \varphi\alpha$, proving that $I_X = \varphi S$, as required. □

We finish this section by providing another version of the Kaplansky-Cohen theorem for modules. To do that, we give the following proposition.

Proposition 5.5. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If every dr-prime submodule of M is M-cyclic, then S is a right principal ideal ring.*

Proof. Assume that S is not a right principal ideal ring. Then there exists a non-principal right ideal I of S. From [6, Corollary 6], I is contained in a maximal non-principal right ideal N and from [6, Proposition 2], N is a dr prime right ideal of S. Let $X = N(M)$. Since M is a quasi-projective finitely generated right R-module, we have $N = I_X$. Now, since $I_{X.}$ is a dr-prime right ideal, we have X is a dr prime submodule. Since $N = I_X$ is a non-principal right ideal, it follows from [7, Lemma 2.3] that X is a dr-prime submodule of M which is not $M-$cyclic. This is a contradiction. Hence, S is a principal right ideal ring. □

We now have the following theorem, that can be considered as a new version of the Kaplansky-Cohen theorem for modules.

Theorem 5.6. *Let M be a quasi-projective finitely generated right R-module which is a self-generator. If every dr-prime submodule of M is M-cyclic, then every submodule of M is M-cyclic.*

Proof. Using Proposition 5.5, we see that S is a right principal ideal ring. Assume that X is a submodule of M. Then we have $I_X(M) = X$. Hence, X is M-cyclic, proving that every submodule of M is M-cyclic. □

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