Thai Journal of Mathematics
Volume 16 (2018) Number 2 : 529–539
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Colorability of Unitary Endo-Cayley Graphs of Cyclic Groups

Chanon Promsakon

Department of Mathematics, Faculty of Applied Science
King Mongkut's University of Technology North Bangkok
Bangkok, 10800, Thailand
Centre of Excellence in Mathematics, CHE Si Ayuttaya Rd.
Bangkok 10400, Thailand
e-mail : chanon.p@sci.kmutnb.ac.th

Abstract : Let n be a positive integer greater than 1, $\mathbb{Z}_n$ the ring of integer modulo n, f an endomorphism on $\mathbb{Z}_n$ and U_n the set of all units in $\mathbb{Z}_n$. **The unitary endo-Cayley digraph**, denoted by $endo-Cay_f(\mathbb{Z}_n, U_n)$, is the digraph whose vertex set is $\mathbb{Z}_n$ and a vertex u is adjacent to v if $v = f(u) + a$ for some $a \in U_n$.

We find conditions for endomorphism f to make sure that $endo-Cay_f(\mathbb{Z}_n, U_n)$ is undirected graph. After that we study about their coloring to investigate bounds of their chromatic numbers. Moreover, we give examples to show the sharpness of each bounds.

Keywords : endo-Cayley graphs; undirected graphs; chromatic numbers.
2010 Mathematics Subject Classification : 46MO5; 05C40; 05C76.

---

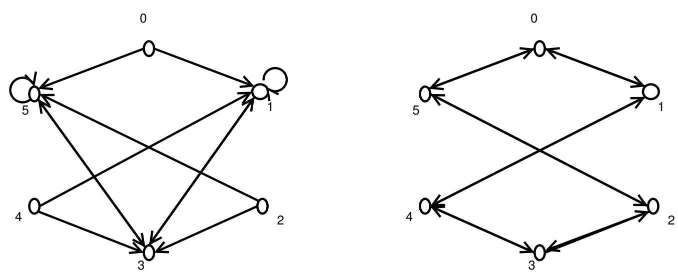
1 Introduction

Let n be a positive integer greater than 1, $\mathbb{Z}_n$ the ring of integer modulo n, f an endomorphism on $\mathbb{Z}_n$ and U_n the set of all units in $\mathbb{Z}_n$. **The unitary**

Thanks! This research was funded by King Mongkut's University of Technology North Bangkok. Contract no. KMUTNB-GOV-58-64.

endo-Cayley digraph, denoted by $endo-Cay_f(\mathbb{Z}_n, U_n)$, is the digraph whose vertex set is $\mathbb{Z}_n$ and a vertex u is adjacent to v if $v = f(u) + a$ for some $a \in U_n$. Sometime we call U_n as a connecting set of $endo-Cay_f(\mathbb{Z}_n, U_n)$. In case that an endomorphism f is an identity map, we have that unitary endo-Cayley digraph is a **unitary Cayley digraph** denoted as $Cay(\mathbb{Z}_n, U_n)$. The properties and structure of unitary Cayley graph have been studied in [1, 2, 3].

To illustrate, let us consider an unitary endo-Cayley digraph of $\mathbb{Z}_6$. Then $U_6 = \{1, 5\}$. Let f and g be endomorphisms on $\mathbb{Z}_6$ defined as $f(x) = 2x$, $g(x) = 5x$ for all $x \in \mathbb{Z}_6$. The resulting digraphs $endo-Cay_f(\mathbb{Z}_6, U_6)$ and $endo-Cay_g(\mathbb{Z}_6, U_6)$ are shown below.



$endo-Cay_f(\mathbb{Z}_6, U_6)$ $endo-Cay_g(\mathbb{Z}_6, U_6)$

Figure 1: Endo-Cayley digraphs of $\mathbb{Z}_6$ with different endomorphisms

We call a digraph D as a undirected graph if arc $xy \in A(D)$ if and only if $yx \in A(D)$. So $endo-Cay_f(\mathbb{Z}_6, U_6)$ is a digraph while $endo-Cay_g(\mathbb{Z}_6, U_6)$ is undirected.

In 2014, C. Promsakon and S. Panma study undirectedness of endo-Cayley digraph of $\mathbb{Z}_n$ for any endomorphisms and connecting sets. They investigate a condition for undirected endo-Cayley digraph of cyclic group of order prime number [4]. That condition involve to an endomorphism and a connecting set showed as follow.

Theorem 1.1 ([4]). *Let p be a prime number, A a subset of $\mathbb{Z}_n$ and f an endomorphism on $\mathbb{Z}_n$. Then $endo-Cay_f(\mathbb{Z}_p, A)$ is undirected if and only if at least one condition below holds*

1. *$f(x) = x$ and $A = A^{-1}$ or*
2. *$f(x) = x^{-1}$ or*
3. *$A = \mathbb{Z}_p$.*

Moreover, they also gave sufficient conditions for undirected endo-Cayley digraph of cyclic group. We restate the results here without proof.

Theorem 1.2 ([4]). *Let m and n be integers and A a subset of $\mathbb{Z}_n$. If $-mA \subseteq A$ and $xm^2 \equiv x \pmod n$ for all $x \in \mathbb{Z}_n$, then $endo-Cay_f(\mathbb{Z}_n, A)$ is undirected where $f(x) = mx$, $1 < m < n-1$ for all $x \in \mathbb{Z}_n$.*

Next, in 2015, C. Promsakon extended his work to study undirected unitary endo-Cayley digraph of $\mathbb{Z}_{p^k}$ where p is a prime number and k is a non-negative integer. We show some Theorems related to this work here. The proof of each Theorems are showed in [5].

We begin by giving a necessary condition for undirected endo-Cayley graph of $\mathbb{Z}_n$ for any $n \in \mathbb{N}$.

Lemma 1.3 (A necessary condition, [5]). *Let n be a positive integer such that $n > 1$, m a positive integer less than n, U_n a set of all units in $\mathbb{Z}_n$ and f an endomorphism on $\mathbb{Z}_n$ defined by $f(x) = mx$ for all $x \in \mathbb{Z}_n$. If $endo-Cay_f(\mathbb{Z}_n, U_n)$ is undirected, then $m \in U_n$.*

When n is 2^k where $k \in \mathbb{N}$, we have a converse a Lemma 1.3 is true.

Corollary 1.4 ([5]). *Let k be a positive integer, m a positive integer less than 2^k, U_{2^k} a set of all units in $\mathbb{Z}_{2^k}$ and f an endomorphism on $\mathbb{Z}_{2^k}$ defined by $f(x) = mx$ for all $x \in \mathbb{Z}_{2^k}$. Then $endo-Cay_f(\mathbb{Z}_{2^k}, U_{2^k})$ is undirected if and only if $m \in U_{2^k}$.*

A condition for endomorphism f to make $endo - Cay_f(\mathbb{Z}_{p^k}, U_{p^k})$ undirected is showed in the next theorm.

Theorem 1.5 ([5]). *Let p be a prime number, k a positive integer, m a positive integer less that p^k, U_{p^k} a set of all units in $\mathbb{Z}_{p^k}$, f an endomorphism on $\mathbb{Z}_{p^k}$ defined by $f(x) = mx$ for all $x \in \mathbb{Z}_{p^k}$. Then $endo - Cay_f(\mathbb{Z}_{p^k}, U_{p^k})$ is undirected if and only if $m^2 \equiv 1 \pmod p$.*

We end this section by giving some definitions and properties about the chromatic number of graphs. The proof in each theorems can found in [6].

A **k-coloring** of a graph G is a labeling of vertices in G, $f : V(G) \to \{1, 2, \ldots, k\}$. We call the image of a coloring as **colors**. A k-coloring is called **proper** if all adjacent vertices in G have different labels. A graph is called **k-colorable** if it has a proper k-coloring. We always use a coloring for a proper coloring. **The chromatic number** of a graph G, denoted by $\chi(G)$, is the least k such that G is k-colorable.

Theorem 1.6 ([6]). *A graph G is a bipartite graph if and only if $\chi(G) = 2$.*

A **clique** of a graph G is a complete subgraph of G. The **clique number** of a graph G, denoted by $\omega(G)$, is the maximum order of cliques of G. Clearly that for each vertices in a complete graph, they are adjacent. So they have different labels in a coloring. Hence $\chi(K_n) = n = \omega(K_n)$.

Proposition 1.7 ([6]). *For any graph G, $\chi(G) \geq \omega(G)$.*

Theorem 1.8 (Seinsehe)**.** *If a graph G has no induced subgraph isomorphic to P_4, then $\chi(G) = \omega(G)$.*

The **greedy algorithm** is an algorithm to find the chromatic number of any graphs. The greedy algorithm takes an order of vertices $v_1, v_2, \ldots, v_n$ and colors vertices in that order, assigning to each vertex the smallest indexed color not already use by it previously colored neighbors. By the greedy algorithm, we have a bound of the chromatic number of a graph.

Theorem 1.9 ([6])**.** *For any graph G, $\chi(G) \leq \Delta(G) + 1$.*

Clearly that an equality in Theorem 1.9 holds in two cases which are odd cycles and complete graphs. Next, R. Leonard Brooks proved that these are the only connected graphs for which the bound is attained.

Theorem 1.10 (Brooks's Theorem)**.** *For every connected graph G that is not an odd cycle or a complete graph, $\chi(G) \leq \Delta(G)$.*

An **independent set** of a graph G is a subset of $V(G)$ such that no two of which are adjacent. An independent set is called **maximum independent set** if it have the largest possible size for a given graph G. That size is called **independent number** of G, denoted by $\alpha(G)$. Since they are no two vertices adjacent in independent set, we can give the same color to each vertex in independent set. Hence we get the lower bound of the chromatic number of a graph G as follow.

Theorem 1.11 ([6])**.** *For any graph G of order n, $\chi(G) \geq \dfrac{n}{\alpha(G)}$.*

In the next section, we explore conditions for an endomorphism f on $\mathbb{Z}_n$ to make $endo - Cay_f(\mathbb{Z}_n, U_n)$ undirected for any $n \in \mathbb{N}$. After that, colorability of undirected $endo{-}Cay_f(\mathbb{Z}_n, U_n)$ is studied. We will show bounds of their chromatic numbers and also give examples to show the sharpness of each bounds.

2 Main Results

We know that for any $n \in \mathbb{N}$ can be rewritten in the from $n = p_1^{k_1} p_2^{k_2} \cdots p_i^{k_i}$ where p_j are distinct prime numbers such that $p_1 < p_2 < \ldots < p_i$ and $k_j \in \mathbb{N}$ for all $j = 1, 2, \ldots, i$. In cases that $n = p$ or $n = p^k$, C. Promsakon and S. Punma characterized undirected unitary endo-Cayley of $\mathbb{Z}_n$. They gave conditions to make $endo - Cay_f(\mathbb{Z}_n, U_n)$ undirected as we mention in introduction. Now we study an undirected for any $n \in \mathbb{N}$. Here is a result.

Theorem 2.1. *Given $n = p_1^{k_1} p_2^{k_2} \cdots p_i^{k_i}$ where p_j are distinct prime numbers such that $p_1 < p_2 < \ldots < p_i$ and $k_j \in \mathbb{N}$ for all $j = 1, 2, \ldots, i$. Let f be an endomorphism on $\mathbb{N}$ defined as $f(x) = mx$ for all $x \in \mathbb{N}$. If $m^2 \equiv 1 \pmod{p_j^{k_j}}$ for all $j = 1, 2, \ldots, i$, then $endo - Cay_f(\mathbb{Z}_n, U_n)$ is a undirected graph.*

Proof. Assume that $m^2 \equiv 1 \pmod{p_j^{k_j}}$ for all $j = 1, 2, \ldots, i$. Fix j. It implies that $m^2 \equiv 1 \pmod{p_j}$ and so $m \equiv 1 \pmod{p_j}$ or $m \equiv -1 \pmod{p_j}$. Hence $(m, n) = 1 = (-m, n)$ and also $m, -m \in U_n$. Because U_n is a group under multiplication, we have $mU_n \subseteq U_n$ and $-mU_n \subseteq U_n$.

Next, Since p_j are distinct prime numbers, we have $(p_{j_1}^{k_{j_1}}, p_{j_2}^{k_{j_2}}) = 1$ for any p_{j_1}, p_{j_2}. Then $m^2 \equiv 1 \pmod{n}$. So $xm^2 \equiv x \pmod{n}$ for all $x \in \mathbb{Z}_n$. By Theorem 1.2, we have $endo - Cay_f(\mathbb{Z}_n, U_n)$ is a undirected graph. □

We know that for any integer $n > 2$, there are at least 2 solutions of equation $x^2 \equiv 1 \pmod{n}$ which are $x \equiv 1 \pmod{n}$ and $x \equiv -1 \pmod{n}$. So there are at least 2^k distinct solutions of equation $x^2 \equiv 1 \pmod{n_1 n_2 \cdots n_k}$ where $(n_i, n_j) = 1$ for all i, j by the Chinese remainder theorem. For example, we find solutions of equation $x^2 \equiv 1 \pmod{45}$. Then $x^2 \equiv 1 \pmod{5}$ and $x^2 \equiv 1 \pmod{9}$ and also $x \equiv 1 \pmod{5}$ or $x \equiv -1 \pmod{5}$ and $x \equiv 1 \pmod{9}$ or $x \equiv -1 \pmod{9}$. By solving the 4 equations systems, the solutions are $x \equiv 1, -1, 19, -19 \pmod{45}$. Therefore integers m for undirected $endo - Cay_f(\mathbb{Z}_{45}, U_{45})$ where $f(x) = mx$ for all $x \in \mathbb{Z}_{45}$ are 1, 19, 26 and 44.

We show next the condition of an endomorphism for loopless unitary endo-Cayley of $\mathbb{Z}_n$.

Theorem 2.2. *Let n be a natural number such that $n > 1$ and m a natural number less than n. If an endomorphisn f on $\mathbb{Z}_n$ is defined as $f(x) = mx$ for all $x \in \mathbb{Z}_n$ where $(m-1, n) \neq 1$, then A graph $endo - Cay_f(\mathbb{Z}_n, U_n)$ is a loopless graph.*

Proof. Assume that $f : \mathbb{Z}_n \to \mathbb{Z}_n$ is defined as $f(x) = mx$ for all $x \in \mathbb{Z}_n$ and $(m-1, n) \neq 1$. Suppose for a contradiction that $endo - Cay_f(\mathbb{Z}_n, U_n)$ has a loop at vertex a. Then $a = f(a) + u = ma + u$ for some $u \in U_n$. So $u = -a(m-1)$. Since $(m-1, n) \neq 1$, we have $(-a(m-1), n) \neq 1$. Hence $u = -a(m-1) \notin U_n$, a contradiction. □

The objectives of this paper is to study about coloring properties of unitary endo-Cayley graph of $\mathbb{Z}_n$ and also find bonds of their chromatic numbers. So we consider only undirected simple $endo - Cay_f(\mathbb{Z}_n, U_n)$. We already showed the case of endomorphisms to make $endo - Cay_f(\mathbb{Z}_n, U_n)$ be undirected simple graph which is $f(x) = mx$ and $m^2 \equiv 1 \pmod{n}$ and $(m-1, n) \neq 1$ for all $x \in \mathbb{Z}_n$. Hence we study only $endo - Cay_f(\mathbb{Z}_n, U_n)$ where $f(x) = mx$ and $m^2 \equiv 1 \pmod{n}$ and $(m-1, n) \neq 1$ for all $x \in \mathbb{Z}_n$. For convenient, we use notation $endo-Cay_m(\mathbb{Z}_n, U_n)$ for undirected simple $endo-Cay_f(\mathbb{Z}_n, U_n)$ where $f(x) = mx$, $m^2 \equiv 1 \pmod{n}$ and $(m-1, n) \neq 1$ for all $x \in \mathbb{Z}_n$. For example, if we mention about $endo - Cay_m(\mathbb{Z}_{45}, U_{45})$, we refer an endomorphism f defined as $f(x) = mx$ where $m = 1, 19$ or 26.

Remark 2.3. *For any undirected loopless graph $endo - Cay_m(\mathbb{Z}_n, U_n)$, we have $m \in U_n$.*

Theorem 2.4. *For any vertices x and y in undirected loopless graph $endo - Cay_m(\mathbb{Z}_n, U_n)$, x is adjacent to y if and only if $(y - mx, n) = 1$.*

Proof. Let x and y be vertices in undirected loopless graph $endo-Cay_m(\mathbb{Z}_n, U_n)$. We assume that x and y are adjacent in $endo-Cay_m(\mathbb{Z}_n, U_n)$. Then $y = mx+u$ for some $u \in U_n$ and also $(y-mx, n) = (u, n) = 1$. Conversely, assume $(y-mx, n) = 1$. So $y - mx \in U_n$ and hence x is adjacent to y. □

There are some basic properties of undirected simple endo-Cayley graphs of $\mathbb{Z}_n$ involving their colorability. We show that properties here.

Theorem 2.5. *A graph* $endo - Cay_m(\mathbb{Z}_n, U_n)$ *is a* $\phi(n)$*-regular graph.*

Proof. Let x be an vertex in $endo-Cay_f(\mathbb{Z}_n, U_n)$. Then x is adjacent to all $f(x)+u$ where $u \in U_n$. Since $endo-Cay_f(\mathbb{Z}_n, U_n)$ be an undirected simple graph, we have $deg(u) = |U_n| = \phi(n)$. Hence $endo - Cay_f(\mathbb{Z}_n, U_n)$ is a $\phi(n)$-regular graph. □

Theorem 2.6. *A graph* $endo-Cay_m(\mathbb{Z}_n, U_n)$ *is a cycle if and only if* $n = 3$ *or* 6.

Proof. In Case that $n = 3$, it is clearly that m which satisfies with $m^2 \equiv 1 \pmod{n}$ and $(m-1, n) \neq 1$ is 1 and $endo - Cay_f(\mathbb{Z}_3, U_n) = Cay(\mathbb{Z}_3, \{1, 2\})$ is C_3. Sufficiency condition is proved. Next, we may assume $endo - Cay_f(\mathbb{Z}_n, U_n)$ is cycle. By Theorem 2.5, we have $\phi(n) = 2$. Hence $n = 3$ or 6. □

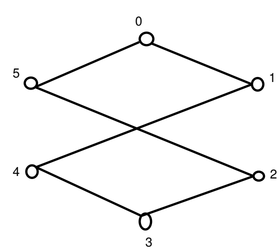


Figure 2: A cycle $endo - Cay_m(\mathbb{Z}_6, U_6)$ where $m = 5$

Now, we know that a graph $endo-Cay_m(\mathbb{Z}_3, U_3)$ is an odd cycle while a graph $endo-Cay_m(\mathbb{Z}_6, U_6)$ is an even cycle. So we instantly have the following Corollary.

Corollary 2.7. $\chi(endo - Cay_m(\mathbb{Z}_3, U_3)) = 3$ *and* $\chi(endo - Cay_m(\mathbb{Z}_6, U_6)) = 2$

Theorem 2.8. *A graph* $endo - Cay_m(\mathbb{Z}_n, U_n)$ *is a complete graph if and only if* n *is a prime number and* $endo - Cay_m(\mathbb{Z}_n, U_n) = Cay(\mathbb{Z}_n, U_n)$.

Proof. It is easy to see that $endo - Cay_1(\mathbb{Z}_2, U_2)$ is a complete graph. So we will prove for any odd prime. We assume that $endo - Cay_m(\mathbb{Z}_n, U_n)$ is a complete graph. Then $\phi(n) = n - 1$ by Theorem 2.5. So n is a prime number. Because $m^2 \equiv 1 \pmod{n}$, it implies that $m \equiv 1$ or $n-1 \pmod{n}$. Since $(n-2, n) = 1$, we have that $m = 1$. Therefore $endo - Cay_m(\mathbb{Z}_n, U_n) = Cay(\mathbb{Z}_n, U_n)$. Clearly that $Cay(\mathbb{Z}_n, U_n)$ is a complete graph where n is a prime number. Therefore the proof is done. □

By Brooks's Theorem, we have a trivial bond of the chromatic number of $endo - Cay_m(\mathbb{Z}_n, U_n)$ where n is not a prime number as follow.

Theorem 2.9. *Let $n \in \mathbb{N}$ be not a prime number. Then $\chi(endo{-}Cay_m(\mathbb{Z}_n, U_n)) \leq \phi(n)$.*

Proof. Since n is not a prime number, we have $endo - Cay_m(\mathbb{Z}_n, U_n)$ is not odd cycle or complete graph. We know that $endo - Cay_m(\mathbb{Z}_n, U_n)$ dose not necessay connected. Than $\chi(endo - Cay_m(\mathbb{Z}_n, U_n)) = max\{\chi(H_i)\} \leq \Delta(H) = \phi(n)$ where H_i is a component of $endo - Cay_m(\mathbb{Z}_n, U_n)$. □

We will show that $endo - Cay_m(\mathbb{Z}_{2n}, U_{2n})$ is a bipartite graph and so its chromatic number is 2 for any positive integer n.

Theorem 2.10. *Let n be a natural number. Then $\chi(endo - Cay_m(\mathbb{Z}_{2n}, U_{2n})) = 2$ and hence $endo - Cay_m(\mathbb{Z}_{2n}, U_{2n})$ is bipartite.*

Proof. We set $\mathbb{Z}_{2n} = \{1, 2, \ldots, 2n\}$. Then all element in U_{2n} are odd integer. So we have m is odd.

Define $\alpha : \mathbb{Z}_{2n} \to \{1, 2\}$ by

$$\alpha(x) = \begin{cases} 1, & \text{if } x \text{ is odd,} \\ 2, & \text{if } x \text{ is even} \end{cases}$$

for all $x \in \mathbb{Z}_{2n}$. To show α is proper, let x and y be elements in $\mathbb{Z}_{2n}$ such that x is adjacent to y. Then $x = f(y) + u = my + u$ for some $u \in U_{2n}$. We can see that x and y have different parity. So $\alpha(x) \neq \alpha(y)$. Hence α is proper and $\chi(endo - Cay_m(\mathbb{Z}_{2n}, U_{2n})) \leq 2$. Cleary that there is an edge in $endo - Cay_m(\mathbb{Z}_{2n}, U_{2n})$ and it implies $\chi(endo{-}Cay_m(\mathbb{Z}_{2n}, U_{2n})) \geq 2$. Therefore $\chi(endo{-}Cay_m(\mathbb{Z}_{2n}, U_{2n})) = 2$ and $endo - Cay_m(\mathbb{Z}_{2n}, U_{2n})$ is bipartite. □

Now we already have the chromatic number of $endo - Cay_m(\mathbb{Z}_n, U_n)$ in cases that n is a prime number or n is an even number. So next we focus to find bounds of the chromatic number of $endo - Cay_m(\mathbb{Z}_{p^k}, U_{p^k})$ where p is an odd prime and k is a natural number.

We recall that for a group G, f an endomorphism on G and A a subset of G, $endo - Cay_f(G, G) \cong endo - Cay_f(G, A) \cup endo - Cay_f(G, A')$ where $A' = G \setminus A$ and $endo - Cay_f(G, G)$ is a complete graph. Hence a complement graph of $endo - Cay_f(G, A)$ is $endo - Cay_f(G, A')$.

Let p be an odd prime number and k be a positive number. Because $endo - Cay_m(\mathbb{Z}_{p^k}, U_{p^k})$ is undirected, so we have that its complement, $endo\text{-}Cay_m(\mathbb{Z}_{p^k}, U'_{p^k})$, is also undirected. We will show in next theorem that $endo - Cay_m(\mathbb{Z}_{p^k}, U'_{p^k})$ contains a complete subgraph of order p^{k-1}

Lemma 2.11. *Let p be an odd prime number and k be a positive number greater than 1. Then $\omega(endo - Cay_m(\mathbb{Z}_{p^k}, U'_{p^k})) \geq p^{k-1}$.*

Proof. We claim that a subgraph induced by U'_{p^k} is complete. Let u and v be elements in U'_{p^k}. It is sufficient to show that u is adjacent to v because $endo-Cay_m(\mathbb{Z}_{p^k}, U'_{p^k})$ is undirected. Since u and v are in U'_{p^k}, we have $p|u$ and $p|v$. So $v-mu \equiv 0 \pmod{p}$ and thus $v-mu \in U'_p$. Since $v = mu+(v-mu)$, we conclude that u is adjacent to v. Therefore a subgraph induced by U'_{p^k} is a complete graph and $\omega(endo-Cay_m(\mathbb{Z}_{p^k}, U'_{p^k})) \geq |U'_{p^k}| = p^{k-1}$. □

Next, we will give a character of $endo-Cay_m(\mathbb{Z}_{p^k}, U'_{p^k})$.

Theorem 2.12. *Let p be an odd prime number such that $p > 3$ and an integer k such that $k > 1$. Then a graph $endo-Cay_m(\mathbb{Z}_{p^k}, U'_{p^k})$ is disconnected.*

Proof. Let x be an element in U'_{p^k}. Then $(x, p^k) \neq 1$ and also $p|x$. For any $u \in U'_{p^k}$, we have $p|u$ and $p|(mx+u)$. Hence $mx+u \in U'_{p^k}$. It follows that x is not adjacent to any vertex in U_{p^k}. Therefore $endo-Cay_m(\mathbb{Z}_{p^k}, U'_{p^k})$ is disconnected. □

Theorem 2.13. *Let p be an odd prime number and k be a positive number greater than 1. Then $\chi(endo-Cay_m(\mathbb{Z}_{p^k}, U'_{p^k})) = p^{k-1} = \omega(endo-Cay_m(\mathbb{Z}_{p^k}, U'_{p^k}))$.*

Proof. It is clearly that $\Delta(endo-Cay_m(\mathbb{Z}_{p^k}, U'_{p^k})) = |U'_{p^k}| = p^{k-1} > 2$. Hence a graph $endo-Cay_m(\mathbb{Z}_{p^k}, U'_{p^k})$ is not odd cycle. By Theorem 2.12, we have $endo-Cay_m(\mathbb{Z}_{p^k}, U'_{p^k})$ is not complete. By Theorem 1.10, $\chi(endo-Cay_m(\mathbb{Z}_{p^k}, U'_{p^k})) \leq |U'_{p^k}| = p^{k-1}$. By Lemma 2.11, we have $\chi(endo-Cay_m(\mathbb{Z}_{p^k}, U'_{p^k})) \geq \omega(endo-Cay_m(\mathbb{Z}_{p^k}, U'_{p^k})) \geq p^{k-1}$. Hence $\chi(endo-Cay_m(\mathbb{Z}_{p^k}, U'_{p^k})) = p^{k-1} = \omega(endo-Cay_m(\mathbb{Z}_{p^k}, U'_{p^k}))$. □

By Theorem 2.13 and 2.12, we can describe character of $endo-Cay_m(\mathbb{Z}_{p^k}, U'_{p^k})$. A graph $endo-Cay_m(\mathbb{Z}_{p^k}, U'_{p^k})$ have exactly two components such as a complete graph induced by U'_{p^k} and an induced subgraph by U_{p^k} and Its chromatic number is equal to its clique number, $\chi(endo-Cay_m(\mathbb{Z}_{p^k}, U'_{p^k})) = p^{k-1} = \omega(endo-Cay_m(\mathbb{Z}_{p^k}, U'_{p^k}))$. Since a complement graph of $endo-Cay_m(\mathbb{Z}_{p^k}, U_{p^k})$ is $endo-Cay_m(\mathbb{Z}_{p^k}, U'_{p^k})$, we conclude that the independent number of $endo-Cay_m(\mathbb{Z}_{p^k}, U_{p^k})$ is the clique number of $endo-Cay_m(\mathbb{Z}_{p^k}, U'_{p^k})$ and the lower bound of $\chi(endo-Cay_m(\mathbb{Z}_{p^k}, U_{p^k}))$ consequently follows.

Theorem 2.14. *Let p be an odd prime number and an integer k such that $k > 1$. Then $\chi(endo-Cay_m(\mathbb{Z}_{p^k}, U_{p^k})) \geq p$.*

Proof. Since a graph $endo-Cay_m(\mathbb{Z}_{p^k}, U_{p^k})$ is a complement graph of $endo-Cay_m(\mathbb{Z}_{p^k}, U'_{p^k})$, we have $\alpha(endo-Cay_m(\mathbb{Z}_{p^k}, U_{p^k})) = \omega(endo-Cay_m(\mathbb{Z}_{p^k}, U'_{p^k})) = p^{k-1}$. Hence $\chi(endo-Cay_m(\mathbb{Z}_{p^k}, U_{p^k})) \geq \dfrac{p^k}{\alpha(endo-Cay_m(\mathbb{Z}_{p^k}, U_{p^k}))} = \dfrac{p^k}{p^{k-1}} = p$ by Theorem 1.11. □

Now we turn to focus conditions of m, an endomorphism on Z_{p^k}. Since we study only an undirected loopless graph $endo{-}Cay_m(\mathbb{Z}_{p^k}, U_{p^k})$, we have conditions of m that $m^2 \equiv 1 \pmod p$ and $(m-1, p^k) \neq 1$. Then $m \equiv 1 \pmod p$ or $m \equiv -1 \pmod p$. In case $m \equiv -1 \pmod p$, we have $p \not| (m-1)$ and hence $(m-1, p^k) = 1$. Therefore $m \equiv 1 \pmod p$ for an undirected loopless graph $endo{-}Cay_m(\mathbb{Z}_{p^k}, U_{p^k})$. The next theorems, we show the independent sets and the chromatic numbers of $endo - Cay_m(\mathbb{Z}_{p^k}, U_{p^k})$.

Theorem 2.15. *Let p be an odd prime number and an integer k such that $k > 1$. For any $i \in \mathbb{Z}_{p^k}$, a set $A_i = \{i, i+p, i+2p, \ldots, i+(p^{k-1}-1)p\}$ is an independent set of $endo - Cay_m(\mathbb{Z}_{p^k}, U_{p^k})$.*

Proof. Fix $i \in \mathbb{Z}_{p^k}$ and let $A_i = \{i, i+p, i+2p, \ldots, i+(p^{k-1}-1)p\}$ be a subset of Z_{p^k}. Suppose $i+sp$ and $i+tp$ are elements in A_i such that they are adjacent. Hence there are $u \in U_{p^k}$ such that $m(i+sp)+u \equiv i+tp \pmod{p^k}$.

$$\begin{aligned} m(i+sp)+u &\equiv i+tp \pmod{p^k} \\ m(i+sp)+u &\equiv i+tp \pmod{p} \\ mi+u &\equiv i \pmod{p} \\ u &\equiv 0 \pmod{p}, \text{ since } m \equiv 1 \pmod{p}. \end{aligned}$$

Thus $u \notin U_{p^k}$, a contradiction. Therefore $i+sp$ and $i+tp$ are not adjacent and $A_i = \{i, i+p, i+2p, \ldots, i+(p^{k-1}-1)p\}$ is an independent set of $endo - Cay_m(\mathbb{Z}_{p^k}, U_{p^k})$. □

Theorem 2.16. *Let p be an odd prime number and integer k such that $k > 1$. Then $\chi(endo - Cay_m(\mathbb{Z}_{p^k}, U_{p^k})) = p$.*

Proof. By following to the proof of Theorem 2.15, we separate Z_{p^k} into p distinct independent sets, say $A_{i_1}, A_{i_2}, \ldots, A_{i_p}$. To get a proper coloring of $endo - Cay_m(\mathbb{Z}_{p^k}, U_{p^k})$, we give color j for every vertices in A_{i_j}. So we have that $\chi(endo{-}Cay_m(\mathbb{Z}_{p^k}, U_{p^k})) \leq p$. By Theorem 2.14, hence $\chi(endo{-}Cay_m(\mathbb{Z}_{p^k}, U_{p^k})) = p$. □

For example, let consider $endo - Cay_4(\mathbb{Z}_9, U_9)$ in figure 3. We can see that $A_1 = \{1, 4, 7\}$, $A_2 = \{2, 5, 8\}$ and $A_3 = \{0, 3, 6\}$ are maximum independent sets. Hence $\chi(endo - Cay_4(\mathbb{Z}_9, U_9)) \geq 3$. It is clearly that $endo - Cay_4(\mathbb{Z}_9, U_9)$ contain K_3. Hence $\chi(endo - Cay_4(\mathbb{Z}_9, U_9)) = 3$.

Finally, We study bound of the chromatic number of $endo - Cay_m(\mathbb{Z}_n, U_n)$ where n is an odd integer, m a positive integer such that $(m-1, n) \neq 1$ and $m^2 \equiv 1 \pmod n$. The trivial two solutions of m are $m \equiv 1 \pmod n$ or $m \equiv 1 \pmod n$. If $m \equiv 1 \pmod n$, we have $endo - Cay_m(\mathbb{Z}_n, U_n) = Cay(\mathbb{Z}_n, U_n)$. Therefore $\chi(endo - Cay_m(\mathbb{Z}_n, U_n)) = \chi(Cay(\mathbb{Z}_n, U_n)) = p$ where p is the smallest prime divisor of n showed in [3].

Theorem 2.17 ([3])**.** *If p is the smallest prime divisor of n, then $\chi(X_n)) = \omega(X_n) = p$ and $\chi(\overline{X_n})) = \omega(\overline{X_n}) = \dfrac{n}{p}$ where X_n is $Cay(\mathbb{Z}_n, U_n)$.*

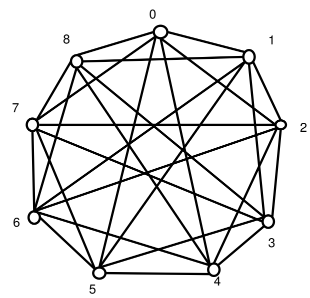


Figure 3: A graph $endo-Cay_4(\mathbb{Z}_9, U_9)$

As we know that n be rewritten as a product of primes as $n = p_1^{k_1} p_2^{k_2} \cdots p_i^{k_i}$ where p_j are distinct prime numbers such that $p_1 < p_2 < \ldots < p_i$ and $k_j \in \mathbb{N}$ for all $j = 1, 2, \ldots, i$. Hence $m^2 \equiv 1 \pmod{p_j}$ and also $m \equiv 1 \pmod{p_j}$ or $m \equiv -1 \pmod{p_j}$ for any prime factor p_j of n.

The next theorem is the chromatic number of $endo-Cay_m(\mathbb{Z}_n, U_n)$ where $m \equiv 1 \pmod{p_1}$. It is clearly that $m \equiv 1 \pmod{p_1}$ implies that $(m-1, n) \geq p_1 > 1$.

Theorem 2.18. *Let n be an odd integer such that $n = p_1^{k_1} p_2^{k_2} \cdots p_i^{k_i}$ where p_j are distinct prime numbers such that $p_1 < p_2 < \ldots < p_i$ and $k_j \in \mathbb{N}$ for all $j = 1, 2, \ldots, i$. For a graph $endo-Cay_m(\mathbb{Z}_n, U_n)$, if $m \equiv 1 \pmod{p_1}$, then $\chi(endo-Cay_m(\mathbb{Z}_n, U_n)) \leq p_1$.*

Proof. We assume that $m \equiv 1 \pmod{p_1}$. Define a coloring $\alpha : \mathbb{Z}_n \to \mathbb{Z}_n$ by

$$\alpha(x) = i, \text{if } x \equiv i \pmod{p_1} \text{ and } 0 \leq i \leq p_1 - 1$$

To show α is proper, suppose x and y be element in $\mathbb{Z}_n$ such that x and y are adjacent. Then $y = mx + u$ for some $u \in U_n$. Then $u \not\equiv 0 \pmod{p_1}$. We have $y = mx + u \equiv x + u \not\equiv x \pmod{p_1}$. Hence $\alpha(x) \neq \alpha(y)$ and α is a proper coloring. Therefore $\chi(endo-Cay_m(\mathbb{Z}_n, U_n)) \leq p_1$. □

In Cayley graph $Cay(\mathbb{Z}_n, U_n)$, we have that a set of vertices $\{0, 1, 2, \ldots, p-1\}$ forms a complete graph but it do not form in $endo-Cay_m(\mathbb{Z}_n, U_n)$. For example, a set $\{0, 1, 2, 3, 4, 5\}$ does not be a complete graph in $endo-Cay_6(\mathbb{Z}_{35}, U_{35})$ because 3 and 4 are not adjacent $[(4-6(3), 35) \neq 1]$. In the next theorem, we show a condition to make $\{0, m, 2m, \ldots, (p_1-1)m\}$ be a complete graph in $endo-Cay_m(\mathbb{Z}_n, U_n)$.

Theorem 2.19. *Let n be an odd integer such that $n = p_1^{k_1} p_2^{k_2} \cdots p_i^{k_i}$ where p_j are distinct prime numbers such that $p_1 < p_2 < \ldots < p_i$ and $k_j \in \mathbb{N}$ for all $j = 1, 2, \ldots, i$. If $p_2 > 2(p_1 - 1)$, $\omega(endo-Cay_m(\mathbb{Z}_n, U_n)) \geq p_1$.*

Proof. Assume that $p_2 > 2(p_1-1)$. We let $A = \{0, m, 2m, \ldots, (p_1-1)m\}$. Clearly that any $x \in A$, we have $(x, n) = 1$. So 0 is adjacent to any x in A. Let im and jm be elements in A such that $0 < i < j \leq p-1$. Then $(jm - m(im), n) = (j - mi, n)$, since $m \in U_n$. Suppose for a contradiction that $p|j-mi$ for some prime factor p of n such that $p \neq p_1$. Then $j \equiv mi \pmod p$ and also $j^2 \equiv i^2 \pmod p$. Since $j \not\equiv i \pmod{p_1}$, we have $j \not\equiv i \pmod p$. So $j \equiv -i \pmod p$. Hence $p_2 > 2(p_1-1) \geq j + i = pk \geq p$. This is a contradiction. Hence there is no prime factor p of n such that $p|j-mi$. It implies that $(j-mi, n) = 1$ and thus im and jm are adjacent. Therefore A forms a complete graph and $\omega(endo-Cay_m(\mathbb{Z}_n, U_n)) \geq p_1$. □

For example, let consider $endo-Cay_4(\mathbb{Z}_{3^i5^j}, U_{3^i5^j})$ where $i, j \in \mathbb{N}$. Since $4 \equiv 1 \pmod 3$, we have $\chi(endo-Cay_4(\mathbb{Z}_{3^i5^j}, U_{3^i5^j})) \leq 3$ by Theorem 2.18. Clearly that $5 > 2(3-1)$ by Theorem 2.19, we have $\omega(endo-Cay_4(\mathbb{Z}_{3^i5^j}, U_{3^i5^j})) \geq 3$. Therefore $\chi(endo-Cay_4(\mathbb{Z}_{3^i5^j}, U_{3^i5^j})) = 3$.

Acknowledgements : I would like to thank the referees for their comments and suggestions on the manuscript. This research was funded by King Mongkut's University of Technology North Bangkok. Contract no. KMUTNB-GOV-58-64.

References

[1] R. Akhtar, M. Boggess, T. Jackson-Henderson, I. Jimenez, R. Karpman, A. Kinzel, D. Pritikin, On unitaty Cayley graphs of finite ring, Electronic Journal of Combinatorics 16 (2009) 1-13.

[2] I. Dejter, R.E. Giudici, On Unitaty Cayley Graphs, J. Combin. Math. Combin. Comput. 18 (1995) 121-124.

[3] W. Klotz, T. Sander, Some properties of unitary Cayley graphs, Electronic Journal of Combinatorics 14 (1) (2007) 1-12.

[4] C. Promsakon, S. Punma, Undirected endo-Cayley digraphs of cyclic groups of order primes, World Academy of Science, Engineering and Technology International Journal of Mathematical and Computational Sciences 5 (1) (2014).

[5] C. Promsakon, Undirectedness of endo-Cayley graphs of cyclic groups of power prime order groups, Journal of Discrete Mathematical Sciences & Cryptography. 19 (2016) 321-330.

[6] B.W. Douglas, Introduction to graph theory, Prentice-Hall Inc., 2001.

(Received 15 August 2017)
(Accepted 27 August 2018)