Thai Journal of Mathematics
Volume 16 (2018) Number 2 : 517–527
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Geraghty's Fixed Point Theorem for Partial-Special Multi-Valued Mappings

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Abstract : In this paper we recall the concept of partial Hausdorff metric. Many authors studied about fixed point theory for multi-valued mappings on a partial metric space using the partial Hausdorff metric. We prove a generalization of Geraghty's fixed point theorem for a type of multi-valued mapping that called partial-special multi-valued mapping in partial metric space.

Keywords : fixed point; partial-special multi-valued mappings.
2010 Mathematics Subject Classification : 47H09; 47H10.

1 Introduction and Preliminaries

Since the Banach's contraction principle, several type of contraction mappings on metric space have appeared in order to prove many fixed point theorems. Various authors gave some generalizations to Banach's contraction principle. One such generalization, in a complete metric space, is due to Geraghty as follows.

Theorem 1.1 ([1]). *Let (X, d) be a complete metric space, let $f : X \to X$ be a mapping such that for each $x, y \in X$, $d(f(x), f(y)) \leq \alpha(d(x, y))d(x, y)$ where $\alpha \in S$, that S is the families of functions from $[0, +\infty)$ into $[0, 1)$ which satisfy the simple condition $\alpha(t_n) \to 1 \Rightarrow t_n \to 0$. Then f has a fixed point $z \in X$, and $\{f^n(x)\}$ converges to z, for each $x \in X$.*

Definition 1.2 ([2])**.** A *partial metric* on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that, for all $x, y, z \in X$

(p1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
(p2) $p(x,x) \leq p(x,y)$,
(p3) $p(x,y) = p(y,x)$,
(p4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A *partial metric space* is a pair (X, p) such that X is a nonempty set and p is a partial metric on X. It is clear that, if $p(x,y) = 0$, then from (p1) and (p2) $x = y$. But if $x = y$, $p(x,y)$ may not be 0.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\epsilon) : x \in X, \epsilon > 0\}$, where

$$B_p(x,\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$$

for all $x \in X$ and $\epsilon > 0$.

Definition 1.3. Let (X, p) be a partial metric space. Then:

1. A sequence $\{x_n\}$ in (X, p) *converges* to a point $x \in X$, with respect to τ_p, if $\lim_{n\to\ +\infty} p(x, x_n) = p(x,x)$.
   This will be denoted as $x_n \to x$, $n \to +\infty$ or $\lim_{n\to+\infty} x_n = x$.
2. A sequence $\{x_n\}$ in (X, p) is called a *Cauchy sequence* if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists (and is finite).
3. The space (X, p) is said to be *complete* if every Cauchy sequence $\{x_n\} \subset X$ converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n, x_m)$.
4. A sequence $\{x_n\} \in (X, p)$ is called 0-*Cauchy* if $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. The space (X, p) is said to be 0-*complete* if every 0-Cauchy sequence in X converges (in τ_p) to a point $x \in X$ such that $p(x,x) = 0$.

If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y),$$

defines a metric on X.

Furthermore, a sequence $\{x_n\}$ converges in (X, p^s) to a point $x \in X$ if and only if

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x) = p(x,x).$$

Example 1.4. Paradigmatic examples of a partial metric space are:

- The pair $(\mathbb{R}^+, p)$, where $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ and

  $$p(x,y) = \max\{x,y\} \quad \text{for all} \quad x, y \in \mathbb{R}^+.$$

  The corresponding metric on X is $p^s(x,y) = 2\max\{x,y\} - x - y = |x - y|$.

- If (X,d) is a metric space and $c \geq 0$ is arbitrary, then $p(x,y) = d(x,y) + c$ defines a partial metric on X and the corresponding metric is $p^s(x,y) = 2d(x,y)$.

Lemma 1.5. *Let (X,p) be a partial metric space;*

(a) *$\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s).*

(b) *The space (X,p) is complete if and only if the metric space (X,p^s) is complete.*

(c) *If $p(x_n,z) \to p(z,z) = 0$ as $n \to \infty$, then $p(x_n,y) \to p(z,y)$ as $n \to \infty$ for each $y \in X$.*

(d) *Every 0-Cauchy sequence in (X,p) is Cauchy in (X,p^s).*

(e) *If (X,p) is complete, then it is 0-complete.*

The converse assertions of (d) and (e) do not hold as the following easy example shows.

Example 1.6 ([2])**.** The space $X = [0,+\infty) \cap \mathbb{Q}$ with the partial metric $p(x,y) = \max\{x,y\}$ is 0-complete, but it is not complete (since $p^s(x,y) = |x-y|$ and (X,p^s) is not complete). Moreover, the sequence $\{x_n\}$ with $x_n = 1$ for each $n \in \mathbb{N}$ is a Cauchy sequence in (X,p), but it is not a 0-Cauchy sequence.

It is easy to see that every closed subset of a 0-complete partial metric space is 0-complete.

Let $CB^p(X)$ denotes the collection of all nonempty closed bounded subset of X. For $A,B \in CB^p(X)$ and $x \in X$, define $p(x,B) = \inf\{p(x,y) : y \in B\}$ and

$$H_p(A,B) = \max\{\delta_p(A,B), \delta_p(B,A)\}$$

where $\delta_p(A,B) = \sup_{a\in A} p(a,B)$.

Lemma 1.7. *Let (X,p) a partial metric space. For all $A,B,C \in CB^p(X)$, we have:*

1. $H_p(A,A) \leq H_p(A,B)$*;*
2. $H_p(A,B) = H_p(B,A)$*;*
3. $H_p(A,B) \leq H_p(A,C) + H_p(C,B) - \inf_{c\in C} p(C,C)$.

Proof. See, [3, Proposition 2.3]. □

In view of Lemma 1.7, we call the mapping $H_p : CB^p(X) \times CB^p(X) \to [0,+\infty)$, a *partial Hausdorff metric* induced by the partial metric p. For details see [3]. Note that a point $x \in X$ is said to be a *fixed point* of a multi-valued mapping $T : X \to CB^p(X)$ if $x \in Tx$.

Remark 1.8 ([3]). *Let (X,p) be a partial metric space and A any nonempty set in (X,p), then $a \in \overline{A}$ if and only if $p(a,A) = p(a,a)$, where $\overline{A}$ denotes the closure of A with respect to the partial metric p.*

Note that A is closed in (X,p) if and only if $A = \overline{A}$.

Throughout this paper, we assume that (X,p) is a complete partial metric space and H_p is the partial Hausdorff metric on $CB^p(X)$ induced by p.

2 Main Results

Now we introduce a notion called partial-special multi-valued mapping. For this type of partial-special multi-valued mappings we have obtained a fixed point theorem that generalizes a Geragth's fixed point theorem for multi-valued mappings.

Definition 2.1. Let (X,p) be a partial metric space, a multi-valued mapping $T : X \to CB^p(X)$ is called *partial-special multi-valued mapping* if

$$\inf_{y\in Tx}\{p(x,y)+p(y,z)\} = p(x,Tx)+p(z,Tx), \quad \forall\ x,z \in X. \tag{2.1}$$

It is clear that every single valued mapping, in a partial metric space, is partial-special multi-valued mapping, also there exist some mappings that are partial-special multi-valued but not single valued.

Example 2.2. Let $X = \left\{\frac{1}{3}, \frac{1}{9}, \cdots, \frac{1}{3^n}, \cdots\right\} \cup \{0,1\}$, $d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y. \end{cases}$ Define $p(x,y) = d(x,y) + c$ with $c \geq 0$ arbitrary. Define mapping $T(x) : X \to$

$CB^p(X)$,

$$T(x) = \begin{cases} \left\{\frac{1}{3^{n+1}}\right\} & \text{if } \ x = \frac{1}{3^n}, \quad n = 1,2,\cdots \\ \{0\} & \text{if } \ x = 0 \\ \left\{0, \frac{1}{3}\right\} & \text{if } \ x = 1 \end{cases}$$

The mapping T is partial-metric multi-valued mapping, it is possible to check (2.1) for every couple $x,z \in X$.

It is clear that above example is partial-special multi-valued mapping but not single valued.

Now we prove our main result in this paper.

Theorem 2.3. *Let (X,p) be a complete partial metric space and let $T : X \to CB^p(X)$ be partial-special multi-valued mapping such that*

$$\begin{aligned} H_p(Tx,Ty) \leq \alpha(p(x,y))p(x,y) &+ \beta(p(x,y))[p(x,Tx)+p(y,Ty)] \\ &+ \gamma(p(x,y))[p(x,Ty)+p(y,Tx)] \end{aligned}$$

for all $x,y \in X$, where α, β, γ are mappings from $[0,+\infty)$ into $[0,1)$ such that $\frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)} \in S$ and $\beta(t) \geq \gamma(t)$ for all $t \in [0,+\infty)$. Then T has a fixed point.

Proof. Define a function α' from $[0,+\infty)$ into $[0,1)$ by $\alpha'(t) = \frac{\alpha(t)+1-2\beta(t)-2\gamma(t)}{2}$ for all $t \in [0,+\infty)$. Then we have

1. $\alpha(t) < \alpha'(t)$ for all $t \in [0,+\infty)$;
2. $\frac{\alpha'+\beta+\gamma}{1-(\beta+\gamma)} \in S$;
3. for $x, y \in X$ and $u \in Tx$, there exists $v \in Ty$ such that

$$p(u,v) \le \alpha'(p(x,y))p(x,y) + \beta(p(x,y))[p(x,Tx)+p(y,Ty)] + \gamma(p(x,y))[p(x,Ty)+p(y,Tx)].$$

   Putting $u = y$ in (3), we obtain that:

4. For $x \in X$ and $y \in Tx$ there exists $v \in Ty$ such that

$$p(v,y) \le \alpha'(p(x,y))p(x,y) + \beta(p(x,y))[p(x,Tx)+p(y,Ty)] + \gamma(p(x,y))[p(x,Ty)+p(y,Tx)].$$

Hence, we can define a sequence $\{x_n\}_{n\in\mathbb{N}}$ which satisfies $x_{n+1} \in Tx_n$, $x_{n+1} \neq x_n$ and

$$\begin{aligned} p(x_{n+2},x_{n+1}) \le\ & \alpha'(p(x_{n+1},x_n))p(x_{n+1},x_n) \\ & + \beta(p(x_{n+1},x_n))[p(x_n,Tx_n)+p(x_{n+1},Tx_{n+1})] \\ & + \gamma(p(x_{n+1},x_n))[p(x_n,Tx_{n+1})+p(x_{n+1},Tx_n)] \end{aligned}$$

for all $n \in \mathbb{N}$. Observing that

$$\begin{aligned} p(x_n,Tx_{n+1})+p(x_{n+1},Tx_n) &\le p(x_n,x_{n+2})+p(x_{n+1},x_{n+1}) \\ &\le p(x_n,x_{n+1})+p(x_{n+1},x_{n+2}), \end{aligned}$$

it follows that

$$p(x_{n+2},x_{n+1}) \le \frac{\alpha'(p(x_{n+1},x_n))+\beta(p(x_{n+1},x_n))+\gamma(p(x_{n+1},x_n))}{1-(\beta(p(x_{n+1},x_n))+\gamma(p(x_{n+1},x_n)))}p(x_{n+1},x_n)$$

for all $n \in \mathbb{N}$. We show that $\{x_n\}$ is a Cauchy sequence. To this end, we break the argument into two Steps.

Step 1: $\lim_{n\to+\infty} p(x_n,x_{n+1}) = 0$. Since $\frac{\alpha'(t)+\beta(t)+\gamma(t)}{1-(\beta(t)+\gamma(t))} < 1$ for all t, $\{p(x_n,x_{n+1})\}$ is decreasing and bounded below, so

$$\lim_{n\to+\infty} p(x_n,x_{n+1}) = r \ge 0.$$

Assume $r > 0$. Then we have

$$\frac{p(x_{n+1},x_{n+2})}{p(x_n,x_{n+1})} \le \frac{\alpha'(p(x_n,x_{n+1}))+\beta(p(x_n,x_{n+1}))+\gamma(p(x_n,x_{n+1}))}{1-(\beta(p(x_n,x_{n+1}))+\gamma(p(x_n,x_{n+1})))} < 1,$$

$n = 1, 2, ...$ By letting $n \to +\infty$, we see that

$$1 \leq \lim_{n\to+\infty} \frac{\alpha'(p(x_n, x_{n+1})) + \beta(p(x_n, x_{n+1})) + \gamma(p(x_n, x_{n+1}))}{1 - (\beta(p(x_n, x_{n+1})) + \gamma(p(x_n, x_{n+1})))} \leq 1.$$

On the other hand, we have $\frac{\alpha'+\beta+\gamma}{1-(\beta+\gamma)} \in S$. Therefore $r = 0$. This is a contradiction, hence, we prove Step 1.

Step 2: $\{x_n\}$ is a 0-Cauchy sequence. Assume $\limsup_{n,m\to+\infty} p(x_n, x_m) > 0$. By triangle inequality for positive integer numbers n, m and for $y \in Tx_m$, we obtain $p(x_n, x_m) \leq p(x_n, y)+p(y, x_m)-p(y, y)$. This means that for every positive integer numbers m, n, with using of relation (2.1), we have

$$\begin{aligned}
p(x_n, x_m) &\leq \inf \{p(x_n, y) + p(y, x_m) - p(y, y)\} \\
&\leq \inf \{p(x_n, y) + p(y, x_m)\} = p(x_m, Tx_m) + p(x_n, Tx_m) \\
&\leq p(x_m, x_{m+1}) + p(x_n, x_{n+1}) + p(x_{n+1}, Tx_m) \\
&\leq H_p(Tx_m, Tx_n) + p(x_n, x_{n+1}) + p(x_m, x_{m+1}) \\
&\leq \alpha(p(x_n, x_m))p(x_n, x_m) + \beta(p(x_n, x_m))[p(x_n, Tx_n) + p(x_m, Tx_m)] \\
&+ \gamma(p(x_n, x_m))[p(x_n, Tx_m) + p(x_m, Tx_n)] + p(x_n, x_{n+1}) + p(x_m, x_{m+1}) \\
&= \alpha(p(x_n, x_m))p(x_n, x_m) + \beta(p(x_n, x_m))[p(x_n, x_{n+1}) + p(x_m, x_{m+1})] \\
&+ \gamma(p(x_n, x_m))[p(x_n, x_{m+1})+p(x_m, x_{n+1})] + p(x_n, x_{n+1}) + p(x_m, x_{m+1}) \\
&\leq \alpha(p(x_n, x_m))p(x_n, x_m) + \beta(p(x_n, x_m))[p(x_n, x_{n+1}) + p(x_m, x_{m+1})] \\
&+ \gamma(p(x_n, x_m))[p(x_n, x_m) + p(x_m, x_{m+1}) - p(x_m, x_m)] \\
&+ \gamma(p(x_n, x_m))[p(x_m, x_n) + p(x_n, x_{n+1}) - p(x_n, x_n)] \\
&+ p(x_n, x_{n+1}) + p(x_m, x_{m+1}) \\
&\leq \alpha(p(x_n, x_m))p(x_n, x_m) + \beta(p(x_n, x_m))[p(x_n, x_{n+1}) + p(x_m, x_{m+1})] \\
&+ \gamma(p(x_n, x_m))[2p(x_n, x_m) + p(x_n, x_{n+1}) + p(x_m, x_{m+1})]+ \\
&+ p(x_n, x_{n+1}) + p(x_m, x_{m+1}).
\end{aligned}$$

Then $p(x_n, x_m) - \alpha(p(x_n, x_m))p(x_n, x_m) - 2\gamma(p(x_n, x_m))p(x_n, x_m) \leq$

$$\begin{aligned}
&\beta(p(x_n, x_m))[p(x_n, x_{n+1}) + p(x_m, x_{m+1})] \\
&+ \gamma(p(x_n, x_m))[p(x_n, x_{n+1}) + p(x_m, x_{m+1})] + p(x_n, x_{n+1}) + p(x_m, x_{m+1})
\end{aligned}$$

and hence:

$$\begin{aligned}
p(x_n, x_m) \leq &\frac{[\beta(p(x_n, x_m)) + \gamma(p(x_n, x_m))][p(x_n, x_{n+1}) + p(x_m, x_{m+1})]}{1 - [\alpha(p(x_n, x_m)) + 2\gamma(p(x_n, x_m))]} \\
&+ \frac{p(x_n, x_{n+1}) + p(x_m, x_{m+1})}{1 - [\alpha(p(x_n, x_m)) + 2\gamma(p(x_n, x_m))]}.
\end{aligned}$$

Under the assumption $\limsup_{n,m\to+\infty} p(x_n, x_m) > 0$, it follows by Step 1, that

$$\limsup_{n,m\to+\infty} \frac{1}{1 - [\alpha(p(x_n, x_m)) + 2\gamma(p(x_n, x_m))]} = +\infty,$$

for which

$$\limsup_{n,m\to+\infty}[\alpha(p(x_n,x_m))+2\gamma(p(x_n,x_m))]=1. \tag{2.2}$$

On the other hand, since

$$\frac{\alpha(t)+\beta(t)+\gamma(t)}{1-(\beta(t)+\gamma(t))}<1, \tag{2.3}$$

then $\beta(t)+\gamma(t)<\frac{1}{2}$, for all $t\in[0,+\infty)$.
Hence, since $\beta(t)\geq\gamma(t)$, for all $t\in[0,+\infty)$, by using (2.2) and (2.3)

$$\begin{aligned}
&\limsup_{n,m\to+\infty}\frac{\alpha(p(x_n,x_m))+\beta(p(x_n,x_m))+\gamma(p(x_n,x_m))}{1-[\beta(p(x_n,x_m))+\gamma(p(x_n,x_m))]}\\
&\quad\geq\limsup_{n,m\to+\infty}\frac{\alpha(p(x_n,x_m))+2\gamma(p(x_n,x_m))}{1-[\beta(p(x_n,x_m))+\gamma(p(x_n,x_m))]}\\
&\quad\geq\limsup_{n,m\to+\infty}[\alpha(p(x_n,x_m))+2\gamma(p(x_n,x_m))]=1.
\end{aligned} \tag{2.4}$$

Now since, $\frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)}\in S$, then using (2.4), we have

$$\limsup_{n,m\to+\infty}\frac{\alpha(p(x_n,x_m))+\beta(p(x_n,x_m))+\gamma(p(x_n,x_m))}{1-[\beta(p(x_n,x_m))+\gamma(p(x_n,x_m))]}=1.$$

It follows that $\limsup_{n,m\to+\infty}p(x_n,x_m)=0$ which is a contradiction. Thus Step 2 is proved.

By completeness of X, there exists $x^*\in X$ such that $\lim_{n\to+\infty}p(x_n,x^*)=p(x^*,x^*)=0$. Now, we have

$$\begin{aligned}
p(x^*,Tx^*)&\leq p(x^*,x_{n+1})+(x_{n+1},Tx^*)\\
&\leq p(x^*,x_{n+1})+H_p(Tx_n,Tx^*)\\
&\leq p(x^*,x_{n+1})+\alpha(p(x_n,x^*))p(x_n,x^*)\\
&\quad+\beta(p(x_n,x^*))[p(x_n,Tx_n)+p(x^*,Tx^*)]\\
&\quad+\gamma(p(x_n,x^*))[p(x_n,Tx^*)+p(x^*,Tx_n)]
\end{aligned}$$

for all $n\in\mathbb{N}$. Therefore

$$\begin{aligned}
p(x^*,Tx^*)&\leq p(x^*,x_{n+1})+\alpha(p(x_n,x^*))p(x_n,x^*)\\
&\quad+[\beta(p(x_n,x^*))+\gamma(p(x_n,x^*))][p(x_n,x_{n+1})+p(x^*,Tx^*)\\
&\quad+p(x_n,Tx^*)+p(x^*,x_{n+1})]
\end{aligned}$$

On the other hand, since $\beta(t)+\gamma(t)<\frac{1}{2}$, for all $t\in[0,+\infty)$, then we have

$$\begin{aligned}
p(x^*,Tx^*)&<p(x^*,x_{n+1})+\alpha(p(x_n,x^*))p(x_n,x^*)\\
&\quad+\frac{1}{2}[p(x_n,x_{n+1})+p(x^*,Tx^*)+p(x_n,Tx^*)+p(x^*,x_{n+1})].
\end{aligned}$$

For $n \to +\infty$ it follows $p(x^*, Tx^*) < p(x^*, Tx^*)$, absurd. Then $p(x^*, Tx^*) = 0 = p(x^*, x^*)$. We know that Tx^* is closed then, by Lemma 1.8, we get $x^* \in Tx^*$. □

Corollary 2.4. *Let (X, p) be a complete partial metric space and let $T : X \to CB^p(X)$ be partial-special multi-valued mapping such that*

$$p(Tx, Ty) \leq \alpha(p(x, y))p(x, y) + \beta(p(x, y))[p(x, Tx) + p(y, Ty)]+ \\ + \gamma(p(x, y))[p(x, Ty) + p(y, Tx)]$$

for all $x, y \in X$, where α, β, γ, are mappings from $[0, +\infty)$ into $[0, 1)$ such that $\frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)} \in S$ and $\beta(t) \geq \gamma(t)$ for all $t \in [0, +\infty)$. Then T has a fixed point.

We observed (Definition 2.1) that every single valued mapping, in a partial metric space, is partial-special multi-valued mapping. Then, putting $\beta = \gamma = 0$ in Theorem 2.3, we obtain the following corollary that is a partial-special multi-valued version of Geraghty's fixed point theorem.

Corollary 2.5. *Let (X, p) be a complete partial metric space and let $T : X \to CB^p(X)$ be partial-special multi-valued mapping such that:*

$$H_p(Tx, Ty) \leq \alpha(p(x, y))p(x, y)$$

for all $x, y \in X$, where $\alpha \in S$. Then T has a fixed point.

Corollary 2.6. *Let (X, p) be a complete partial metric space and let $T : X \to CB^p(X)$ be partial-special multi-valued mapping such that*

$$H_p(Tx, Ty) \leq \beta(p(x, y))[D(x, Tx) + D(y, Ty)]$$

for all $x, y \in X$, where β is a mapping from $[0, +\infty)$ into $[0, \frac{1}{2})$ such that $\frac{\beta}{1-\beta} \in S$. Then T has a fixed point.

Example 2.7. Let $X = \left\{0, \frac{1}{3}, \frac{1}{9}\right\}$. Define $p(x, y) = \frac{1}{2}|x - y| + \frac{1}{2}\max\{x, y\}$. Define mapping $T(x) : X \to CB^p(X)$,

$$T(x) = \begin{cases} \{0\} & \text{if } x = 0, \frac{1}{9} \\ \left\{0, \frac{1}{9}\right\} & \text{if } x = \frac{1}{3}. \end{cases}$$

First: We prove that T is partial-special multi-valued mapping, indeed:

$$\inf_{y \in Tx} \{p(x, y) + p(y, z)\} = p(x, Tx) + p(z, Tx), \quad \forall\ x, z \in X.$$

If $x \in \{0, \frac{1}{9}\}$ and $z = \frac{1}{3}$ then $Tx = \{0\}$ and:

$$\inf_{y \in Tx} \{p(x, y) + p(y, z)\} = \left\{p(x, 0) + p(0, \frac{1}{3})\right\} = (x + \frac{1}{3});$$

$$p(x, Tx) + p(z, Tx) = p(x, 0) + p(\frac{1}{3}, 0) = (x + \frac{1}{3}).$$

If $x = \frac{1}{3}$ and $z = 0$ then $Tx = \{0, \frac{1}{9}\}$ and:

$$\begin{aligned}
\inf_{y\in Tx}\{p(x,y)+p(y,z)\} &= \inf_{y\in Tx}\left\{p(\frac{1}{3},y)+\inf\{p(0,0),p(\frac{1}{9},0)\}\right\}\\
&= \inf_{y\in Tx}\left\{p(\frac{1}{3},Tx)+0\right\}\\
&= \inf\{p(\frac{1}{3},0),\ p(\frac{1}{3},\frac{1}{9})\}\\
&= \inf\{\frac{1}{3},\ \frac{5}{18}\} = \frac{5}{18};\\
p(x,Tx)+p(z,Tx) &= p(\frac{1}{3},Tx)+p(0,Tx)\\
&= \inf\{p(\frac{1}{3},0),p(\frac{1}{3},\frac{1}{9})\}+\inf\{p(0,0),p(0,\frac{1}{9})\} = \{\frac{5}{18}+0\}\\
&= \frac{5}{18}.
\end{aligned}$$

If $x = \frac{1}{9}$ and $z = 0$ then $Tx = 0$ and:

$$\inf_{y\in Tx}\{p(x,y)+p(y,z)\} = \left\{p(\frac{1}{9},0)+p(0,0)\right\} = \frac{1}{9};$$

$$p(x,Tx)+p(z,Tx) = p(\frac{1}{9},0)+p(0,0) = \frac{1}{9}.$$

Second: We prove that $\{0\}, \{0, \frac{1}{9}\} \in CB^p(X)$. (See Remark 1.8).
We show that $a \in \overline{\{0\}}$ if and only if $p(a,\{0\}) = p(a,a) \Leftrightarrow a = 0$ (trivial), and $a \in \overline{\{0, \frac{1}{9}\}}$ if and only if $p(a,\{0,\frac{1}{9}\}) = p(a,a)$.
If $a = 0 \ \Rightarrow\ p(a,\{0,\frac{1}{9}\}) = \inf\{p(0,0),p(0,\frac{1}{9})\} = \inf\{0,\frac{1}{9}\} = 0 = p(a,a)$.
If $a = \frac{1}{9} \ \Rightarrow\ p(a,\{0,\frac{1}{9}\}) = \inf\{p(\frac{1}{9},0),p(\frac{1}{9},\frac{1}{9})\} = \inf\{\frac{1}{9},\frac{1}{18}\} = \frac{1}{18} = p(a,a)$.
If $a = \frac{1}{3} \ \Rightarrow\ p(a,\{0,\frac{1}{9}\}) = \inf\{p(\frac{1}{3},0),p(\frac{1}{3},\frac{1}{9})\} = \inf\{\frac{1}{3},\frac{5}{18}\} = \frac{5}{18} \neq p(a,a)$.

Third: If $\beta(t) = \frac{1}{t^2+2} \ \Rightarrow\ \frac{\beta}{1-\beta} = \frac{1}{t^2+1} \in S$ then it is possible to test the contractive condition:

$$H_p(Tx,Ty) \le \beta(p(x,y))[D(x,Tx)+D(y,Ty)]$$

for all $x, y \in X$. Indeed:
If $x = \frac{1}{3}$ and $y \in \{0, \frac{1}{9}\}$ then $Tx = \{0, \frac{1}{9}\}$ and $Ty = \{0\}$, it follows:

$$H_p(Tx,Ty) = H_p(\{0,\frac{1}{9}\},0) = \sup\{p(0,0),p(0,\frac{1}{9})\} = \frac{1}{9}.$$

If $y = 0$ then

$$\begin{aligned}\beta(p(x,y))[D(x,Tx)+D(y,Ty)] &= \beta(p(\frac{1}{3},0))[D(\frac{1}{3},\{0,\frac{1}{9}\})\}+D(0,0)]\\ &= \beta(\frac{1}{3})[\inf\{p(\frac{1}{3},0),p(\frac{1}{3},\frac{1}{9})\}+0]\\ &= \frac{1}{\frac{1}{9}+2}[\inf\{\frac{1}{3},\frac{5}{18}\}] = \frac{5}{38},\end{aligned}$$

and the contractive condiction becomes $\frac{1}{9} \leq \frac{5}{38}$ (true).
If $y = \frac{1}{9}$ then

$$\begin{aligned}\beta(p(x,y))[D(x,Tx)+D(y,Ty)] &= \beta(p(\frac{1}{3},\frac{1}{9}))[D(\frac{1}{3},\{0,\frac{1}{9}\})\}+D(\frac{1}{9},0)]\\ &= \beta(\frac{5}{18})[\inf\{\frac{1}{3},\frac{5}{18}\}+\frac{1}{9}]\\ &= \frac{324}{673}[\frac{5}{18}+\frac{1}{9}] = \frac{324}{673}[\frac{7}{18}],\end{aligned}$$

and the contractive condiction becomes $\frac{1}{9} \leq \frac{324}{673}[\frac{7}{18}]$ (true).
If $x = y = 0$ then the contractive condition becomes $0 \leq 0$ (true).
If $x = y = \frac{1}{3}$ then $Tx = Ty = \{0, \frac{1}{9}\}$

$$H_p(Tx,Ty) = \sup\{p(0,0),p(0,\frac{1}{9}),p(\frac{1}{9},\frac{1}{9})\} = \frac{1}{9}$$

$D(x,Tx) = D(y,Ty) = \inf\{\frac{1}{3},\{0,\frac{1}{9}\}\} = \inf\{p(\frac{1}{3},0),p(\frac{1}{3},\frac{1}{9})\} = \frac{5}{18}$
and the contractive condiction becomes $\frac{1}{9} \leq \beta(p(\frac{1}{3},\frac{1}{3}))[\frac{5}{18}+\frac{5}{18}] = \beta(\frac{1}{6})[\frac{10}{18}] = \frac{36}{73}[\frac{10}{18}]$ (true).
If $x = y = \frac{1}{9}$ then $Tx = Ty = \{0\}$, it follows that $H_p(Tx,Ty) = 0$ and

$$\beta(p(x,y))[D(x,Tx)+D(y,Ty)] = \beta(p(\frac{1}{9},\frac{1}{9}))[p(\frac{1}{9},0)+p(\frac{1}{9},0)] > 0.$$

All conditions of Corollary 2.6 are verified and $x = 0$ is the fixed point.

Corollary 2.8. *Let (X,p) be a complete partial metric space and let $T : X \to CB^p(X)$ be partial-special multi-valued mapping such that*

$$H_p(Tx,Ty) \leq \alpha(p(x,y))p(x,y)+\beta(p(x,y))[D(x,Tx)+D(y,Ty)]$$

for all $x,y \in X$, where α,β are mappings from $[0,+\infty)$ into $[0,1)$ such that $\frac{\alpha+\beta}{1-\beta} \in S$. Then T has a fixed point.

Acknowledgement(s) : The author would like to thank the referees for their comments and suggestions on the manuscript. The author is supported by Università degli Studi di Palermo (Local University Project R.S. ex 60%).

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(Received 26 March 2014)
(Accepted 21 March 2015)