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Convergence Theorems for Common Fixed Points of Two G-Nonexpansive Mappings in a Banach Space with a Directed Graph

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Abstract : The purpose of this paper is to prove weak and strong convergence theorems of a new iterative scheme for common fixed points of two G-nonexpansive mappings in a Banach space endowed with a directed graph. Also, we give an example for numerical result of our main theorem and compare the rate of convergence of our iteration and Ishikawa iteration. Furthermore, we give some consequences of those theorems for two monotone nonexpansive mappings in a Banach space.

Keywords : fixed point theorems; G-nonexpansive mapping; edge-preserving; directed graph.
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1 Introduction

Let B be a real Banach space and C be a nonempty subset of B. A mapping $T : C \to C$ is called *nonexpansive* if $\|Tu - Tv\| \leq \|u - v\|$ for all $u, v \in C$. A point

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$x \in C$ is said to be a fixed point of T if $x = Tx$. We use the notation $F(T)$ for the set of fixed points of T.

Let $V(G)$ be a set of vertices of a directed graph G and $E(G)$ be a set of edges of G which contains all the loops, that is $(u, u) \in E(G)$ for any $u \in V(G)$). We can identify G with $(V(G), E(G))$, where G has no paralled edges. The notation G^{-1} is stand for the graph obtained from G by reversing the direction of edges, that is $E(G^{-1}) = \{(u, v) \mid (v, u) \in E(G)\}$. Let $\hat{G}$ be the undirected graph obtained from G by ignoring the direction of edges (that is $E(G) \cup E(G^{-1}) = E(\hat{G})$.

Many authors have investigated fixed point theorems for nonexpansive mappings on both Hilbert spaces and Banach spaces.

Browder [1], by using Banach contraction principle, proved a strong convergence theorem for a fixed point of a nonexpansive mapping in a Hilbert space.

Jachymski [2] was the first who introduced a fixed point theory with a graph and studied a fixed point theorem of G-contraction mappings.

In 2015, Alfuraidan [3] showed the existence of a fixed point of G-monotone pointwise contraction mappings on a Banach space with a graph.

Recently, Tiammee *et al.* [4] proved Browder's convergence theorem for a G–nonexpansive mapping $T : C \to C$ in a Hilbert space H with a directed graph. They also used the Halpern iterative scheme for approximating a fixed point of G-nonexpansive mappings. As far as we know, no attention paid on how to approximating a commom fixed point of two G-nonexpansive mappings in a setting of a Banach space with a directed graph.

Motivated by those previous works, in this paper, we introduce a new iterative scheme for finding a common fixed point of two G-nonexpansive mappings in Banach spaces endowed with a directed graph and we also prove weak and strong convergence theorems of the propose method for this type of mappings.

2 Preliminaries

In this section, we recall some useful definitions, concepts and results which are needed to prove our main results.

Jachymski [2] was the first who combined two concepts of fixed point theory and graph theory to study fixed point theorm of G-contraction in a complete metric space endowed with a directed graph.

Let (X, d) be a complete metric space and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $E(G)$ contains all loops, that is $\triangle = \{(u, u) : u \in X\} \subseteq E(G)$. A mapping $T : X \to X$ is a *G-contraction* if T preserves edges of G, i.e., for all $u, v \in X$,

$$(u, v) \in E(G) \Rightarrow (Tu, Tv) \in E(G)$$

and there exists $\beta \in (0, 1)$ such that for any $u, v \in X$,

$$(u, v) \in E(G) \Rightarrow d(Tu, Tv) \leq \beta d(u, v).$$

Also, he proved under some conditions that let (X, d) be a complete metric space, and a triple (X, d, G) has the following property: for any $(u_n)_{n\in\mathbb{N}} \in X$, if $u_n \to u$ and $(u_n, u_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then there is a subsequence $(u_{k_n})_{n\in\mathbb{N}}$ with $(u_{k_n}, u) \in E(G)$ for $n \in \mathbb{N}$. Then $T : X \to X$ has a fixed point if and only if $X_T := \{u \in X : (u, Tu)\} \neq \emptyset$.

The above result has been improved and extended in many ways (see [5, 6, 7]). Next, we will give the notion of G-continuous. (see [2])

A mapping $T : X \to X$ is called *G-continuous* if given $u \in X$ and a sequence $(u_n)_{n\in\mathbb{N}}$, for $n \in \mathbb{N}$,

$$u_n \to u \quad \textit{and} \quad (u_n, u_{n+1}) \in E(G) \quad \text{imply} \quad Tu_n \to Tu.$$

Let B be a Banach space and C be a nonempty convex subset of B. Let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $T : C \to C$. A mapping T is called *G-nonexpansive* if the following conditions hold:

(i) T is edge-preserving;

(ii) $\|Tu - Tv\| \leq \|u - v\|$, whenever $(u, v) \in E(G)$ for any $u, v \in C$.

Recall that a Banach space B satisfies *Opial's condition* [8], that is, for any sequence $\{u_n\}$ with $u_n \rightharpoonup u$, the inequality

$$\liminf_{n\to\infty} \|u_n - u\| < \liminf_{n\to\infty} \|u_n - v\|$$

holds for every $v \in X$ with $v \neq u$.

Let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$. A mapping $T : C \to C$ is called *G-demiclosed* at 0 if for any sequence $\{x_n\} \subseteq C$, $(x_n, x_{n+1}) \in E(G)$, $x_n \rightharpoonup x$ and $Tx_n \to 0$, then $Tx = 0$.

Lemma 2.1. [9] *Let B be a uniformly convex Banach space and $\{\beta_n\}$ a sequence in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$. Suppose $\{u_n\}$ and $\{v_n\}$ are sequences in X such that* $\limsup_{n\to\infty} \|u_n\| \leq r$, $\limsup_{n\to\infty} \|v_n\| \leq r$ *and* $\limsup_{n\to\infty} \|\beta_n u_n + (1 - \beta_n)v_n\| = r$ *hold for some $r \geq 0$. Then* $\lim_{n\to\infty} \|u_n - v_n\| = 0$.

Lemma 2.2. [10] *Let B be a Banach space which satisfices Opial's condition and let $\{u_n\}$ be a sequence in B. Let $p, q \in B$ be such that* $\lim_{n\to\infty} \|u_n - p\|$ *and* $\lim_{n\to\infty} \|u_n - q\|$ *exist. If $\{u_{n_k}\}$ and $\{u_{n_j}\}$ are subsequences of $\{u_n\}$ which converges weakly to p and q, respectively, then $p = q$.*

To have the main results, we need the following property .

Property *G*: Let X be a normed space and $\emptyset \neq C \subseteq X$. Let $G = (V(G), E(G))$ be a directed graph with $V(G) = C$. We said that C have the Property G if for every sequence $\{u_n\}$ in C with $(u_n, u_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ and weakly convergence to $u \in C$, there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $(u_{n_k}, u) \in E(G)$ for all $n \in \mathbb{N}$.

Next, we need some basic definitions of a domination in a graph to obtain results for G-nonexpansive mappings (see [11, 12]).

Let $G = (V(G), E(G))$ be a directed graph. A set $D \subseteq V(G)$ is said to be a *dominating set* if for every $v \in V(G)\backslash D$ there exists $d \in D$ such that $(d, v) \in E(G)$. Let $v \in V(G)$ and set $D \subseteq V(G)$. We say that v is dominated by D if $(d, v) \in E(G)$ for any $d \in D$. Let $A, B \subseteq V(G)$. If $(a, b) \in E(G)$ for all $a \in A$ and all $b \in B$, then we say that A dominates B. In this paper, we assume that $E(G)$ contains all loops.

A graph G is called *transitive* if for any $u, v, w \in V(G)$ such that (u, v) and (v, w) are in $E(G)$, then $(u, w) \in E(G)$.

3 Main Results

In this section, we present some weak and strong convergence theorems of a new iterative process for two G-nonexpansive mappings in a Banach space endowed with a directed graph. To this end, the following iterative schemes are introduced:

$$\begin{cases} x_0 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n T_2 x_n, \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T_1 y_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{\alpha_n\} \in [a, 1-a]$, $\{\beta_n\} \in [b, 1-b]$ for some $a, b \in (0, 1)$ and C is a closed convex subset of a real Banach space X and T_1, $T_2 : C \to C$ are two G-nonexpansive mappings.

The following results are needed for proving our main results.

Proposition 3.1. *Let X be a normed space and C be a convex subset of X. Let $G = (V(G), E(G))$ be a directed graph and transitive such that $V(G) = C$ and $E(G)$ is convex. Let $T_1, T_2 : C \to C$ be edge-preserving and let $\{x_n\}$ and $\{y_n\}$ be sequences defined by the recursion (3.1). Suppose that (x_0, y_0), (y_0, x_0), (x_0, T_1y_0), (T_1y_0, x_0), (y_0, T_2x_0), (y_0, y_0), (y_0, T_1y_0), (T_1y_0, y_0), (T_2x_0, y_0), (y_0, T_2y_0), (T_1x_0, x_0) and (T_2x_0, x_0) are in $E(G)$. Then (x_n, x_{n+1}), (x_{n+1}, x_n), (x_n, x_0), (x_0, x_n), (y_0, y_n), (y_n, y_0), (y_0, x_n), (x_n, y_0), (y_0, T_1y_n), (y_0, T_2x_n), (T_2x_n, y_0), (T_2x_n, x_0), (x_n, y_n) and (x_n, T_1y_n) are in $E(G)$ for all $n \in \mathbb{N}$.*

Proof. We shall prove the results by using induction. Since (x_0, y_0), (x_0, T_1y_0) are in $E(G)$ and $E(G)$ is convex, we obtain $(x_0, x_1) \in E(G)$. Thus $(T_2x_0, T_2x_1) \in E(G)$ because T_2 is edge-preserving. Since (y_0, T_2x_0), (T_2x_0, T_2x_1) are in $E(G)$ and G is transitive, we get $(y_0, T_2x_1) \in E(G)$. Since (y_0, y_0), (y_0, T_1y_0) are in $E(G)$ and $E(G)$ is convex, we obtain $(y_0, x_1) \in E(G)$. Since (y_0, T_2x_1), (y_0, x_1) are in $E(G)$ and $E(G)$ is convex, we get $(y_0, y_1) \in E(G)$. Since (y_0, y_0), (T_1y_0, y_0) are in $E(G)$ and $E(G)$ is convex, we get $(x_1, y_0) \in E(G)$. Since (y_0, y_1), (x_1, y_0) are in $E(G)$ and G is transitive, we get $(x_1, y_1) \in E(G)$. Given (y_0, x_0), (T_1y_0, x_0) are in $E(G)$ and since $E(G)$ is convex, we get $(x_1, x_0) \in E(G)$. Then $(T_2x_1, T_2x_0) \in E(G)$ because T_2 is edge-preserving. Since (T_2x_1, T_2x_0), (T_2x_0, y_0) are in $E(G)$ and G is transitive, we obtain $(T_2x_1, y_0) \in E(G)$. Since (x_1, y_0), (T_2x_1, y_0) are in $E(G)$ and $E(G)$ is convex, we get $(y_1, y_0) \in E(G)$. Since (T_2x_1, y_0), (y_0, T_1y_0) are in $E(G)$

and G is transitive, we get $(T_2x_1, T_1y_0) \in E(G)$. Since (T_2x_1, T_1y_0), (T_1y_0, T_1y_1) are in $E(G)$ and G is transitive, we get $(T_2x_1, T_1y_1) \in E(G)$. Similarly, we obtain (x_1, T_1y_1), (y_0, T_1y_1) and $(T_2x_1, x_0) \in E(G)$.

Next, assume that (x_k, x_{k+1}), (x_{k+1}, x_k), (y_0, y_k), (y_k, y_0), (x_k, y_k), (x_k, T_1y_k), (y_0, T_1y_k), (x_k, x_0), (T_2x_k, x_0), (x_0, x_k), (y_0, x_k), (x_k, y_0), (y_0, T_2x_k) and (T_2x_k, y_0) are in $E(G)$. From (y_0, y_k), (y_0, T_1y_k) are in $E(G)$ and $E(G)$ is convex, we get $(y_0, x_{k+1}) \in E(G)$. Since (y_0, T_2y_0), (T_2y_0, T_2x_{k+1}) are in $E(G)$ and G is transitive, we get $(y_0, T_2x_{k+1}) \in E(G)$. Since (y_0, x_{k+1}), (y_0, T_2x_{k+1}) are in $E(G)$ and $E(G)$ is convex, we get $(y_0, y_{k+1}) \in E(G)$. Since (y_k, y_0), (y_0, y_{k+1}) are in $E(G)$ and G is transitive, we get $(y_k, y_{k+1}) \in E(G)$. Since (x_k, x_0), (T_2x_k, x_0) are in $E(G)$ and $E(G)$ is convex, we get $(y_k, x_0) \in E(G)$. Since (T_1y_k, T_1x_0), (T_1x_0, x_0) are in $E(G)$ and G is transitive, we get $(T_1y_k, x_0) \in E(G)$. Since (y_k, x_0), (T_1y_k, x_0) are in $E(G)$ and $E(G)$ is convex, we get $(x_{k+1}, x_0) \in E(G)$. Since (x_{k+1}, x_0), (x_0, y_0), (y_0, y_{k+1}) are in $E(G)$ and G is transitive, we get $(x_{k+1}, y_{k+1}) \in E(G)$. Since (y_0, T_1y_0), (T_1y_0, T_1y_{k+1}) are in $E(G)$ and G is transitive, we get $(y_0, T_1y_{k+1}) \in E(G)$. Since (x_{k+1}, x_k), (x_k, y_k), (y_k, y_0), (y_0, T_1y_{k+1}), are in $E(G)$ and G is transitive, we get $(x_{k+1}, T_1y_{k+1}) \in E(G)$. Since (x_{k+1}, y_{k+1}), (x_{k+1}, T_1y_{k+1}), are in $E(G)$ and $E(G)$ is convex, we obtain $(x_{k+1}, x_{k+2}) \in E(G)$. Since (T_2x_{k+1}, T_2x_0), (T_2x_0, x_0) are in $E(G)$ and G is transitive, we get $(T_2x_{k+1}, x_0) \in E(G)$. Since (x_{k+1}, x_0), (x_0, y_0) are in $E(G)$ and G is transitive, we get $(x_{k+1}, y_0) \in E(G)$. Since $(T_2x_{k+1}, x_0),)$ (x_0, y_0) are in $E(G)$ and G is transitive, we get $(T_2x_{k+1}, y_0) \in E(G)$. Since (x_{k+1}, y_0), (T_2x_{k+1}, y_0) are in $E(G)$ and $E(G)$ is convex, we get $(y_{k+1}, y_0) \in E(G)$. Since (x_0, x_k), (x_k, x_{k+1}) are in $E(G)$ and G is transitive, we get $(x_0, x_{k+1}) \in E(G)$. Since (y_{k+1}, x_{k+1}), (T_1y_{k+1}, x_{k+1}) are in $E(G)$ and $E(G)$ is convex, we get $(x_{k+2}, x_{k+1}) \in E(G)$. Thus, by induction, we conclude that (x_n, x_{n+1}), (x_{n+1}, x_n), (x_n, x_0), (x_0, x_n), (y_0, y_n), (y_n, y_0), (y_0, x_n), (x_n, y_0), (y_0, T_1y_n), (y_0, T_2x_n), (T_2x_n, y_0), (T_2x_n, x_0), (x_n, y_n) and (x_n, T_1y_n) are in $E(G)$ for all $n \in \mathbb{N}$. □

Proposition 3.2. *Let X be a normed space and $G = (V(G), E(G))$ a directed graph and transitive with $V(G) = X$ and $E(G)$ is convex. Let $T : X \to X$ be a G-nonexpansive mapping. If X has the Property G, then T is G-continuous.*

Proof. Let $\{x_n\}$ be a sequence in X such that $x_n \to x$. We shall show that $Tx_n \to Tx$. To show this, let $\{Tx_{n_k}\}$ be a subsequence of $\{Tx_n\}$. Since $(x_n, x_{n+1}) \in G$ and G is a transitive, we obtain $(x_{n_k}, x_{n_k+1}) \in E(G)$. Since $x_{n_k} \to x$ and $(x_{n_k}, x_{n_k+1}) \in E(G)$, by Property G, there is a subsequence $\{x_{n_{k'}}\}$ of $\{x_{n_k}\}$ such that $(x_{n_{k'}}, x) \in E(G)$ for all $n \in \mathbb{N}$. Since T is a G-nonexpansive mapping and $(x_{n_{k'}}, x) \in E(G)$, we obtain

$$\|Tx_{n_{k'}} - Tx\| \leq \|x_{n_{k'}} - x\| \to 0$$

as $k' \to \infty$. Thus $Tx_{n_{k'}} \to Tx$. By the double extract subsequence principle we conclude that $Tx_n \to Tx$. Therefore T is G-continuous. □

Lemma 3.3. *Let X be a real uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $E(G)$ is convex. Let T_1, $T_2 : C \to C$ be two G-nonexpansive mappings with $z_0 \in F := F(T_1) \cap F(T_2) \neq \emptyset$, $F(T_1) \times F(T_1) \subset E(G)$ and $F(T_2) \times F(T_2) \subset E(G)$. Let $\{x_n\}$ and $\{y_n\}$ be sequences defined by the recursion (3.1). Assume that $(y_0, z_0) \in E(G)$ and z_0 is dominated by $\{x_n\}$ and $\{y_n\}$. Then $\lim_{n\to\infty} \|x_n - z_0\|$ exists.*

Proof. Since $z_0 \in F$ and z_0 is dominated by $\{x_n\}$ and $\{y_n\}$, we obtain (x_n, z_0) and $(y_n, z_0) \in E(G)$. By the definition of x_n, we obtain

$$\begin{aligned}
\|x_{n+1} - z_0\| &\leq (1-\alpha_n)\|y_n - z_0\| + \alpha_n\|T_1 y_n - z_0\| \\
&\leq (1-\alpha_n)\|y_n - z_0\| + \alpha_n\|y_n - z_0\| \\
&= \|y_n - z_0\| \\
&\leq (1-\beta_n)\|x_n - z_0\| + \beta_n\|T_2 x_n - T_2 z_0\| \\
&\leq (1-\beta_n)\|x_n - z_0\| + \beta_n\|x_n - z_0\| \\
&= \|x_n - z_0\|.
\end{aligned}$$

Thus $\|x_{n+1} - z_0\| \leq \|x_n - z_0\|$ and hence $\lim_{n\to\infty} \|x_n - z_0\|$ exists, as desired. □

Lemma 3.4. *Let X be a real uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $G = (V(G), E(G))$ be a directed graph and transitive such that $V(G) = C$ and $E(G)$ is convex. Let T_1, $T_2 : C \to C$ be two G-nonexpansive mappings with $z_0 \in F := F(T_1) \cap F(T_2) \neq \emptyset$, $F(T_1) \times F(T_1) \subset E(G)$ and $F(T_2) \times F(T_2) \subset E(G)$. Let $\{x_n\}$ and $\{y_n\}$ be sequences defined by the recursion (3.1). Assume that $(y_0, z_0) \in E(G)$ and z_0 is dominated by $\{x_n\}$ and $\{y_n\}$. Suppose (x_0, y_0), (y_0, x_0), $(x_0, T_1 y_0)$, $(T_1 y_0, x_0)$, $(y_0, T_2 x_0)$, (y_0, y_0), $(y_0, T_1 y_0)$, $(T_1 y_0, y_0)$, $(y_0, T_2 y_0)$, $(T_1 x_0, x_0)$ and $(T_2 x_0, x_0)$ are in $E(G)$. Then*

(*i*) $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0$;

(*ii*) $\lim_{n\to\infty} \|x_n - T_2 x_n\| = 0$.

Proof. Let $z_0 \in F$. By Proposition 3.1, we obtain (x_n, x_{n+1}), (x_{n+1}, x_n), (x_n, x_0), (x_0, x_n), (y_0, y_n), (y_n, y_0), (y_0, x_n), (x_n, y_0), $(y_0, T_1 y_n)$, $(y_0, T_2 x_n)$, $(T_2 x_n, y_0)$, $(T_2 x_n, x_0)$, (x_n, y_n) $(x_n, T_1 y_n)$ and are in $E(G)$ for all $n \in \mathbb{N}$. Since $z_0 \in F$ and $(y_0, z_0) \in E(G)$ and z_0 is dominated by $\{x_n\}$ and $\{y_n\}$ and by Lemma 3.3, we obtain $\lim_{n\to\infty} \|x_n - z_0\|$ exists. Assume that $\lim_{n\to\infty} \|x_n - z_0\| = c$. For each $i = 1, 2$, we have

$$\begin{aligned}
\|x_n - T_i x_n\| &\leq \|x_n - z_0\| + \|z_0 - T_i x_n\| \\
&= \|x_n - z_0\| + \|T_i z_0 - T_i x_n\| \\
&\leq \|x_n - z_0\| + \|z_0 - x_n\|.
\end{aligned}$$

If $c = 0$, then $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$. Let $c > 0$. From (3.1), we obtain

$$\begin{aligned}\|y_n - z_0\| &\le (1-\beta_n)\|x_n - z_0\| + \beta_n\|T_2 x_n - T_2 z_0\| \\ &\le (1-\beta_n)\|x_n - z_0\| + \beta_n\|x_n - z_0\| \\ &= \|x_n - z_0\|.\end{aligned}$$

Thus

$$\limsup_{n\to\infty} \|y_n - z_0\| \le c. \tag{3.2}$$

Since $\|T_2 x_n - z_0\| \le \|x_n - z_0\|$, we obtain

$$\limsup_{n\to\infty} \|T_2 x_n - z_0\| \le c. \tag{3.3}$$

Since $\|T_1 y_n - z_0\| \le \|y_n - z_0\|$ and by using (3.2), we obtain

$$\limsup_{n\to\infty} \|T_1 y_n - z_0\| \le c.$$

Since $c = \lim_{n\to\infty} \|x_{n+1} - z_0\| = \lim_{n\to\infty} \|(1-\alpha_n)(y_n - z_0) + \alpha_n(T_1 y_n - z_0)\|$, by Lemma 2.1, we obtain

$$\lim_{n\to\infty} \|T_1 y_n - y_n\| = 0. \tag{3.4}$$

By the definition of x_{n+1}, we have

$$\begin{aligned}\|x_{n+1} - z_0\| &\le \alpha_n\|T_1 y_n - z_0\| + (1-\alpha_n)\|y_n - z_0\| \\ &\le \alpha_n\|y_n - z_0\| + (1-\alpha_n)\|y_n - z_0\| \\ &= \|y_n - z_0\|.\end{aligned}$$

Thus $c \le \liminf_{n\to\infty} \|y_n - z_0\|$. From (3.2) and the definition of y_n, we obtain

$$c = \lim_{n\to\infty} \|y_n - z_0\| = \lim_{n\to\infty} \|\beta_n(T_2 x_n - z_0) + (1-\beta_n)(x_n - z_0)\|.$$

By (3.3), Lemma 3.3 and Lemma 2.1, we obtain

$$\lim_{n\to\infty} \|T_2 x_n - x_n\| = 0. \tag{3.5}$$

Since $\|y_n - x_n\| = \beta_n\|T_2 x_n - x_n\|$ and by (3.5), we have

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.6}$$

Since $\|x_{n+1} - x_n\| \le \|y_n - x_n\| + \alpha_n\|T_1 y_n - y_n\|$ and by using (3.4) and (3.6), we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.7}$$

Since $\|x_{n+1}-T_1y_n\| \leq \|x_{n+1}-x_n\|+\|x_n-y_n\|+\|y_n-T_1y_n\|$, and by (3.7), (3.4) and (3.6), we obtain

$$\lim_{n\to\infty}\|x_{n+1}-T_1y_n\|=0. \tag{3.8}$$

Notice that

$$\begin{aligned}\|x_n-T_1x_n\| &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-T_1y_n\|+\|T_1y_n-T_1x_n\|\\ &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-T_1y_n\|+\|y_n-x_n\|.\end{aligned} \tag{3.9}$$

From (3.9) and by (3.7), (3.8) and (3.6), we obtain

$$\lim_{n\to\infty}\|x_n-T_1x_n\|=0,$$

as desired. □

The following result is crucial for our main results.

Proposition 3.5. *Let X be a Banach space satisfying the Opial's condition, and C be a nonempty closed subset of X having property G. Let $G=(V(G),E(G))$ be a directed graph and transitive such that $V(G)=C$ and $E(G)$ is convex. Let $T_1,T_2:C\to C$ be two G-nonexpansive mappings with $F:=F(T_1)\cap F(T_2)\neq\emptyset$, $F(T_1)\times F(T_1)\subset E(G)$ and $F(T_2)\times F(T_2)\subset E(G)$. Let $\{x_n\}$ and $\{y_n\}$ be sequences defined by the recursion (3.1). Assume that (x_0,y_0), (y_0,x_0), (x_0,T_1y_0), (T_1y_0,x_0), (y_0,T_2x_0), (y_0,y_0), (y_0,T_1y_0), (T_1y_0,y_0), (y_0,T_2y_0), (T_1x_0,x_0) and (T_2x_0,x_0) are in $E(G)$. If F dominates sequences $\{x_n\}$ and $\{y_n\}$, then $(I-T_1)$ and $(I-T_2)$ are demiclosed at zero.*

Proof. Suppose that $\{x_n\}$ is a sequence in C converges weakly to x and $(x_n,x_{n+1})\in E(G)$ for all $n\in\mathbb{N}$. Since C has property G, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k},x)\in E(G)$ for all $k\in\mathbb{N}$. From Lemma 3.4, we obtain

$$\lim_{k\to\infty}\|x_{n_k}-T_1x_{n_k}\|=0 \quad\text{and}\quad \lim_{k\to\infty}\|x_{n_k}-T_2x_{n_k}\|=0.$$

Suppose, for contradiction, that $x\neq T_1x$ and $x\neq T_2x$. Then, by the Opial's condition, we obtain

$$\begin{aligned}\limsup_{k\to\infty}\|x_{n_k}-x\| &< \limsup_{k\to\infty}\|x_{n_k}-T_1x\|\\ &\leq \limsup_{k\to\infty}(\|x_{n_k}-T_1x_{n_k}\|+\|T_1x_{n_k}-T_1x\|)\\ &\leq \limsup_{k\to\infty}\|x_{n_k}-x\|\end{aligned}$$

a contradiction. This proves that $(I-T_1)x=0$. Similary, $(I-T_2)x=0$. □

The mappings $T_1, T_2 : C \to C$ with $F := F(T_1) \cap F(T_2) \neq \emptyset$ are said to satisfy the *condition* (B) (see in [13]) if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\max\{\|x - T_1 x\|, \|x - T_2 x\|\} \geq f(d(x, F))$$

for all $x \in C$. A mapping $T_1 : C \to C$ is called *demicompact* if for every bounded sequence $\{x_n\}$ in C such that $\{x_n - T_1 x_n\}$ converges has a convergent subsequence. A mapping $T_2 : C \to C$ is called *semicompact* if for every bounded sequence $\{x_n\}$ in C satisfying $\|x_n - T_2 x_n\| \to 0$ as $n \to \infty$ has a convergent subsequence.

Theorem 3.6. *Let X be a real uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $G = (V(G), E(G))$ be a directed graph and transitive such that $V(G) = C$ and $E(G)$ is convex. Let $T_1, T_2 : C \to C$ be two G-nonexpansive mappings with $F := F(T_1) \cap F(T_2) \neq \emptyset$, $F(T_1) \times F(T_1) \subset E(G)$ and $F(T_2) \times F(T_2) \subset E(G)$. Suppose that C has the Property G and T_1 and T_2 satisfy the condition (B). Let $\{x_n\}$ and $\{y_n\}$ be sequences defined by the recursion (3.1). Assume that (x_0, y_0), (y_0, x_0), $(x_0, T_1 y_0)$, $(T_1 y_0, x_0)$, $(y_0, T_2 x_0)$, (y_0, y_0), $(y_0, T_1 y_0)$, $(T_1 y_0, y_0)$, $(y_0, T_2 y_0)$, $(T_1 x_0, x_0)$ and $(T_2 x_0, x_0)$ are in $E(G)$. If F dominates sequences $\{x_n\}$ and $\{y_n\}$, then $\{x_n\}$ converges strongly to $x \in F$.*

Proof. Let $w \in F$. Then (x_n, w) and $(y_n, w) \in E(G)$ for all $n \in \mathbb{N}$. By Lemma 3.3, we have $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - w\|$ exists. Also

$$\|x_{n+1} - w\| \leq \|x_n - w\|$$

for all $n \geq 1$. This implies that $d(x_{n+1}, F) \leq d(x_n, F)$ and so $\lim_{n\to\infty} d(x_n, F)$ exists. Also, by Lemma 3.4, we obtain

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0 = \lim_{n\to\infty} \|x_n - T_2 x_n\|.$$

Since T_1 and T_2 satisfy the condition (B), we have

$$\lim_{n\to\infty} f(d(x_n, F)) = 0.$$

This implies that $\lim_{n\to\infty} d(x_n, F) = 0$. So we can find a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ and a sequence $\{w_l\} \subset F$ satisfying $\|x_{n_l} - w_l\| \leq 2^{-l}$. Put $n_{l+1} = n_l + m$ for some $m \geq 1$. Then

$$\|x_{n_{l+1}} - w_l\| \leq \|x_{n_l+m-1} - w_l\| \leq \|x_{n_l} - w_l\| \leq \frac{1}{2^l}$$

and so we have $\|w_{l+1} - w_l\| \leq \frac{3}{2^{l+1}}$. Thus $\{w_l\}$ is a cauchy sequence. Therefore there exists $y \in C$ such that $w_l \to y$ as $l \to \infty$. Thus $y \in F$ because F is closed. As a result, we have $x_{n_l} \to y$ as $l \to \infty$. Since $\lim_{n\to\infty} \|x_n - y\|$ exists by Lemma 3.3, the conclusion follows. □

Theorem 3.7. *Let X be a real uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $G = (V(G), E(G))$ be a directed graph and transitive such that $V(G) = C$ and $E(G)$ is convex. Let $T_1, T_2 : C \to C$ be two G-nonexpansive mappings with $F := F(T_1) \cap F(T_2) \neq \emptyset$, $F(T_1) \times F(T_1) \subset E(G)$ and $F(T_2) \times F(T_2) \subset E(G)$. Suppose that C has the Property G and one of T_1 and T_2 is semicompact. Let $\{x_n\}$ and $\{y_n\}$ be sequences defined by the recursion (3.1). Assume that (x_0, y_0), (y_0, x_0), (x_0, T_1y_0), (T_1y_0, x_0), (y_0, T_2x_0), (y_0, y_0), (y_0, T_1y_0), (T_1y_0, y_0), (y_0, T_2y_0), (T_1x_0, x_0) and (T_2x_0, x_0) are in $E(G)$. If F dominates sequences $\{x_n\}$ and $\{y_n\}$, then $\{x_n\}$ converges strongly to $x \in F$.*

Proof. We may assume that T_2 is semicompact. By Lemma 3.3, we have $\{x_n\}$ is bounded. From Lemma 3.4, we obtain $\lim_{n\to\infty} \|x_n - T_1x_n\| = 0 = \lim_{n\to\infty} \|x_n - T_2x_n\|$. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to z_0 \in C$ as $n_k \to \infty$. Thus $\lim_{n_k\to\infty} \|x_{n_k} - T_1x_{n_k}\| = 0 = \lim_{n_k\to\infty} \|x_{n_k} - T_2x_{n_k}\|$. By Proposition 3.2, we obtain $T_1$and T_2 are G-continuous. It follows that

$$\|z_0 - T_1z_0\| = \lim_{k\to\infty} \|x_{n_k} - T_1x_{n_k}\| = 0$$

and

$$\|z_0 - T_2z_0\| = \lim_{k\to\infty} \|x_{n_k} - T_2x_{n_k}\| = 0.$$

Therefore $z_0 \in F$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, we obtain $\{x_n\}$ converges strongly to $x \in F$. □

Theorem 3.8. *Let X be a real uniformly convex Banach space which satisfies the Opial's condition. Let C be a nonempty closed convex subset of X. Let $G = (V(G), E(G))$ be a directed graph and transitive such that $V(G) = C$ and $E(G)$ is convex. Let T_1, $T_2 : C \to C$ be two G-nonexpansive mappings with $F := F(T_1) \cap F(T_2) \neq \emptyset$, $F(T_1) \times F(T_1) \subset E(G)$ and $F(T_2) \times F(T_2) \subset E(G)$. Suppose that C has the Property G. Let $\{x_n\}$ be a sequence defined by the recursion (3.1). Assume that $x_0, y_0 \in C$ such that (x_0, y_0), (y_0, x_0), (x_0, T_1y_0), (T_1y_0, x_0), (y_0, T_2x_0), (y_0, y_0), (y_0, T_1y_0), (T_1y_0, y_0), (y_0, T_2y_0), (T_1x_0, x_0) and (T_2x_0, x_0) are in $E(G)$. If F dominates sequences $\{x_n\}$ and $\{y_n\}$, then $\{x_n\}$ converges weakly to a common fixed point in F.*

Proof. By Lemma 3.3, we have $\lim_{n\to\infty} \|x_n - z_0\|$ exists for all $z_0 \in F$. To complete the proof, we have to show that a sequence $\{x_n\}$ has a unique weak subsequential limit in F. Let u and v be weak limits of subsequence $\{x_{n_k}\}$ and $\{x_{n_j}\}$, respectively. By Lemma 3.4 and Proposition 3.5, we obtain $u, v \in F$. Thus $x_{n_k} \rightharpoonup u$ and $x_{n_j} \rightharpoonup v$. By Lemma 2.2, we get $u = v$. Thus $\{x_n\}$ converges weakly to a common fixed point in F. □

4 Numerical Example

In this section, we give numerical example for our main Theorem 3.6 and we use the idea of Rhoades [14] to compare the rate of convergence between Ishikawa iteration and our iteration. The following definition is need for our numerical example.

Definition 4.1. [14] Let X be a Banach space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be a mapping. Suppose that $\{x_n\}$ and $\{y_n\}$ are two iterations which converge to a fixed point w of T. If $\|x_n - w\| \leq \|y_n - w\|$ then $\{x_n\}$ *converges faster than* $\{y_n\}$ for all $n \geq 1$.

Example 4.2. Let $X = \mathbb{R}, C = [0, 2]$ and $\{x_n\}$ be generated by (3.1). Assume that $(x, y) \in E(G)$ if and only if $0.75 \leq x, y \leq 1.7$ and $x, y \in \mathbb{Q}$. Define two mappings $T_1, T_2 : C \to C$ by

$$T_1x = \begin{cases} \frac{5}{8}\arcsin(x-1)+1 & \text{if } x \neq \sqrt{3} \\ 0 & \text{if } x = \sqrt{3} \end{cases}$$

and

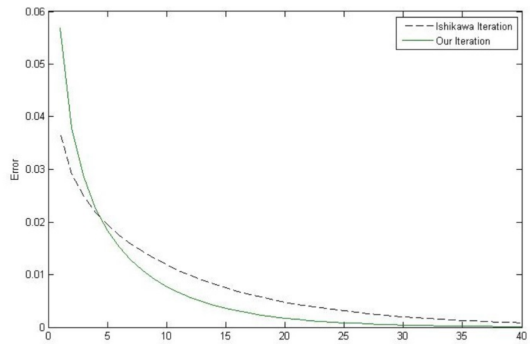
$$T_2x = \begin{cases} x^{\log(3x)} & \text{if } x \neq \sqrt{2} \\ 2 & \text{if } x = \sqrt{2} \end{cases}$$

for all $x \in C$. It is easy to see that T_1, T_2 are G- nonexpansive mappings. Let $x = \sqrt{3}, u = \sqrt{2}$ and $y = 1 = v$. Since $\|T_1x - T_1y\| > 0.75 > \|x - y\|$ and $\|T_2u - T_2v\| > 0.75 > \|u - v\|$, we obtain T_1, T_2 are not nonexpansive mappings. Let $\alpha_n = \frac{n+1}{5n+3}$ and $\beta_n = \frac{n+4}{10n+7}$. Choose $x_0 = 1.4$ We obtain the numerical result of Theorem 3.6

Table 1 Numerical result of Example 4.2.

	Ishikawa Iteration		Our Iteration		Rate of convergence	
n	k_n	$\|k_n - k_{n-1}\|$	x_n	$\|x_n - x_{n-1}\|$	$\|k_n - 1\|$	$\|x_n - 1\|$
1	1.33118	0.06882	1.26768	0.13232	0.33118	0.26768
2	1.29419	0.03699	1.21089	0.05679	0.29419	0.21089
3	1.26503	0.02917	1.17297	0.03792	0.26503	0.17297
4	1.24015	0.02488	1.14445	0.02852	0.24015	0.14445
5	1.21829	0.02186	1.12186	0.02259	0.21829	0.12186
6	1.19881	0.01948	1.10348	0.01838	0.19881	0.10348
7	1.18130	0.01751	1.08827	0.01521	0.18130	0.08827
⋮	⋮	⋮	⋮	⋮	⋮	⋮
24	1.04020	0.00367	1.00743	0.00112	0.04020	0.00743

Fig.1 Comparision of errors of the Ishikawa iteration and our iteration process.



From Table 1 and Fig. 1, we shall see that Ishikawa iteration and our iteration are converge to the fixed point 1, however our iteration converges faster than Ishikawa iteration.

5 Consequence Results for Monotone Nonexpansive Mappings

In this section, we discuss the common fixed points of two monotone nonexpansive mappings on a Banach space. Let $(X, \|\cdot\|)$ be a Banach space and $\preceq$ be a preorder on X. The linear structure of X is assumed to be compatible with the order structure in the following:

(i) $x \preceq y \Rightarrow x + z \preceq y + z$ for all $x, y, z \in X$;

(ii) $x \preceq y \Rightarrow \alpha x \preceq \alpha y$ for all $x, y \in X$ and $\alpha \in \mathbb{R}^+$;

Let $G = (V(G), E(G))$ be defined by $V(G) = C$ and $E(G) = \{(x, y) : x \preceq y\}$. This graph G is called *the graph generated by* $\preceq$.

Proposition 5.1. *Let G be a graph generated by $\preceq$. Then $E(G)$ is convex.*

Proof. To prove $E(G)$ is convex. Let $(x, y), (u, v) \in E(G)$ and $\alpha \in \mathbb{R}^+$. Then

$$\alpha x \preceq \alpha y \quad \text{and} \quad (1-\alpha)u \preceq (1-\alpha)v$$

Since $\alpha x + (1-\alpha)u \preceq \alpha y + (1-\alpha)u \preceq \alpha y + (1-\alpha)v$ and $E(G)$ is transitive, we obtain $\alpha x + (1-\alpha)u \preceq \alpha y + (1-\alpha)v$. Thus $(\alpha x + (1-\alpha)u, \alpha y + (1-\alpha)v) \in E(G)$ and hence $E(G)$ is convex. □

Definition 5.2. Let $(X, \|\cdot\|)$ be a Banach space and $\preceq$ be a preorder on X and let $\emptyset \neq C \subseteq X$. A mapping $T : C \to C$ is said to be *monotone nonexpansive* if for any $x, y \in C$,

(i) $x \preceq y \Rightarrow Tx \preceq Ty$;

(ii) $x \preceq y \Rightarrow \|Tx - Ty\| \le \|x - y\|$.

The following property is useful for the next results.

Property A: Let X be a normed space, C a nonempty subset of X and let $\preceq$ be a preorder on C. Then C is said to have the Property A if for every sequence $\{u_n\}$ in C with $u_n \preceq u_{n+1}$ for all $n \in \mathbb{N}$ and weakly converges to $u \in C$, there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \preceq u$ for all $k \in \mathbb{N}$.

Theorem 5.3. *Let X be a Banach space which is a real uniformly convex and satisfies the Opial's condition. Let C be a closed convex subset of X with $C \neq \emptyset$. Let $\preceq$ be a compatible preorder on X. Let T_1, $T_2 : C \to C$ be two monotone nonexpansive mappings with $F := F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that C has the Property A. Let $\{x_n\}$ and $\{y_n\}$ be sequences defined by the recursion (3.1). Assume that $x_0 \preceq y_0$, $y_0 \preceq x_0$, $x_0 \preceq T_1y_0$, $T_1y_0 \preceq x_0$, $y_0 \preceq T_2x_0$, $y_0 \preceq y_0$, $y_0 \preceq T_1y_0$, $T_1y_0 \preceq y_0$, $y_0 \preceq T_2y_0$, $T_1x_0 \preceq x_0$ and $T_2x_0 \preceq x_0$. Suppose that $z \preceq x_n$ and $z \preceq y_n$ for all $n \in \mathbb{N}$ and all $z \in F$. Then $\{x_n\}$ converges weakly to a common fixed point in F.*

Proof. Let $G = (V(G), E(G))$ be the directed graph generated by $\preceq$. By Proposition 5.1, we obtain $E(G)$ is convex. Then the result is obtained directly by Theorem 3.8. □

Theorem 5.4. *Let X be a Banach space which is a real uniformly convex and C be a closed convex subset of X with $C \neq \emptyset$. Let $\preceq$ be a compatible preorder on X. Let T_1, $T_2 : C \to C$ be two monotone nonexpansive mappings with $F := F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that C has the Property A and one of T_1 and T_2 is semicompact. Let $\{x_n\}$ and $\{y_n\}$ be sequences defined by the recursion (3.1). Assume that $x_0 \preceq y_0$, $y_0 \preceq x_0$, $x_0 \preceq T_1y_0$, $T_1y_0 \preceq x_0$, $y_0 \preceq T_2x_0$, $y_0 \preceq y_0$, $y_0 \preceq T_1y_0$, $T_1y_0 \preceq y_0$, $y_0 \preceq T_2y_0$, $T_1x_0 \preceq x_0$ and $T_2x_0 \preceq x_0$. Suppose that $z \preceq x_n$ and $z \preceq y_n$ for all $n \in \mathbb{N}$ and all $z \in F$. Then $\{x_n\}$ converges strongly to $x \in F$.*

Proof. Let $G = (V(G), E(G))$ be the directed graph generated by $\preceq$. By Proposition 5.1, we obtain $E(G)$ is convex. Then the result is obtained directly by Theorem 3.7. □

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